Josef Kolomý

New methods for solving linear functional equations with bounded operators

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NEW METHODS FOR SOLVING LINEAR FUNCTIONAL EQUATIONS WITH BOUNDED OPERATORS

JOSEF KOLOMÝ, Praha

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1. In this note some new methods concerning the solution of the functional equations with linear bounded operators in Hilbert spaces are given.

Let X be real or complex Hilbert space, $A : X \to X$ a linear continuous operator from X into X. We shall solve the equation

$$Ax = f \tag{1}$$

by the successive approximations

$$x_{n+1} = Pf + \beta_n(I - PA)\,x_n\,, \quad (n = 0, 1, 2, \ldots)\,, \tag{2}$$

where P is a linear continuous mapping of X into X having a bounded inverse P^{-1}. The parameters β_n $(n = 0, 1, 2, \ldots)$ are to be determined either from

$$\|f - \beta_n A x_n\|^2 = \text{Min} \tag{3}$$

or from the conditions

$$\|f - A x_{n+1}\|^2 = \text{Min}\,, \quad (n = 0, 1, 2, \ldots)\,. \tag{4}$$

From (3), (4) we obtain that

$$\beta_n = \frac{\text{Re}\,(f, Ax_n)}{\|Ax_n\|^2}\,, \tag{5}$$

$$\tilde{\beta}_n = \frac{\text{Re}\,(Lf, LAx_n)}{\|LAx_n\|^2}\,, \tag{6}$$

where $L = I - PA$. The following theorem is valid:

Theorem 1. ([1], [2].) *Let A, P be linear bounded commutative operators in X such that P^{-1} exists and is bounded, and $\|I - PA\| = q < 1$. Then the equation (1) has a unique solution x^* in X. The sequences $\{x_n\}$, $\{\tilde{x}_n\}$ defined by (2), (5); (2), (6) converge in the norm topology of X to the solution x^* of (1) and their errors are bounded by*

$$\|x^* - x_n\| \leqq kq\|f - Ax_{n-1}\|, \quad \|x^* - \tilde{x}_n\| \leqq kq\|f - A\tilde{x}_{n-1}\|,$$

where $k = \|A^{-1}\| \leqq \|P\|/(1 - q)$.

Let us set $A = I - \lambda K$, where K is a linear bounded operator from X into X, λ is a complex parameter.

Theorem 2. *Let one of the following conditions be fulfilled:*

1) $P = I$, $\|\lambda K\| < 1$.

2) $P = \vartheta I$, *A is a self-adjoint operator in X,* $mI \leqq A \leqq MI$, $0 < m \leqq M$, $\vartheta = 2/(M + m)$, *where* $m = \inf_{\|x\|=1} (Ax, x)$, $M = \sup_{\|x\|=1} (Ax, x)$.

3) $P = \vartheta I$, $\operatorname{Re}(Ax, x) \geqq m\|x\|^2$ *for every* $x \in X$, $(m > 0)$ *and* $0 < \vartheta < 2m/\|A\|^2$.

4) $P = \vartheta^2(I - \bar{\lambda}K^*)$, *where $\bar{\lambda}$ is the complex conjugate number to λ, K^* is an adjoint operator to K, K is normal,* $\|Ax\| \geqq k\|x\|$ *holds for every* $x \in X$, $(k > 0)$ *and* $0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2$.

5) $P = I + J$, *where J is a linear bounded operator in X, commutative with K and such that* $\|\lambda G - J\| < 1/(1 + \|\lambda K\|)$, *where G is the resolvent operator of K.*

Then the equation (1) has a unique solution x^* in X. The sequences $\{x_n\}$, $\{\tilde{x}_n\}$ converge in the norm topology of X to x^* at least with speed of a geometric sequence.

Proof. 1) It follows immediately from Theorem 1.

2) Assuming that $P = \vartheta I$, $\vartheta = 2/(M + m)$, we shall prove $\|I - PA\| = q < 1$. The operator $I - PA$ is self-adjoint, its upper bound is $1 - \vartheta m$ lower one $1 - \vartheta M$. The number ϑ is to be determined from the condition that the norm $\|I - \vartheta A\|$ is to assume its minimum value, i.e. from the equality $1 - \vartheta m = -(1 - \vartheta M)$.

