Czechoslovak Mathematical Journal

Branko Grünbaum; Ernest Jucovič

On non-inscribable polytopes

*Czechoslovak Mathematical Journal*, Vol. 24 (1974), No. 3, 424–429

Persistent URL: http://dml.cz/dmlcz/101257

**Terms of use:**

© Institute of Mathematics AS CR, 1974

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

# ON NON-INSCRIBABLE POLYTOPES

BRANKO GRÜNBAUM, Seattle and ERNEST JUCOVIČ, Košice

(Received May 16, 1973)

**1. Introduction.** The purpose of this note is to establish the existence of polytopes which behave very poorly with respect to the possibility of inscribing any isomorphic polytope into a sphere. To formulate the result and its background more precisely we need some definitions.

Let $P$ be a 3-polytope (that is, 3-dimensional convex polytope) with $v(P)$ vertices. We shall denote by $s(P)$ the largest integer $s$ with the property: There exists a 3-polytope $P'$ isomorphic to $P$, and a sphere $S$ that encloses $P'$, such that $s$ vertices of $P'$ are on $S$.

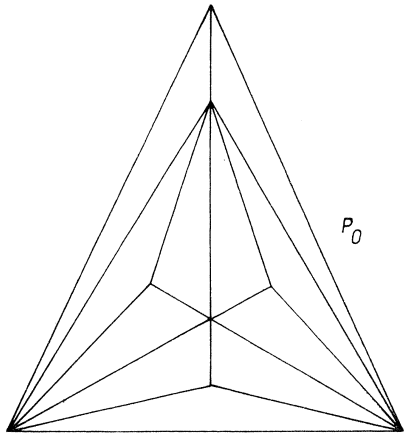


Fig. 1

STEINER [7] asked whether all 3-polytopes are inscribable, where a 3-polytope $P$ is called *inscribable* provided $s(P) = v(P)$. BRÜCKNER [1, p. 163, footnote 4] remarked that all simplicial 3-polytopes are necessarily inscribable; we recall that a 3-polytope $P$ is *simplicial* if all the faces of $P$ are triangles. However, as was observed by STEINITZ [8], this remark of Brückner's is mistaken; indeed, Steinitz proceeded to prove by

a peculiarly elegant geometric argument that there are infinitely many types of 3-polytopes (among them simplicial ones) that are non-inscribable. The non-inscribable simplicial 3-polytope $P_0$ with fewest vertices obtainable by Steinitz's method is represented in Figure 1 by its Schlegel diagram. Note that $P_0$ may be interpreted as the polytope arising from the tetrahedron by placing a sufficiently flat 3-sided pyramid on each of the four faces of the tetrahedron.

Steinitz's method was extended by JUCOVIČ [6]; a corollary of that paper which we shall use here is: If $S$ is a sphere that contains a polytope $P_0'$ isomorphic to $P_0$, then not all four 3-valent vertices of $P_0'$ are on $S$.

Another step in investigating how badly non-inscribable may 3-polytopes be was taken in Jucovič [5]. His result may be interpreted as asserting the existence of simplicial 3-polytopes $P$ with an arbitrarily prescribed number of vertices such that

$$s(P) \leqq [(2v(P) + 13)/3] .$$

The main aim of the present note is to improve upon those results by establishing the following

**Theorem.** *If $P$ ranges over all types of simplicial 3-polytopes then*

$$\liminf \frac{\log s(P)}{\log v(P)} \leqq \frac{\log 2}{\log 3} = 0.630930 \ldots .$$

In other words, for every $\alpha > \log 2/\log 3$ there exists a constant $\beta$, and a sequence of non-isomorphic simplicial 3-polytopes $P_n$ such that

$$s(P_n) \leqq \beta \,.\, (v(P_n))^\alpha \quad \text{for all} \quad n .$$

We shall present the proof of the Theorem in Section 2; various remarks, open problems and conjectures are presented in Section 3.

**2. Proof of the Theorem.** Before starting the actual proof, we shall establish a lemma. In order to ease its formulation, we shall first explain the terminology and notation.

