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# A NOTE ON THE CONDITIONAL EXPECTATIONS

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It seems reasonable to expect that the conditional expectation of a function $h(\xi, \eta)$ (where $\xi$ and $\eta$ are random variables) under the hypothesis that $\eta = y$ (where $y$ is a fixed number) is the same, as the conditional expectation of the function $h(\xi, y)$ under the hypothesis that $\eta = y$.

This assertion is trivial, when the probability of the event $\eta = y$ is not zero, but in the contrary case, this assertion requires some modification, in order to get the precise meaning, and the validity of the assertion so modified is not so obvious.

We shall prove three theorems of this kind under some conditions on the basic spaces, that are satisfied when $X$ and $Y$ are $n_1$ resp. $n_2$ dimensional Euklidean spaces and $\mathbf{X}$ and $\mathbf{Y}$ are $\sigma$-algebras of the *Borel* sets (for the meaning of the symbols $X$, $\mathbf{X}$, $Y$, $\mathbf{Y}$ see later).

In our consideration we use some of the principal properties of the conditional expectations (defined by *Kolmogorov* 1933), which the reader can find in the books of *Halmos* (Measure theory, New York, 1950) and *Doob* (Stochastic processes, New York, 1953).

Troughout the discussion, let $(X, \mathbf{X}, \mu)$, $(Y, \mathbf{Y}, \nu)$ be probability spaces, $T$ a measurable transformation of $X$ into $Y$ (i. e. $B \in \mathbf{Y} \Rightarrow T^{-1}(B) \in \mathbf{X}$) and $\nu(B) = \mu(T^{-1}(B))$ for every $B \in \mathbf{Y}$. By $c_A$ we denote the characteristic function of the set $A$. When a function $h$ is defined on a cartesian product $A \times B$, then $h(x, y)$ is the value of $h$ at the point $[x, y] \in A \times B$, for $y \in B$ $h(*, y)$ is the function defined on $A$, which acquires at the point $x \in A$ the value $h(x, y)$; similarly for $h(x, *)$. When a function $g$ is defined on $Y$, then we denote $gT$ the function defined on $X$ by the relation $gT(x) = g(T(x))$.

If $f$ is an integrable function defined on $X$, then the set of all measurable functions $g$ on $Y$, which satisfies the condition

$$B \in \mathbf{Y} \Rightarrow \int_{T^{-1}(B)} f \, d\mu = \int_B g \, d\nu \tag{1}$$

is not empty; we call it the conditional expectation of $f$ relative to $T$ and denote $e_T(f)$.

When $f = c_A$ is the characteristic function of $A \in \mathbf{X}$, we call $e_T(c_A)$ the conditional probability of $A$ relative to $T$ and we write $e_T(c_A) = p_T(A)$.

If $g_1 \in e_T(f)$, then $(g_2 \in e_T(f) \Longleftrightarrow g_1(y) = g_2(y)$ for almost all $y$ in $Y$ and $g_2$ s measurable $(\mathbf{Y})$).

In this note we suppose, that there exists a function $P_T$, defined on $\mathbf{X} \times Y$, such that:

(2) for every $A \in \mathbf{X}$, $P_T(A, *)$ as a function defined on $Y$ satisfies $P_T(A, *) \in \in p_T(A)$

(3) for every $y \in Y$ $P_T(*, y)$ as a set function is a probability on $\mathbf{X}$.

(For the existence of such a function $P_T$, it is sufficient but not necessary, that $X$ be the $n$-dimensional Euclidean space and $\mathbf{X}$ the class of all Borel sets. [See Doob]).

It is known, that if $f$ is an integrable function on $X$, $g \in e_T(f)$, then $g(y) = = \int_X f \, \mathrm{d}P(*, y)$ for almost all $y \in Y$.

**Lemma.** *Let $B \in \mathbf{Y}$. Then there exists a set $N_B \in \mathbf{Y}$, $\nu(N_B) = 0$ and for every $y \in Y - N_B$*

$$P_T(T^{-1}(B), y) = c_B(y)\,, \tag{4}$$

*where $c_B$ is the characteristic function of $B$.*

Proof: By the definition of $P_T$ [see (1) and (2)]

$$C \in \mathbf{Y} \Rightarrow \int_{T^{-1}(C)} c_{T^{-1}(B)} \, \mathrm{d}\mu = \int_C P_T(T^{-1}(B), *) \, \mathrm{d}\nu$$

but

$$\int_{T^{-1}(C)} c_{T^{-1}(B)} \, \mathrm{d}\mu = \mu(T^{-1}(C) \cap T^{-1}(B)) = \nu(C \cap B) = \int_C c_B \, \mathrm{d}\nu$$

consequently

$$C \in \mathbf{Y} \Rightarrow \int_C c_B \, \mathrm{d}\nu = \int_C P_T(T^{-1}(B), *) \, \mathrm{d}\nu\,.$$

Hence follows, that $P_T(T^{-1}(B), y) = c_B(y)$ for almost all $y \in Y$, q. e. d.