Then

$$\|I - \vartheta A\| = 1 - \vartheta m = \vartheta M - 1 = (M - m)/(M + m) < 1.$$

3) We have

$$\|I - \vartheta A\|^2 = \sup_{\|x\|=1} (x - \vartheta Ax, x - \vartheta Ax) \leqq$$

$$\leqq \sup_{\|x\|=1} (1 - 2\vartheta \operatorname{Re}(Ax, x) + \vartheta^2\|A\|^2) \leqq 1 - 2m\vartheta + \vartheta^2\|A\|^2.$$

Let us put $f(\vartheta) = 1 - 2m\vartheta + \vartheta^2\|A\|^2$. Then $f(0) = 1, f(2m/\|A\|^2) = 1$. The function $f(\vartheta)$ assumes for $\vartheta_0 = m/\|A\|^2$ the minimum value $f(\vartheta_0) = 1 - m^2/\|A\|^2$. It is evident that $f(\vartheta) < 1$ for every $\vartheta \in (0, 2m/\|A\|^2)$. Hence $\|I - \vartheta A\| < 1$ for every $\vartheta \in (0, 2m/\|A\|^2)$.

4) We have $\|Ax\| \geqq k\|x\|$ for every $x \in X$, $(k > 0)$. Hence there exist the bounded operators $A^{-1}, (A^{-1})^*, (A^*)^{-1}$ and $(A^{-1})^* = (A^*)^{-1}$. Therefore P has the bounded inverse P^{-1}. Because K is normal, then A is normal and hence A is commutative with P. From the definition of the norm in X it follows that we have to prove

$$\sup_{\|x\|=1} (\vartheta^2\|A^*A\|^2 - 2\|Ax\|^2) < 0\,.$$

Because $\|A^*A\| = \|AA^*\| = \|A\|^2$, the above inequality will be satisfied if $\|A\|^4\vartheta^2 - 2k^2 < 0$. But this inequality is fulfilled, since

$$0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2 \leqq k\sqrt{2}/\|A\|^2\,.$$

5) We shall prove that $q = \|I - (I + J)A\| < 1$. According to the definition of the resolvent operator G we conclude that

$$PA = [(I + \lambda G) - (\lambda G - J)](I - \lambda K) = I - (\lambda G - J)A\,.$$

Hence

$$q = \|I - (I + J)A\| = \|(\lambda G - J)A\| \leqq \|\lambda G - J\|\,\|A\| \leqq$$
$$\leqq \|\lambda G - J\|(1 + \|\lambda K\|) < 1\,.$$

To show that $P^{-1} = (I + J)^{-1}$ exists and is bounded we use the following lemma: Let T_0, T_1 be linear continuous mappings of X into X such that T_0^{-1} exists and $\|T_1\| < 1/\|T_0^{-1}\|$. Then $T = T_0 + T_1$ has an inverse T^{-1} and

$$\|T^{-1}\| \leqq \|T_0^{-1}\|(1 - \|T_0^{-1}\|\,\|T_1\|)^{-1}, \quad (\|T_0^{-1}\| < +\infty)\,.$$

That P^{-1} exists is now clear. It is sufficient to set $T_0 = I + \lambda G$, $T_1 = P - (I + \lambda G) = J - \lambda G$. This concludes the proof.

Remark. It is useful to estimate the number q. The theorems 1,2 are valid if the number q is replaced by q' such that $q \leqq q' < 1$. We may set in the theorem 2: 1) $q' = \|\lambda K\|$, 2) $q' = (M - m)/(M + m)$, 3) if $\vartheta = m/\|A\|^2$, then $q' = (1 - (m/\|A\|)^2)^{\frac{1}{2}}$, 4) $q' = (1 - 2\vartheta^2k^2 + \vartheta^4\|A\|^4)^{\frac{1}{2}}$, 5) $q' = \|\lambda G - J\|\,\|I - \lambda K\|$.