Let $Q$ be a simplicial 3-polytope in which we have singled out a set $T$ of 3-valent vertices no two of which share an edge. Although the lemma would be valid also without this assumption, we shall simplify its formulation by assuming that $T$ is the set of all 3-valent vertices of $Q$, the number of which is denoted by $v_3(Q)$. We denote by $W$ the set of $v(Q) - v_3(Q)$ vertices of $Q$ that have valence 4 or more. We shall denote by $t(Q)$ and by $w(Q)$ the maximal possible numbers of 3-valent vertices, and of 4 or higher valent vertices, of any 3-polytope $Q'$ isomorphic to $Q$, that belong to a sphere that contains $Q'$. Therefore $s(Q) \leqq t(Q) + w(Q)$. Note that if $V \in T$, the three faces of $Q$ incident with $V$ form a complex isomorphic to the complex $N$ in-

dicated in Figure 2. The vertices $U_1$, $U_2$, $U_3$ of $N$ belong to $W$. We shall denote by $Q^*$ the simplicial 3-polytope obtained from $Q$ by replacing every vertex $V \in T$ by a copy of the complex $H$ shown in Figure 3. It follows easily that the polytopes $Q$ and $Q^*$

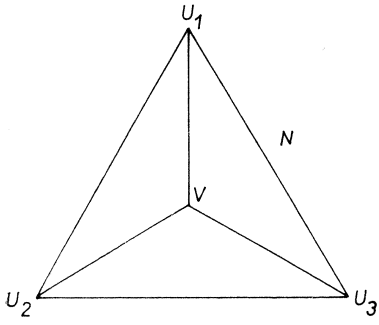


Fig. 2

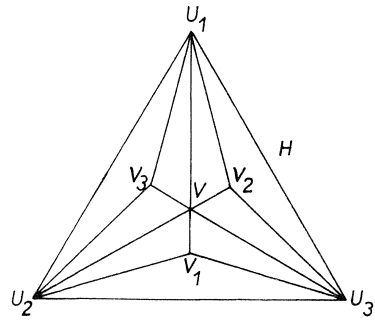


Fig. 3

satisfy $v(Q^*) = v(Q) + 3\, v_3(Q)$ and $v_3(Q^*) = 3\, v_3(Q)$. Furthermore, we shall establish the following

**Lemma.** *$t(Q^*) \leqq 2\, t(Q)$ and $w(Q^*) \leqq w(Q) + t(Q)$.*

Proof. If any one of the four "interior" vertices $V, V_1, V_2, V_3$ of a copy of $H$ belongs to a sphere $S$ containing a polytope $Q^{*\prime}$ isomorphic to $Q^*$, the three triangles determined by that vertex and by the three "outer" vertices $U_1$, $U_2$, $U_3$ of $H$ yield a copy of $N$, and the convexity of $Q^{*\prime}$ is not affected by the replacement of that copy of $H$ by the just constructed copy of $N$. Therefore, if in more than $t(Q)$ copies of $H$ a vertex would belong to a containing sphere $S$, we should be able to form a polytope isomorphic to $Q$ in which more than $t(Q)$ of its 3-valent vertices belong to the sphere $S$ that contains it. As this is impossible, we must have $w(Q^*) \leqq w(Q) + t(Q)$. On the other hand, in each of the at most $t(Q)$ copies of $H$ that have any of the interior vertices belonging to $S$, at most 2 of the three 3-valent vertices present may belong to the sphere (unless *all* the vertices of $Q^{*\prime}$ that belong to $S$ are among the seven vertices of that one copy of $H$). Indeed, if for some copy of $H$ the three vertices $V_1, V_2, V_3$ were on the sphere $S$ that contains $Q^{*\prime}$, the convex hull of that copy of $H$ and of an arbitrary vertex of $Q^{*\prime}$ outside of $H$ that belongs to $S$ would yield a polytope isomorphic to the polytope $P_0$ of Section 1, which has its four 3-valent vertices on a sphere that contains $P_0$. As that is impossible, we have $t(Q^*) \leqq 2\, t(Q)$, and the proof of the Lemma is completed.