**Theorem I.** *Suppose there exists a denumerable basis $\mathfrak{N}$ of the $\sigma$-algebra $\mathbf{Y}$. Then there exists a set $N \in \mathbf{Y}$, $\nu(N) = 0$ and for every $B \in \mathbf{Y}$ and every $y \in Y - N$.*

$$P_T(T^{-1}(B), y) = c_B(y) \tag{5}$$

Proof: Let $N = \bigcup_{B \in \mathfrak{N}} N_B$, where $N_B$ are sets from the preceding lemma. Clearly $\nu(N) = 0$.

Let $y \in Y - N$. Then $P_T(T^{-1}(*), y)$ and $c_*(y)$ are measures on $\mathbf{Y}$ and (5) holds for every $B \in \mathfrak{N}$. But since $\mathfrak{N}$ is the basis of $\mathbf{Y}$, (5) holds for every $B \in \mathbf{Y}$, q. e. d.

**Theorem II.** *Suppose there exists a denumerable basis of the $\sigma$-algebra $\mathbf{Y}$. Let for every $y \in Y$ the set $\{y\}$ containing one single element $y$ be measurable ($\mathbf{Y}$).*

*Let $h$ be a function defined on the cartesian product $X \times Y$. Let, for every $y \in Y$, the function $h(*, y)$ be integrable over $X$, and the function $h(*, T(*))$ be integrable over $X$.*

*Then, if* $g \in e_T(h(*, T(*))$, *so*

$$g(y) = \int_X h(*, y)\, \mathrm{d}P_T(*, y) \tag{6}$$

*for almost all* $y \in Y$.

Proof: Let $N$ be the set from the preceding theorem, i. e. $\nu(N) = 0$, and $P_T(T^{-1}(B), y) = c_B(y)$ for every $B \in \mathbf{Y}$ and for every $y \in Y - N$. Therefore, for every $y \in Y - N$, since $\{y\} \in \mathbf{Y}$,

$$P_T(T^{-1}(\{y\}), y) = 1$$

and

$$P_T(X - T^{-1}(\{y\}), y) = 0\,. \tag{7}$$

Now, since $g \in e_T(h(*, T(*))$, there exists a set $M \in \mathbf{Y}$, $\nu(M) = 0$ and

$$g(y) = \int_X h(*, T(*))\, \mathrm{d}P_T(*, y) \text{ for every } y \in Y - M.$$

Thus, for $y \in Y - M - N$,

$$g(y) = \int_X h(*, T(*))\, \mathrm{d}P_T(*, y) = \int_{T^{-1}(\{y\})} h(*, T(*))\, \mathrm{d}P_T(*, y) =$$

$$= \int_{T^{-1}(\{y\})} h(*, y)\, \mathrm{d}P_T(*, y) = \int_X h(*, y)\, \mathrm{d}P_T(*, y)\,,$$

and the theorem is proved, since $\nu(M \cup N) = 0$.

**Theorem III.** *Let* $(Y, \mathbf{Y}, \nu)$ *be the probability spaces with the same properties, as in the preceding theorem. Let* $(Y_1, \mathbf{Y}_1, \nu_1)$ *be a probability space,* $\mathbf{Y}_1$ *a* $\sigma$*-algebra with a denumerable basis* $\mathfrak{M}$, $T_1$ *a measurable transformation of* $X$ *into* $Y_1$, $\nu_1(C) = \mu(T_1^{-1}(C))$ *for every* $C \in \mathbf{Y}_1$.

*Let* $h$ *be a function defined on* $Y_1 \times Y$. *Let for every* $y \in Y$ *the function* $h(*, y)$ *be integrable over* $Y_1$ *and the function* $h(T_1(*), T(*))$ *integrable over* $X$.