2. Under the assumptions of Theorem 1, let A be a self-adjoint operator in X. Let us denote by x^* the unique solution of (1) in X. We shall solve the equation (1) by an iterative process (2). The real parameters β_n $(n = 0, 1, 2, \ldots)$ are to be determined now either from

$$\|x^* - \beta_n x_n\|^2 = \text{Min}\,, \tag{7}$$

or from

$$(8) \qquad \|x^* - x_{n+1}\|^2 = \text{Min}\,, \quad (n = 0, 1, 2, \ldots)\,.$$

First of all we shall determine β_n $(n = 0, 1, 2, \ldots)$ from the conditions (7). According to (2)

$$\|x^* - \beta_n x_n\|^2 = \|x^* - x_{n+1} + Pf - \beta_n PAx_n\|^2 =$$
$$= \|x^* - x_{n+1} + Pf\|^2 - 2\beta_n \operatorname{Re}(x^* - x_{n+1} + Pf, PAx_n) + \beta_n^2 \|PAx_n\|^2\,.$$

From (7) it follows that

$$\beta_n = \frac{\operatorname{Re}(x^* - x_{n+1} + Pf, PAx_n)}{\|PAx_n\|^2}\,.$$

Hence from the above equality and according to (2) we have

$$\beta_n \|PAx_n\|^2 = \operatorname{Re}(x^* - Pf - \beta_n x_n + \beta_n PAx_n + Pf, PAx_n) =$$
$$= \operatorname{Re}(x^*, PAx_n) - \beta_n \operatorname{Re}(x_n, PAx_n) + \beta_n \|PAx_n\|^2\,.$$

Therefore

$$\beta_n = \frac{\operatorname{Re}(x^*, PAx_n)}{\operatorname{Re}(x_n, PAx_n)}\,.$$

The operators A, P are commutative and A is self-adjoint. Hence

$$(9) \qquad \beta_n = \frac{\operatorname{Re}(f, Px_n)}{\operatorname{Re}(x_n, PAx_n)}\,.$$

Now we shall determinate β_n $(n = 0, 1, 2, \ldots)$ from the conditions (8). From (2) we obtain

$$\|x^* - x_{n+1}\|^2 = \|L(x^* - \beta_n x_n)\|^2 = L(x^* - x_{n+1} + Pf - \beta_n PAx_n)\|^2 =$$
$$= \|L(x^* - x_{n+1} + Pf) - \beta_n LPAx_n\|^2 =$$
$$= \|L(x^* - x_{n+1} + Pf)\|^2 - 2\beta_n \operatorname{Re}(L(x^* - x_{n+1} + Pf), LPAx_n) +$$
$$+ \beta_n^2 \|LPAx_n\|^2\,,$$

where $L = I - PA$. From (8) we get

$$\operatorname{Re}(L(x^* - x_{n+1} + Pf), LPAx_n) = \beta_n \|LPAx_n\|^2\,.$$

From the above equality and from (2) we see that

$$(10) \qquad \beta_n = \frac{\operatorname{Re}(Lf, PLx_n)}{\operatorname{Re}(Lx_n, PLAx_n)}\,.$$

The formulae (2), (9); (2), (10) give new methods for solving linear functional equations in Hilbert space X. The methods (2), (9); (2), (10) are more simple for computation in comparison with (2), (5); (2), (6). In a real Hilbert space X formulae (9), (10) are as follows:

$$\beta_n = (f, P_n)(x_n, PAx_n)^{-1}, \quad \beta_n = (Lf, PLx_n)(Lx_n, PLAx_n)^{-1}.$$

Taking $P = \vartheta I$, where $\vartheta > 0$, then formulae (9), (10) have the simple form:

$$\beta_n = \frac{(f, x_n)}{(x_n, Ax_n)}, \quad \beta_n = \frac{(Lf, Lx_n)}{(Lx_n, LAx_n)}.$$

