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Czechoslovak Mathematical Journal, Vol. 17 (1967), No. 3, 313–316

Persistent URL: <http://dml.cz/dmlcz/100780>

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ON THE POLAR DECOMPOSITION OF SCALAR-TYPE OPERATORS

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(Received October 14, 1965)

In [3] it is proved that for an arbitrary scalar-type operator S we have $S = A + iB$, where A and B are commuting scalar-type operators with a real spectrum and i is the imaginary unit. This decomposition is analogous to the expressing a complex number in the form $\lambda = \alpha + i\beta$, where α and β are real numbers.

The present paper deals with the so-called polar decomposition $S = RU$ of the scalar-type operator S , where R and U are commuting scalar-type operators, R has a real spectrum and the spectrum of U lies on the boundary of the unit circle. Thus, an analog of expressing a complex number in the form $\lambda = \rho e^{i\varphi}$ is under consideration.

The polar decomposition of a normal operator (thus a scalar-type operator) in a Hilbert space is well-known [2].

The concepts and results from the theory of spectral operators used here are published in [1].

Notation. Denote p the complex plane, K the boundary of the unit circle in p , P the set of all non-negative real numbers, and \tilde{P} the set of all positive real numbers. \mathcal{B}_K will be the algebra of Borel sets on K and \mathcal{B}_P the algebra of Borel sets on P .

Definition. The scalar-type operator S is said to be pseudo-unitary, if $\sigma(S) \subset K$. Before proving our theorem we shall prove the following lemma.

Lemma. Let A be a set and \mathcal{B} be an algebra of its subsets. Let E be the spectral measure defined on \mathcal{B} . Let f and g be two bounded \mathcal{B} -measurable complex functions defined on A . Then

$$(1) \quad \int_A f(z) g(z) dE(z) = \int_A f(z) dE(z) \int_A g(z) dE(z).$$

The integral is that defined in [1].

Proof. Let f, g be \mathcal{B} -measurable finite-valued functions on A . Due to the multiplicative property of the spectral measure $E(1)$ is valid.

If f, g are bounded \mathcal{B} -measurable functions on A , then there exist two sequences f_n, g_n of \mathcal{B} -measurable finite-valued functions such that $f_n \rightarrow f, g_n \rightarrow g$ uniformly on A . According to (1) and definition of the integral we have

$$\begin{aligned} \int_A f(z) g(z) dE(z) &= \lim_n \int_A f_n(z) g_n(z) dE(z) = \\ &= \lim_n \int_A f_n(z) dE(z) \int_A g_n(z) dE(z) = \int_A f(z) dE(z) \int_A g(z) dE(z). \end{aligned}$$

Theorem. Let \mathfrak{X} be a Banach space and $S \in B(\mathfrak{X})$ be a scalar type operator with the resolution of identity E .

Then there exist operators R and U such that

- (i) $S = RU$ where R is scalar-type and U is pseudo-unitary
- (ii) R commutes with U and the operators R and U commute with S
- (iii) $\sigma(R) = \{\varrho : \varrho = |\lambda|, \lambda \in \sigma(S)\}$

(iv) if in addition $S^{-1} \in B(\mathfrak{X})$, then $E = E_R \otimes E_U$, where E_R and E_U are resolutions of identity for R and U , respectively. Under this assumption the decomposition $S = RU$ is unique and the spectrum of the operator U is given by

$$\sigma(U) = \{\lambda : \lambda = \eta/|\eta|, \eta \in \sigma(S)\}.$$

Proof. If $\sigma \subset P$ and $\delta \subset K$, then $\sigma \times \delta$ denotes the set of all ordered pairs of numbers (ϱ, η) , where $\varrho \in \sigma, \eta \in \delta$. Since every complex number λ can be written in the form $\lambda = \varrho e^{i\varphi}$, $\sigma \times \delta$ can be identified with the set of complex numbers of the form $\{\lambda : \lambda = \varrho e^{i\varphi}, \varphi \in \sigma, e^{i\varphi} \in \delta\}$. Then the set $\tilde{P} \times K$ represents the entire complex plane without the origin.

We define

$$(2) \quad \begin{aligned} E_R(\sigma) &= E(\sigma \times K) && \text{for } \sigma \in \mathcal{B}_P, \\ E_U(\delta) &= E(\tilde{P} \times \delta) && \text{for } 1 \notin \delta, \delta \in \mathcal{B}_K, \\ E_U(\delta) &= E(\tilde{P} \times \delta) + E(\{0\}) && \text{for } 1 \in \delta, \delta \in \mathcal{B}_K. \end{aligned}$$

Thus, E_K is a spectral measure on \mathcal{B}_P . Let us prove that E_U is a spectral measure on \mathcal{B}_K . We have

$$\begin{aligned} E_U(\emptyset) &= E(\tilde{P} \times \emptyset) = E(\emptyset) = 0, \\ E_U(K) &= E(\tilde{P} \times K) + E(\{0\}) = E(p) = I. \end{aligned}$$