*Let* $T_1$ *and* $T$ *be independent (i. e.* $B \in \mathbf{Y}$, $C \in \mathbf{Y}_1 \Rightarrow \mu(T^{-1}(B) \cap T_1^{-1}(C)) = \nu(B)\, \nu_1(C)$).

*Then, if* $g \in e_T(h(T_1(*), T(*)))$, *then*

$$g(y) = \int_X h(T_1(*), y)\, \mathrm{d}\mu$$

*for almost all* $y \in Y$.

Proof: (i) We observe first, that if $f$ is a function integrable over $X$ a independent of $T$, then $\int_X f\, \mathrm{d}\mu \in e_T(f)$. (See Halmos, Doob).

(ii) Further, when $g$ is a function integrable over $Y_1$, so is the function $gT_1$ integrable over $X$ and for every $C \in \mathbf{Y}_1$

$$\int_{T_1^{-1}(C)} gT_1\, \mathrm{d}\mu = \int_C g\, \mathrm{d}\nu_1\,. \quad \text{(See Halmos).}$$

Consequently, the function $h(T_1(*), *)$ satisfies the conditions of the preceding theorem and we thus have

$$g(y) = \int_X h(T_1(*), y)\, \mathrm{d}P_T(*, y) \quad \text{for} \quad y \in Y - M \quad \text{and} \quad \nu(M) = 0\,. \tag{8}$$

Let now $D \in \mathfrak{M}$. We have

$$P_T(T_1^{-1}(D), *) \in p_T(T_1^{-1}(D)) = e_T(c_{T_1^{-1}(D)}) \tag{9}$$

and (since $c_{T_1^{-1}(D)} = c_D\, T_1$)

$$P_T(T_1^{-1}(D), *) \in e_T(c_D T_1)\,. \tag{10}$$

Further, since $c_D T_1$ is a function independent of $T$, we get from (i), that

$$\nu_1(D) = \mu(T_1^{-1}(D)) = \int_X c_{T_1^{-1}(D)}\, \mathrm{d}\mu = \int_X c_D T_1\, \mathrm{d}\mu \in e_T(c_D T_1)\,.$$

Hence and from (10) it follows, that $P_T(T_1^{-1}(D), y) = \nu_1(D)$ for $y \in Y - N_D$, where $N_D \in \mathbf{Y}$, $\nu(N_D) = 0$. Put $N = \bigcup_{D \in \mathfrak{M}} N_D$. Then $\nu(N) = 0$ and $P_T(T_1^{-1}(D), y) = \nu_1(D)$ for every $D$ and every $y \in Y - N$. $\mathfrak{M}$ however is the basis of $\mathbf{Y}_1$, $\nu_1$ and $P_T(T_1^{-1}(*), y)$ are measures on $\mathbf{Y}_1$ and so

$$\nu_1(D) = P_T(T_1^{-1}(D), y) \quad \text{for every} \quad D \in \mathbf{Y}_1, y \in Y - N\,. \tag{11}$$

Consequently, applying (ii) and (8), we see that for every $y \in Y - N - M$

$$g(y) = \int_X h(T_1(*), y)\, \mathrm{d}P_T(*, y) = \int_{Y_1} h(*, y)\, \mathrm{d}P_T(T_1^{-1}(*), y) =$$
$$= \int_{Y_1} h(*, y)\, \mathrm{d}\nu_1 = \int_X h(T_1(*)\, y)\, \mathrm{d}\mu\,.$$

The theorem is proved, since $\nu(N \cup M) = 0$.


Резюме

## ЗАМЕТКА К УСЛОВНЫМ МАТЕМАТИЧЕСКИМ ОЖИДАНИЯМ

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Интуитивно можно ожидать, что условное математическое ожидание функции $h(\xi, \eta)$ (где $\xi$ и $\eta$ — случайные величины) при условии $\eta = y$ (где $y$ — фиксированное число), будет равным условному математическому ожиданию функции $h(\xi, y)$ при условии $\eta = y$.

Это утверждение становится тривиальным, если вероятность явления $\eta = y$ положительна, но в противном случае утверждение требует некоторых дополнительных пояснений, чтобы оно имело вообще точный смысл,

а справедливость переделанного таким образом утверждения уже не столь очевидна.

В статье доказаны три теоремы такого характера при достаточно общих условиях, налагаемых на основные пространства; эти условия выполняются, если $X$ и $Y$ являются $n_1$- или $n_2$-мерными евклидовыми пространствами и **X** и **Y** — $\sigma$-алгебры борелевских множеств.

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