Theorem 3. *Under the assumptions of Theorem* 1, *let A be a self-adjoint operator in X. Then the equation* (1) *has a unique solution x^* in X. The sequences $\{x_n\}$, $\{\tilde{x}_n\}$ defined by* (2), (9); (2), (10) *are convergent in the norm topology of X to the solution x^* at least with the speed of a geometric sequence.*

Proof. Because

$$A^{-1} = \sum_{j=0}^{\infty} (I - PA)^j P,$$

then $\|A^{-1}\| \leqq \|P\|/(1 - q)$. Hence the equation (1) has a unique solution x^* in X. We shall prove that $\|x^* - x_n\| \to 0$ whenever $n \to \infty$. According to (7) we conclude that $\|\beta_n x_n - x^*\| \leqq \|x^* - x_n\|$ for every n ($n = 0, 1, 2, \ldots$). Further

$$\begin{aligned} \|x^* - x_{n+1}\| &= \|Pf + \beta_n x_n - \beta_n PAx_n - x^*\| = \\ &= \|PAx^* - x^* + \beta_n x_n - \beta_n PAx_n\| = \\ &= \|(I - PA)(x^* - \beta_n x_n)\| \leqq q\|x^* - \beta_n x_n\| \leqq q\|x^* - x_n\|. \end{aligned}$$

Now it is evident that $\|x^* - x_n\| \to 0$ whenever $n \to \infty$. We shall show that $\|x^* - \tilde{x}_n\| \to 0$ when $n \to \infty$. Since the following equalities

$$F(\tilde{x}_{n+1}) = \|x^* - \tilde{x}_{n+1}\|^2 = \|L(x^* - \beta_n \tilde{x}_n)\|^2$$

hold (β_n is defined by the equality (10)), the element $\tilde{x}_{n+1}$ defined by (2), (10) minimizes the functional $F(x) = \|x^* - x\|^2$ on the set of all the elements of the form $Pf + \alpha L\tilde{x}_n$, $\alpha \in \mathfrak{R}$ (where $\mathfrak{R}$ is the set of all real numbers), then the element $\beta_n \tilde{x}_n$ gives the minimum value of $\tilde{F}(x) = \|L(x^* - x)\|^2$ on the set of all points $\alpha\tilde{x}_n$, where $\alpha \in \mathfrak{R}$. Hence

$$F(\tilde{x}_{n+1}) = \tilde{F}(\beta_n \tilde{x}_n) = \min_{\alpha \in \mathfrak{R}} \tilde{F}(\alpha \tilde{x}_n) \leqq \tilde{F}(\tilde{x}_n).$$

Observe that

$$\tilde{F}(\tilde{x}_n) = \|L(x^* - \tilde{x}_n)\|^2 \leqq q^2 \|x^* - \tilde{x}_n\|^2.$$

Then, clearly, $\|x^* - \tilde{x}_{n+1}\| \leqq q\|x^* - \tilde{x}_n\|$. Thus $\|x^* - \tilde{x}_n\| \to 0$ whenever $n \to \infty$. The proof is complete.

The analogical theorem to theorem 2 is valid. Let us set again $A = I - \lambda K$, where K is a linear bounded operator from X into X, λ is a complex parameter generally.

Theorem 4. *Let one of the following conditions be fulfilled:*

1) $P = I$, $\|\lambda K\| < 1$, *K is a self-adjoint operator.*

2) $P = \vartheta I$, *A is self-adjoint,* $mI \leqq A \leqq MI$, $0 < m \leqq M$, *where* $m = \inf_{\|x\|=1} (Ax, x)$, $M = \sup_{\|x\|=1} (Ax, x)$, $\vartheta = 2/(M + m)$.

3) $P = \vartheta I$, *A is self-adjoint,* $(Ax, x) \geqq m\|x\|^2$ *holds for every* $x \in X$, $(m > 0)$ *and* $0 < \vartheta < 2m/\|A\|^2$.

4) $P = \vartheta^2(I - \bar{\lambda}K^*)$, *where $\bar{\lambda}$ is a complex conjugate number to λ, K^* is an adjoint operator to normal mapping K,* $\|Ax\| \geqq k\|x\|$ *holds for every* $x \in X$, $(k > 0)$ *and* $0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2$.

