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A CHARACTERIZATION OF VERY k-SPACES

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We shall be concerned here only with Hausdorff spaces. In this case the definition of a k-space runs as follows:

Definition 1. (See [1], [2].) A topological space X is said to be a *k-space* if and only if all subsets of X having bicompact intersection with an arbitrary bicompact subspace of the space X are closed in X.

Thus the topology of a k-space is completely determined by the array of all bicompact subsets of this space. The class of k-spaces is very wide. Not only metric spaces and locally bicompact spaces belong to this class, but also all G_δ-spaces (i.e. spaces complete in the sense of E. Čech) do.

Unfortunately, a subspace of a k-space need not be a k-space: each completely regular T_1-space can be embedded into a bicompact Hausdorff space, and the latter is surely a k-space. The purpose of this note is to investigate which spaces are "very k-spaces".

Definition 2. A topological space X is said to be a *very k-space* if and only if each subspace of the space X is a k-space.

Remark 1. Obviously, each very k-space X must satisfy the following condition:

(k_1) If M is a subset of X and x is a point such that $x \in [M]$, then there exists a bicompact subspace Φ of the space X such that

$$x \in [\Phi \cap M] .$$

It seems quite natural to expect that this condition characterizes the k-spaces, but this is not true. There are k-spaces which do not satisfy this condition (an example can be found in [3]). For the full treatment of the subject see [4]; a classification of k-spaces, based on condition k_1, is given there.

Remark 2. Here is an obvious reformulation of definition 2.

Proposition 1. *A topological space X is a very k-space if and only if for each subset $M \subset X$ and for each point $x \in [M] \setminus M$ there exists a bicompact subset $\Phi \subset M \cup \{x\}$ such that $x \in [\Phi \setminus \{x\}]$.*

Now we shall state the main theorem.

Theorem 1. *A space X is a very k-space if and only if for each subset $M \subset X$ and for each point $x \in [M]$ there exists a sequence $\{x_n : n = 1, 2, \dots\}$ of points in M such that* $\lim x_n = x$.

Proof. Let $M \subset X$ and let x be any point of the set $[M] \setminus M$. Evidently we can find a set $L \subseteq M$ such that the two following conditions are fulfilled: 1) $x \in [L]$; 2) if $L' \subset M$ and $x \in [L']$, then the cardinality of L is less or equal to the cardinality of L'. Proposition 1 enables us to find a bicompactum $\Phi \subset L \cup \{x\}$ with the property $x \in [\Phi \setminus \{x\}]$. It follows from the choice of the set L that the cardinality of Φ and the cardinality of L are equal. We denote it by τ. Let us show that $\tau = \aleph_0$. Then the theorem will follow. The point x is not isolated in Φ; moreover, the character of the point x in the space Φ is equal to τ. Consider some base $\{U_\alpha : \alpha \in A\}$ of x in Φ, such that card $A = \tau$. We can suppose that the index set A is well ordered as the smallest ordinal corresponding to the cardinal number τ. Now we are in need of some transfinite construction.

Let $O_1 x$ be some neighbourhood of the point x such that $[O_1 x] \subset U_1$ and let x_1 be some point from $O_1 x \setminus \{x\}$. Suppose that we have defined, for all $\alpha < \beta$, $\beta \in A$, neighbourhoods $O_\alpha x$ of the point x as well as points $x_\alpha \in \Phi \setminus \{x\}$. The cardinality of the set $\bigcup_{\alpha<\beta} \{x_\alpha\}$ is less than τ, hence $[\bigcup_{\alpha<\beta} \{x_\alpha\}] \not\ni x$. Take for $O_\beta x$ any neighbourhood of x such that $[\bigcup_{\alpha<\beta} \{x_\alpha\}] \cap [O_\beta x] = \Lambda$ and $[O_\beta x] \subset U_\beta$.

