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A CHARACTERIZATION OF SEMILATTICES OF LEFT OR RIGHT GROUPS

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In this note we give a characterization of right regular or periodic semigroups, which are semilattices of left groups.

Let S be a semigroup. The mapping $\boldsymbol{U} : \exp S \to \exp S$ is said to be a *$\mathscr{C}$-closure operation* if the mapping $\boldsymbol{U}$ satisfies the following conditions:

i) $$\boldsymbol{U}(\emptyset) = \emptyset\,;$$

ii) $$A \subset B \subset S \Rightarrow \boldsymbol{U}(A) \subset \boldsymbol{U}(B)\,;$$

iii) $$A \subset \boldsymbol{U}(A) \quad \text{for each} \quad A \subset S\,;$$

iv) $$\boldsymbol{U}(\boldsymbol{U}(A)) = \boldsymbol{U}(A) \quad \text{for each} \quad A \subset S\,.$$

For $x \in S$ we write simply $\boldsymbol{U}(x)$ instead of $\boldsymbol{U}(\{x\})$. The set of all $\mathscr{C}$-closure operations for a semigroup S will be denoted by $\mathscr{C}(S)$.

Let $\boldsymbol{U} \in \mathscr{C}(S)$. A subset A of S will be called *$\boldsymbol{U}$-closed* if $\boldsymbol{U}(A) = A$. Let $\mathscr{F}(\boldsymbol{U})$ denote the set of all $\boldsymbol{U}$-closed subsets of S.

We recall the following notion introduced in [1]. If $\boldsymbol{U} \in \mathscr{C}(S)$, $\boldsymbol{V} \in \mathscr{C}(S)$ we define $\boldsymbol{U} \varrho \boldsymbol{V}$ if and only if the following holds: For any $\boldsymbol{U}$-closed (non-empty) subset $A \subset S$ and any $\boldsymbol{V}$-closed (non-empty) subset $B \subset S$, we have

(1) $$A \cap B = AB\,.$$

Let $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}(S)$, then we define $\boldsymbol{U} \leqq \boldsymbol{V}$ if and only if $\boldsymbol{U}(A) \subset \boldsymbol{V}(A)$ for any subset $A \subset S$. The ordered set $\mathscr{C}(S)$ is a lattice $(\wedge, \vee)$. If $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}(S)$, then there holds:

(2) $$\boldsymbol{U} \leqq \boldsymbol{V} \Leftrightarrow \mathscr{F}(\boldsymbol{V}) \subset \mathscr{F}(\boldsymbol{U})\,;$$

(3) $$\mathscr{F}(\boldsymbol{U} \wedge \boldsymbol{V}) = \{A \cap B / A \in \mathscr{F}(\boldsymbol{U}),\ B \in \mathscr{F}(\boldsymbol{V})\}\,;$$

(4) $$\mathscr{F}(\boldsymbol{U} \vee \boldsymbol{V}) = \mathscr{F}(\boldsymbol{U}) \cap \mathscr{F}(\boldsymbol{V})\,.$$

(See [1].)

From (1) and (2) there follows:

Lemma 1. *Let* $\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{V}_1, \boldsymbol{V}_2 \in \mathscr{C}(S)$ *and let* $\boldsymbol{U}_1 \leqq \boldsymbol{U}_2$, $\boldsymbol{V}_1 \leqq \boldsymbol{V}_2$. *If* $\boldsymbol{U}_1 \varrho \boldsymbol{V}_1$, *then* $\boldsymbol{U}_2 \varrho \boldsymbol{V}_2$.

Let $A \subset S$, $A \neq \emptyset$. Put $\boldsymbol{L}(A) = SA \cup A$ and $\boldsymbol{R}(A) = AS \cup A$. Finally $\boldsymbol{L}(\emptyset) = \emptyset = \boldsymbol{R}(\emptyset)$. It is clear that $\boldsymbol{L}, \boldsymbol{R} \in \mathscr{C}(S)$ and $\mathscr{F}(\boldsymbol{L})$ is the set of all left ideals of S (including $\emptyset$), $\mathscr{F}(\boldsymbol{R})$ is the set of all right ideals of S (including $\emptyset$). Put $\boldsymbol{M} = \boldsymbol{L} \vee \boldsymbol{R}$, $\boldsymbol{H} = \boldsymbol{L} \wedge \boldsymbol{R}$. Evidently $\boldsymbol{M}, \boldsymbol{H} \in \mathscr{C}(S)$. It follows from (3) and (4) that $\mathscr{F}(\boldsymbol{M})$ is the set of all two-sided ideals of S (including $\emptyset$) and $\mathscr{F}(\boldsymbol{H})$ is the set of all quasi-ideals of S (including $\emptyset$).