5) $P = I + J$, *where J is a linear bounded operator in X, commutative with K and such that* $\|\lambda G - J\| < 1/(1 + \|\lambda K\|)$ *holds, where G is the resolvent operator of the self-adjoint mapping K.*

Then the equation (1) *has a unique solution x^* in X. The sequences* $\{x_n\}$, $\{\tilde{x}_n\}$ *defined by* (2), (9); (2), (10) *respectively, converge in the norm topology of X to x^* at least with the speed of the geometric sequence.*

Remark. If $x_n \to x^*$ (or $\tilde{x}_n \to x^*$) in the norm topology of X, then the sequences $\{\beta_n\}$ defined by (9), (10) respectively converge to one, i.e. $\lim_{n\to\infty} \beta_n = 1$. It follows immediately from the continuity of the inner product of X. Under the assumptions of theorem 3, $F(\beta_n x_n) \searrow 0$, $F(\tilde{x}_n \beta_n) \searrow 0$ (here β_n is defined by (10)). In fact

$$F(x_{n+1}) \leqq q^2\, F(\beta_n x_n) \leqq q^2\, F(x_n) < F(x_n)\,,$$
$$F(\beta_n x_n) \leqq q^2\, F(\beta_{n-1} x_{n-1}) < F(\beta_{n-1} x_{n-1})\,, \quad (n = 0, 1, 2, \ldots)\,.$$

3. The purpose of this paragraph is to remove the condition that A be a self-adjoint operator in X. We shall solve again the equation (1) with linear bounded operator A. Let us suppose that A maps X into itself. The equation (1) is equivalent to the equation

$$B_1 x = f_1\,, \tag{11}$$

where $B_1 = A^*A$, $f_1 = A^*f$. But the equation (11) is equivalent to

$$x = (I - \vartheta B_1)\, x + \vartheta f_1\,, \tag{12}$$

where ϑ is an arbitrary positive number. Suppose that A^{-1} exists and is bounded operator in X. Let us denote by x^* the unique solution of (1). Hence x^* is the unique solution of (12) in X. We solve (12) by an iterative process

$$x_{n+1} = \vartheta f_1 + \alpha_n(I - \vartheta B_1)\, x_n\,, \tag{13}$$

where the parameters α_n $(n = 0, 1, 2, \ldots)$ are to be determined from the conditions that $\|x^* - \alpha_n x_n\|^2 = \text{Min}$. Since B_1 is self-adjoint, then according to (9)

$$\alpha_n = \frac{\operatorname{Re}(f_1, x_n)}{(x_n, B_1 x_n)}\,. \tag{14}$$

Finally we have

$$\alpha_n = \frac{\operatorname{Re}(f, Ax_n)}{\|Ax_n\|^2}\,. \tag{15}$$

It is easy to show now that the following theorem is valid.

Theorem 5. *Let A be a linear bounded operator which maps X into itself such that A^{-1} exists and is bounded in X. Furthermore let the inequality $0 < \vartheta < 1/\|A\|^2$ be fulfilled. Define the sequence $\{x_n\}$ as follows:*

$$x_{n+1} = \vartheta A^* f + \alpha_n(I - \vartheta A^* A)\, x_n\,,$$
$$\alpha_n = \operatorname{Re}(f, Ax_n)/\|Ax_n\|^2\,, \quad (n = 0, 1, 2, \ldots)\,.$$

Then $\|x^* - x_n\| \to 0$ whenever $n \to \infty$, $\|x^* - x_n\| \leqq q^n(1 - q)^{-1}\,\|x_1 - x_0\|$, where $q = \|I - \vartheta A^* A\|$. Moreover $\|x_{n+1} - x^*\| < \|x_n - x^*\|$ for every n $(n = 0, 1, 2, \ldots)$.