If $\delta_1, \delta_2 \in \mathcal{B}_K$ and $1 \in \delta_1 \cap \delta_2$, then

$$\begin{aligned} E_U(\delta_1 \cap \delta_2) &= E(\tilde{P} \times \delta_1 \cap \delta_2) + E(\{0\}) = E(\tilde{P} \times \delta_1) E(\tilde{P} \times \delta_2) + E(\{0\}) = \\ &= E(\tilde{P} \times \delta_1) E(\tilde{P} \times \delta_2) + E(\{0\}) E(\tilde{P} \times \delta_2) + E(\tilde{P} \times \delta_2) E(\{0\}) + \\ &\quad + E^2(\{0\}) = (E(\tilde{P} \times \delta_1) + E(\{0\}))(E(\tilde{P} \times \delta_2) + E(\{0\})) = \\ &= E_U(\delta_1) E_U(\delta_2). \end{aligned}$$

An analogous calculation shows that E_U has all properties of a spectral measure. The uniform boundedness of E_U on \mathcal{B}_K is obvious.

Define

$$\varphi(\lambda) = \begin{cases} \lambda/|\lambda| & \text{for } \lambda \neq 0, \quad \lambda \in p \\ 1 & \text{for } \lambda = 0 \end{cases}$$

and

$$(3) \quad R = \int_p \varrho = \varrho \, dE_R(\varrho) \quad \text{and} \quad U = \int_K z \, dE_U(z).$$

According to Lemma 6 in [1] R and U are scalar-type operators with resolutions of identity E_R and E_U , respectively. U is pseudo-unitary and $\sigma(R) \subset P$. From the definition of R and U it follows that

$$R = \int_p |\lambda| \, dE(\lambda) \quad \text{and} \quad U = \int_p \varphi(\lambda) \, dE(\lambda).$$

Using the lemma we have

$$\begin{aligned} RU &= \int_p |\lambda| \, dE(\lambda) \int_p \varphi(\lambda) \, dE(\lambda) = \int_{\sigma(S)} |\lambda| \, dE(\lambda) \int_{\sigma(S)} \varphi(\lambda) \, dE(\lambda) = \\ &= \int_{\sigma(S)} |\lambda| \varphi(\lambda) \, dE(\lambda) = \int_{\sigma(S) - \{0\}} |\lambda| \lambda/|\lambda| \, dE(\lambda) + \int_{\{0\}} \lambda \, dE(\lambda) = \\ &= \int_p \lambda \, dE(\lambda) = S. \end{aligned}$$

Therefore (i) is proved. From the lemma there follows the commutativity of the operator R with U and of the operators R and U with S , and hence (ii). To prove (iii) consider that R is a continuous function of S .

Let now $S^{-1} \in B(\mathfrak{X})$. Then $E(\{0\}) = 0$ and in (2) we have $E_U(\delta) = E(\tilde{P} \times \delta)$ for every $\delta \in \mathcal{B}_K$.

$$(4) \quad \begin{aligned} E_R(\sigma) E_U(\delta) &= E(\sigma \times K) E(\tilde{P} \times \delta) = E((\sigma \times K) \cap (\tilde{P} \times \delta)) + E(\{0\}) = \\ &= E((\sigma \cap \tilde{P}) \times \delta) \cup \{0\}) = E(\sigma \times \delta). \end{aligned}$$

Let pairs of operators R_1, U_1 and R_2, U_2 satisfy (i) to (iii) and let their resolutions of identity be E_R, E_U and F_R, F_U , respectively. Due to (4) we have $E = E_R \otimes E_U$ and $E = F_R \otimes F_U$.

If $\sigma \in \mathcal{B}_P$ and $\delta \in \mathcal{B}_K$, the identities

$$E(\sigma \times K) = E_R(\sigma) E_U(K) = E_R(\sigma)$$

$$E(\sigma \times K) = F_R(\sigma) F_U(K) = F_R(\sigma)$$

$$E(P \times \delta) = E_R(P) E_U(\delta) = E_U(\delta)$$

$$E(P \times \delta) = F_R(P) F_U(\delta) = F_U(\delta)$$

are valid. This implies $E_R(\sigma) = F_R(\sigma)$ and $E_U(\delta) = F_U(\delta)$; thus $R_1 = R_2$ and $U_1 = U_2$.

Finally,

$$U = \int_K \lambda dE(\lambda) = \int_P \varphi(\lambda) dE(\lambda) = \int_P \lambda/|\lambda| dE(\lambda).$$

The function $\varphi(\lambda)$ is continuous on some neighborhood of the spectrum of the operator S and therefore $U = \varphi(S)$, $\sigma(U) = \sigma(\varphi(S)) = \varphi(\sigma(S)) = \{\lambda : \lambda = \eta/|\eta|, \eta \in \sigma(S)\}$, and the proof is completed.

Note. As we can see in the proof of the theorem, the operator U need not be uniquely determined if $S^{-1} \notin B(\mathfrak{X})$. Defining the resolution of identity E_U as in the proof of the theorem we have

$$\sigma(U) = \{\lambda : \lambda = \eta/|\eta|, \eta \in \sigma(S) - \{0\}\} \cup \{1\}.$$

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