Lemma 2. *Let* $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}(S)$. *Then* $\boldsymbol{U} \varrho \boldsymbol{V}$ *if and only if* $\boldsymbol{R} \leqq \boldsymbol{U}$, $\boldsymbol{L} \leqq \boldsymbol{V}$ *and* $x \in \boldsymbol{U}(x)\, \boldsymbol{V}(x)$ *for every* $x \in S$.

Proof. See Theorem 9 [1].

A semigroup S is called *left* (*right*) *regular* if $x \in \boldsymbol{L}(x^2)$ ($x \in \boldsymbol{R}(x^2)$) for every x of S (see Lemma 3 [1]). A semigroup S is said to be *left* (*right*) *cancellative* if in S the left (right) cancellation law holds, that is $ax = ay$ ($xa = ya$) implies $x = y$ for all $a, x, y \in S$. A semigroup S is called *left* (*right*) *simple* if S does not contain a left (right) ideal different from S. A semigroup S is called a *left* (*right*) *group* if it is left (right) simple and right (left) cancellative.

Lemma 3. *The following conditions on a semigroup* S *are equivalent*:

1. S *is a semilattice of left groups*;
2. S *is a union of groups and* $\boldsymbol{R} \leqq \boldsymbol{L}$;
3. S *is a right regular and* $\boldsymbol{R} \leqq \boldsymbol{L}$.

Proof. $1 \Leftrightarrow 2$. This follows from Theorem 11 [2].

$2 \Rightarrow 3$. Evident.

$3 \Rightarrow 2$. Let S be right regular and $\boldsymbol{R} \leqq \boldsymbol{L}$. We show that S is left regular, which implies (see Theorem 8 [2]) that S is a union of groups. Let x be an arbitrary element of S. Then $x \in \boldsymbol{R}(x^2) \subset \boldsymbol{L}(x^2)$. Hence, S is left regular.

From Remark 1 [2] we obtain the following:

Lemma 4. *The following conditions on a periodic semigroup* S *are equivalent*:

1. S *is a union of groups*;
2. S *is right regular*;
3. S *is left regular*.

Theorem 1. *The following conditions on a right regular or periodic semigroup S are equivalent:*

1. $\boldsymbol{H} \varrho \boldsymbol{M}$;
2. $\boldsymbol{H} \varrho \boldsymbol{L}$;
3. $\boldsymbol{L} \varrho \boldsymbol{L}$;
4. $\boldsymbol{L} \varrho \boldsymbol{M}$;
5. *S is a semilattice of left groups.*

Proof. $2 \Rightarrow 3 \Rightarrow 4$. This follows from Lemma 1.

$4 \Rightarrow 5$. Let $\boldsymbol{L} \varrho \boldsymbol{M}$ and let S be a right regular semigroup. By Lemma 2, we have $\boldsymbol{R} \leqq \boldsymbol{L}$. Hence, by Lemma 3, S is a semilattice of left groups.

Let $\boldsymbol{L} \varrho \boldsymbol{M}$ and let S be a periodic semigroup. It follows from Theorem 11 [1] that S is left regular and $\boldsymbol{R} \leqq \boldsymbol{L}$. According to Lemma 4 and Lemma 3, S is a semilattice of left groups.

$5 \Rightarrow 1$. Let S be a semilattice of left groups. By Lemma 3, S is a union of groups and $\boldsymbol{R} \leqq \boldsymbol{L}$. Since S is regular, by Theorem 10 [1], we have $\boldsymbol{R} \varrho \boldsymbol{L}$. This implies that $\boldsymbol{H} \varrho \boldsymbol{M}$.

$1 \Rightarrow 2$. Let $\boldsymbol{H} \varrho \boldsymbol{M}$. By Lemma 2 we have $\boldsymbol{R} \leqq \boldsymbol{H} \leqq \boldsymbol{L}$ and so $\boldsymbol{M} = \boldsymbol{L}$. This implies $\boldsymbol{H} \varrho \boldsymbol{L}$.

Dually we have the following:

Theorem 2. *The following conditions on a left regular or periodic semigroup S are equivalent:*

1. $\boldsymbol{M} \varrho \boldsymbol{H}$;
2. $\boldsymbol{R} \varrho \boldsymbol{H}$;
3. $\boldsymbol{R} \varrho \boldsymbol{R}$;
4. $\boldsymbol{M} \varrho \boldsymbol{R}$;
5. *S is a semilattice of right groups.*

References

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[2] *B. Pondělíček:* A certain equivalence on a semigroup, Czechoslovak Math. J. *21* (*96*), (1971), 109—117.

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