Proof. Since A has the bounded A^{-1}, then there exists a positive constant k such that $\|Ax\| \geqq k\|x\|$ holds for every $x \in X$. Because $I - \vartheta A^* A$ is self-adjoint, then

$$\|I - \vartheta A^* A\| = \sup_{\|x\|=1} |((I - \vartheta A^* A)\, x, x)| = \sup_{\|x\|=1} |\|x\|^2 - \vartheta\|Ax\|^2| =$$
$$= \sup_{\|x\|=1} (1 - \vartheta\|Ax\|^2)\,,$$

since

$$1 - \vartheta\|Ax\|^2 \geqq 1 - \vartheta\|A\|^2\,\|x\|^2 = 1 - \vartheta\|A\|^2 > 0$$

for every $x \in X$ with $\|x\| = 1$. Further

$$q = \|I - \vartheta A^* A\| \leqq \sup_{\|x\|=1} (1 - k^2\vartheta\|x\|^2) = 1 - \vartheta k^2 < 1\,,$$

since $0 < \vartheta k^2 < 1$. Hence, we infer that conditions of theorem 3 are satisfied and the theorem is proved.

Remark. Let us solve the equation (1) again by the successive approximations

$$x_{n+1} = \vartheta f_1 + \gamma_n(I - \vartheta B_1)\, x_n\,, \quad (\vartheta > 0,\ P = \vartheta I)\,, \tag{16}$$

where $f_1 = A^*f$, $B_1 = A^*A$. If the constants γ_n $(n = 0, 1, 2, \ldots)$ are to be determined from the conditions that $\|x^* - x_{n+1}\|^2 = \text{Min}$ $(n = 0, 1, 2, \ldots)$ then

$$\gamma_n = \frac{\text{Re}\,(L_1^2\, f_1, x_n)}{\text{Re}\,(B_1 L_1 x_n, L_1 x_n)}, \tag{17}$$

where $L_1 = I - \vartheta B_1$. Under the assumptions of theorem 5, the sequence $\{x_n\}$ defined by (16), (17) is convergent in the norm topology of X to x^* and $\|x^* - x_n\| = O(q^n)$, where $q = \|I - \vartheta A^*A\|$. But formulae (16), (17) are rather complicated for practical computation.

It was shown that the iterative processes (2), (9); (2), (10) are convergent at least with the rate of the geometric sequence with the quotient $q = \|I - PA\|$. Hence it is natural to choose the operator P such that the norm $\|I - PA\|$ assumes its minimum value.

Let $P = \vartheta I$, A is a self-adjoint operator in X, $mI \leqq A \leqq MI$, $m > 0$. The constant ϑ (Theorem 4) is chosen such that $\|I - \vartheta A\|$ catches its minimum value. The lower bound of $I - \vartheta A$ is $1 - \vartheta M$, the upper one is $1 - \vartheta m$. The spectrum of $I - \vartheta A$ lies on the segment $\langle 1 - \vartheta M, 1 - \vartheta m\rangle$. Suppose that B is a linear self-adjoint operator in X with the lower and the upper bounds b, N respectively. Then $\|B\| = \max(|b|, |N|)$. The norm $\|I - \vartheta A\|$ will be smallest, if $1 - \vartheta M = -(1 - \vartheta m)$, i.e. if $\vartheta = 2/(M + m)$. Then $\|I - \vartheta A\| = (M - m)/(M + m)$. Hence the processes (2), (9); (2), (10) are convergent (under the best choice of ϑ) at least with the speed of the geometric sequence with the quotient $q = (M - m)/(M + m)$. The constant ϑ satisfies in 3) of theorem 4 the inequality $0 < \vartheta < 2m/\|A\|^2$. The estimate of the speed of the convergence is best when $\vartheta = \vartheta_0 = m/\|A\|^2$. Then $\|I - \vartheta A\| \leqq [1 - (m/\|A\|^2)^2]^{\frac{1}{2}}$. Taking in 4) of theorem 4 $\vartheta = \vartheta_1 = k\sqrt{2}/(2\|A\|^2)$, then $\|I - \vartheta A\| \leqq [1 - 1/2(k/\|A\|)^4]^{\frac{1}{2}}$. The parameter ϑ of theorem 5 satisfies the inequality $0 < \vartheta < 1/\|A\|^2$. Then the norm $\|I - \vartheta A^*A\|$ will be smaller when ϑ is nearer to $1/\|A\|^2$. The conclusions of theorem 5 remain valid if ϑ satisfies $0 < \vartheta < 2k/\|A\|^4$, where $k > 0$ is the number from the inequality $\|Ax\| \geqq k\|x\|^2$ (the existence of a bounded A^{-1}). The best choice of ϑ is $\vartheta = k/\|A\|^4$. Then $\|I - \vartheta A^*A\| \leqq [1 - (k/\|A\|)^2]^{\frac{1}{2}}$.