We shall now prove our Theorem by exhibiting a sequence $P_n$, $n = 0, 1, \ldots$, of simplicial 3-polytopes such that $v(P_n) = 2 + 6.3^n$ and $s(P_n) \leqq 1 + 6.2^n$. It is obvious that

$$\liminf \frac{\log s(P)}{\log v(P)} \leqq \lim_{n\to\infty} \frac{\log s(P_n)}{\log v(P_n)} \leqq \frac{\log 2}{\log 3}\,.$$

We construct the sequence $P_n$ by induction, taking as $P_0$ the polytope described in Section 1 and shown in Figure 1. If any $P_n$ is already constructed, we form $P_{n+1}$ as $P_n^*$, by replacing the configuration $N$ around each 3-valent vertex of $P_n$ by a copy of the complex $H$. Since $v(P_0) = 8$ and $v_3(P_0) = 4$, the construction clearly implies that $v(P_n) = 2 + 6.3^n$ and $v_3(P_n) = 4.3^n$. Using the Lemma and $t(P_0) = 3$, $w(P_0) = 4$, it then follows at once that $t(P_n) \leqq 3.2^n$, and $w(P_n) \leqq 1 + 3.2^n$, so that $s(P_n) \leqq 1 + 6.2^n$, as asserted.

This completes the proof of the Theorem.

**3. Remarks.** (1) Following the terminology introduced in GRÜNBAUM-WALTHER [4], the number

$$s^{(d)} = \liminf \frac{\log s(P)}{\log v(P)}$$

where $P$ ranges over all $d$-polytopes, could be called the "*inscribability exponent*" of the family of all $d$-polytopes. Our Theorem asserts that $s^{(3)} \leqq \log 2/\log 3$; we make the following

*Conjecture.* $s^{(3)} = \log 2/\log 3$.

It should be pointed out that our conjecture could be wildly wrong. Not only do we ignore whether $s^{(3)} > 0$, but all our present knowledge is even compatible with the following rather outrageous statement: There exists an absolute constant $k$ and 3-polytopes $P$ with arbitrarily large $v(P)$ such that $s(P) \leqq k$.

(2) The results of Steinitz [8] and Jucovič [6] mentioned in Section 1 may easily be extended to higher dimensions. It follows, for example, that $P_0^{(d)}$, the "Kleetope" (see GRÜNBAUM. [3, p. 217]) over the $d$-dimensional simplex has the following property: If a polytope $P'$ isomorphic to $P_0^{(d)}$ is contained in the ball determined by some $(d-1)$-dimensional sphere $S$, then not all $d+1$ of the $d$-valent vertices of $P'$ can be on $S$. By a simple extension of the proof of the Theorem in Section 2 we may construct a sequence $P_n^{(d)}$, $n = 0, 1, \ldots$, of simplicial $d$-polytopes such that

$$\lim_{n\to\infty} \frac{\log s(P_n^{(d)})}{\log v(P_n^{(d)})} = \frac{\log (d-1)}{\log d}.$$

We venture:

*Conjecture.* The inscribability exponent of the family of simplicial $d$-polytopes, $d \geqq 3$,

$$\bar{s}^{(d)} = \liminf \frac{\log s(P)}{\log v(P)}$$

where $P$ ranges over the simplicial $d$-polytopes, satisfies

$$\bar{s}^{(d)} = \log (d-1)/\log d\,.$$

It should be noted that, as shown by the 4-dimensional pyramids based on the 3-polytopes $P_n$ constructed in Section 2, the inscribability exponent $s^{(4)}$ of *all* 4-polytopes satisfies $s^{(4)} \leqq \log 2/\log 3$; similarly $s^{(d)} \leqq \log 2/\log 3$ for *all* $d \geqq 3$. It is well possible that strict inequality holds in this relation, at least for sufficiently large $d$; this is in contrast to the conjectured relation $s^{(3)} = \bar{s}^{(3)}$.

(3) It was shown in Grünbaum [2] that there exist simple 3-polytopes $P$ which are not inscribable. (A $d$-polytope is called *simple* if each vertex is incident with precisely $d$ different edges.) It would be of interest to investigate the inscribability exponent $\hat{s}^{(d)}$ of simple $d$-polytopes. So far we were unable to show that $\hat{s}^{(d)} < 1$ for any $d \geqq 3$.