Now, $\bigcap_{\alpha \leqq \beta} O_\alpha x \setminus \{x\} \neq \Lambda$. For the proof we need only to mention that the cardinality of the family $\{O_\alpha x : \alpha \leqq \beta\}$ is less than τ if the character of x in Φ is equal τ. For x_β we choose any point from the set $\bigcap_{\alpha \leqq \beta} O_\alpha x \setminus \{x\}$. In such a way we can define x_α and $O_\alpha x$ for all $\alpha \in A$. Consider the subspace $X^* = \bigcup_{\alpha \in A} \{x_\alpha\} \cup \{x\}$ of the space X. Clearly, x is not isolated in X^*. On the other hand, the set $X \setminus ([\bigcup_{\alpha<\beta} \{x_\alpha\}] \cup [O_{\beta+1} x])$ is a neighbourhood of x_β which does not intersect the set $X^* \setminus \{x_\beta\}$. Hence all points of the set $X^* \setminus \{x\}$ are isolated in X^*. By Proposition 1 we can find a bicompactum F in X^* such that x is a non-isolated point of this bicompactum. Now, $F \setminus \{x\} \subset M$. By the definition of the cardinal number τ, the cardinality of F is equal to τ. Let P be an infinite countable subset of the set $F \setminus \{x\}$. No point of the set $F \setminus \{x\}$ is an accumulation point of this subset. It follows from the bicompactness of F that $[P] \ni x$. Now, $P \subseteq M$. Hence, $\tau = \aleph_0$. The theorem is proved.

Remark 3. In fact, the following general lemma is established by the argument:

Lemma. *Let X be a bicompact space and let x be any point of X. Denote the character of x in X by τ. We shall call the point x "λ-achievable", for some cardinal number λ, iff there exists a set $P \subseteq X \setminus \{x\}$ of the power*[1] *λ such that $x \in [P]$. If x is not λ-achievable for any $\lambda < \tau$, we can find the standard subspace $X^* \subset X$ of the power τ, only one point of which is not isolated in X^*, such that the neighbourhoods of the point in X are complements to arbitrary subsets of cardinality less than τ.*

Remark 4. The topological spaces in which the sequential closure of a set coincides with the closure of this set are called Frechet-Urysohn spaces (FU-spaces). So the theorem established may be formulated as follows: The class of all very k-spaces coincides with the class of all FU-spaces (among Hausdorff spaces!).

Now we will show how very k-spaces are related to metric spaces.

Definition 3. A map $f : X \to Y$ is called *pseudoopen* if for each point $y \in Y$ and for each open neighbourhood U of the set $f^{-1}y$ the interior of the set fU contains y.

In [4] FU-spaces we characterized as pseudoopen continuous images of metric spaces. So we have

Theorem 2. *A topological space X is a very k-space if and only if it is a pseudoopen continuous image of some (locally bicompact) metric space.*

Remark 5. The k_2-spaces [4] have an obvious characterization as pseudoopen continuous images of locally bicompact spaces (see [4]).

From the main result of this paper, together with the main result of [7, § 7], the following theorem can be deduced.

Theorem 3. *Let X be a topological group such that the space of this group is a p-space*[2]*. Then either of the two following conditions is fulfilled:*

1) *X is metrizable;*

2) *X contains a subspace, which is not a k-space.*

Remark 6. This result is new and non-trivial even in the case when the space of the group under consideration is bicompact. In fact, a more general result holds: each dyadic bicompactum in which every subspace is a k-space must be metrizable.

In conclusion we will discuss another phenomena which can occur when dealing with k-spaces. The fact is that the product of two k-spaces need not be a k-space. This may happen even with very k-spaces. Theorem 2 enables us to give an indirect description of a wide class of FU-spaces, which is closed with respect to the product.

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[1]) "The power" means the same as "the cardinality".

[2]) For the definition of a p-space see [5] or [7]. In particular, any space which is G_δ in its bicompactification, as well as any metric space, is a p-space.

The elements of the class are pseudoopen bicompact continuous images of metric spaces. It would be fine to know more about the topological structure of these spaces. I conjecture that all paracompact spaces, belonging to the class, are metrizable. If so, it would be a considerable generalization of the theorem on metrizability of all paracompact spaces which are open continuous bicompact images of metric spaces (see [6]).

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