Remark. Some results of this paper were published without proofs in [3], [4].

References

[1] *J. Kolomý:* O konvergenci a užití iteračních metod. Čas. pěst. mat. *86* (1961), 148—177.

[2] *J. Kolomý:* K metodě podobné iterace. Čas. pěst. mat. *86* (1961), 308—313.

[3] *J. Kolomý:* Новые методы решения линейных функциональных уравнений в пространствах гильберта. Apl. mat. *10* (1965), 246—248.

[4] *J. Kolomý:* On the solution of functional equations with linear bounded operators. Comment. Math. Univ. Carolinae *6* (1965), 141—143.

[5] *L. B. Rall:* Error bounds for iterative solutions of Fredholm integral equations. Pac. Journ. Math. *5* (1955), 977—986.
[6] *M. A. Красносельский:* О решении методом последовательных приближений уравнений с самосопряженными операторами. Усп. мат. наук *15* (1960), 161—165.

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU)

Резюме

НОВЫЕ МЕТОДЫ РЕШЕНИЯ ЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ С ОГРАНИЧЕННЫМИ ОПЕРАТОРАМИ

ЙОСЕФ КОЛОМЫ (Josef Kolomý), Прага

Пусть дано уравнение (1) в действительном или комплексном гильбертовом пространстве X. Пусть A, P – линейные ограниченные перестановочные операторы в X, и P таков, что существует ограниченный P^{-1} в X и выполнено условие $\|I - PA\| = q < 1$. Пусть, далее, A – самосопряженный оператор в X. Обозначим через x^* единственное в X решение уравнения (1). Последовательные приближения вычисляются по формуле (2), где действительные коэффициенты β_n ($n = 0, 1, 2, \ldots$) определим так, чтобы выполнялись условия (7), (8). Формулы (2), (9); (2), (10) определяют новые итерационные процессы, обеспечивающие на каждомш аге наибольшее уменшение норм ошибок $\|x^* - \beta_n x_n\|$, $\|x^* - x_{n+1}\|$. Если выбрать P подходящим образом (напр., $P = \vartheta I$, где $\vartheta > 0$), то формулы (9), (10) более просты, чем (5), (6), соответственно.

Теорема. *Пусть выполнены выше сформулированные предположения. Тогда последовательности* $\{x_n\}$, $\{\tilde{x}_n\}$, *определенные равенствам* (2), (9); (2), (10) *сходятся по норме X к единственному решению x^* уравнения* (1) *с быстротой геометрической прогрессии* $\{q^n\}$.

В случае несамосопряженного оператора A имеет место следующия теорема.

Теорема. *Пусть A – линейный ограниченный оператор в X такой, что существует ограниченный A^{-1}. Пусть выполнено неравенство* $0 < \vartheta < 1/\|A\|^2$. *Определим последовательность $\{x_n\}$ следующим образом*:

$$x_{n+1} = \vartheta A^* f + \alpha_n (I - \vartheta A^* A)\, x_n\,, \quad \alpha_n = \operatorname{Re}\,(f, Ax_n)/\|Ax_n\|^2\,, \quad (n = 0, 1, 2, \ldots)\,.$$

Тогда $\|x^* - x_n\| \to 0$ $(n \to \infty)$ *и имеет место оценка* $\|x^* - x_n\| \leqq q^n (1 - q)^{-1} \cdot \|x_1 - x_0\|$, *где* $q = \|I - \vartheta A^* A\|$.