(4) If $P$ is a $d$-polytope, let $c(P)$ denote the maximal number of co-spherical vertices in any polytope isomorphic with $P$. Then clearly $c(P) \geqq s(P)$, but if $P$ is non-inscribable then $c(P) < v(P)$. It would be of interest to investigate whether

$$\liminf \frac{\log c(P)}{\log v(P)} < 1$$

when $P$ ranges over all $d$-polytopes, or over certain classes of them.

(5) If $P$ is a $d$-polytope with $f(P)$ facets (that is, $(d-1)$-dimensional faces), let $i(P)$ denote the greatest integer $i$ such that there exists a polytope $P'$ isomorphic with $P$, with the property that $i$ facets of $P'$ are inscribable $(d-1)$-spheres. Clearly $i(P) = f(P)$ whenever $P$ is a simplicial polytope; on the other hand, if $P$ is simple then $s(P) < v(P)$ if and only if $i(P) < f(P)$. The 3-dimensional case of this observation was used in Grünbaum [2] to derive the existence of simple non-inscribable 3-polytopes from the existence of 3-polytopes "without circumcircles". Although it is easy to find a sequence of 3-polytopes $P_n$ such that

$$\lim_{n\to\infty} \frac{i(P_n)}{f(P_n)} = 2/3$$

it is not known whether other sequences would give smaller limits. Similarly, the higher dimensional analogues are completely unexplored.

(6) It is probably true that if $P$ is a 3-polytope with $v(P) \leqq 7$ then $P$ is inscribable; this is easily checked for simplicial $P$. However, in general we have no method of determining whether a given 3-polytope is inscribable or not. Several criteria establishing non-inscribability of 3-polytopes are known (Steinitz [8], Grünbaum [2]) but they certainly do not cover all cases.

(7) As explained in detail in Steinitz [8], the question whether a polytope $P$ is *circumscribable* (that is, isomorphic to a polytope all facets of which are tangent to a fixed sphere) is equivalent to the question whether the polytope $P^*$ dual to $P$ is inscribable. However, the following problems are still open:

(i) Does every 3-polytope $P$ have an isomorphic polytope $P'$ such that every facet of $P'$ is a polygon with an incircle?

(ii) Is every 3-polytope isomorphic to a cage for the sphere? (Here a polytope $Q$ is called a *cage* for the sphere provided there exists a sphere that touches all the edges of $Q$.)

(iii) Is every 3-polytope isomorphic to a normal one? (We call a 3-polytope $Q$ *normal* if there exists a point $O$ such that for each face $F$ of $Q$ the foot of the perpendicular from $O$ to the plane of $F$ belongs to the interior of $F$.)

Clearly, such and similar questions may be formulated also for higher dimensional polytopes, and in many other variations.

*References*


[1] *M. Brückner:* Vielecke und Vielfache. Teubner, Leipzig 1900.

[2] *B. Grünbaum:* On Steinitz's theorem about non-inscribable polyhedra. Proc. Ned. Akad. Wetenschap. Ser. A, *66* (1963), 452—455.

[3] *B. Grünbaum:* Convex Polytopes. Interscience, New York 1967.

[4] *B. Grünbaum* and *H. Walther:* Shortness exponents of families of graphs. J. Combinat. Theory, *14* (1973), 364—385.

[5] *E. Jucovič:* O mnohostenoch bez opísanej guľovej plochy. II. Mat.-fyz. časopis *16* (1966), 229—234.

[6] *E. Jucovič:* Bemerkungen zu einem Satz von Steinitz. Elem. der Math. *22* (1967), 39.

[7] *J. Steiner:* Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander. Fincke, Berlin 1832.
(=Gesammelte Werke, Vol. 1, Reimer, Berlin 1881, 229—458.)

[8] *E. Steinitz:* Über isoperimetrische Probleme bei konvexen Polyedern. J. reine angew. Math. *159* (1928), 133—143.



*Authors' adresses:* B. Grünbaum, University of Washington, Seattle (USA); E. Jucovič, P. J. Šafárik University, Košice (ČSSR).