Joji Kajiwara; Yasuko Mori

On the existence of global holomorphic solutions of differential equations with complex parameters

Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 3, 444–454

Persistent URL: http://dml.cz/dmlcz/101259

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

ON THE EXISTENCE OF GLOBAL HOLOMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH COMPLEX PARAMETERS

JOJI KAJIWARA, Fukuoka and YASUKO MORI, Naha

(Received June 1, 1973)

1. INTRODUCTION AND PRELIMINARIES

Let S be a Stein manifold, Ω be a Stein subdomain of the product manifold $C \times S$ of the complex plane C and S, $\mathbf{O}$ be the sheaf of germs of holomorphic functions in Ω and m be a positive integer. Let $a_{jk}(z, x)$ be holomorphic functions in Ω for $j, k = 1, 2, \ldots, m$. We define a homomorphism T of $\mathbf{O}^m$ in $\mathbf{O}^m$ by putting

$$T(u_1, u_2, \ldots, u_m) = \left(\frac{\partial u_1}{\partial z} + \sum_{j=1}^{m} a_{1j}(z, x)\, u_j, \; \frac{\partial u_2}{\partial z} + \sum_{j=1}^{m} a_{2j}(z, x)\, u_j, \ldots, \frac{\partial u_m}{\partial z} + \sum_{j=1}^{m} a_{mj}(z, x)\, u_j\right) \tag{1}$$

for $(u_1, u_2, \ldots, u_m) \in \mathbf{O}^m$. Let $\mathbf{A}$ be the kernel of T. The short exact sequence

$$0 \to \mathbf{A} \to \mathbf{O}^m \xrightarrow{T} \mathbf{O}^m \to 0 \tag{2}$$

of sheaves over Ω gives the long exact sequence

$$\ldots \to H^0(\Omega, \mathbf{O}^m) \xrightarrow{T} H^0(\Omega, \mathbf{O}^m) \to H^1(\Omega, \mathbf{A}) \to H^1(\Omega, \mathbf{O}^m) \to \ldots \tag{3}$$

of cohomology groups. Since Ω is a Stein manifold, we have $H^1(\Omega, \mathbf{O}^m) = 0$. Hence we have

$$H^1(\Omega, \mathbf{A}) = H^0(\Omega, \mathbf{O}^m)/TH^0(\Omega, \mathbf{O}^m) \tag{4}$$

by (3). In other words, the necessary and sufficient condition that for any $v \in H^0(\Omega, \mathbf{O}^m)$ there is $u \in H^0(\Omega, \mathbf{O}^m)$ satisfying the inhomogeneous equation

(5) $$Tu = v$$

is that there holds

(6) $$H^1(\Omega, \mathbf{A}) = 0 .$$

Recently SUZUKI [6] has obtained the necessary and sufficient condition for (6) in case that $m = 1$ and $Tu = \partial u/\partial z$. On the other hand, one of the authors KAJIWARA [4] obtained the necessary and sufficient condition for (6) in case that S consists of a single point. In the present paper we shall study the condition for (6) making use of the methods of the above Suzuki [6] and Kajiwara [4].

2. ONE DIMENSIONAL SECTION OF Ω

For any point x of S and for any analytic set M in S, we put

(7) $$\Omega(x) = (C \times \{x\}) \cap \Omega , \quad \Omega(M) = (C \times M) \cap \Omega .$$

For any point (z, x) of Ω we put

(8) $$\Omega(z, x) = \text{the connected component of } \Omega(x) \text{ containing } (z, x).$$

Lemma 1. *If $H^1(\Omega, \mathbf{A}) = 0$, we have $H^1(\Omega(M), \mathbf{A}) = 0$ for any analytic set M in S.*

Proof. By the exact sequence (2) of sheaves over $\Omega(M)$ we have the long exact sequence

$$\ldots \to H^0(\Omega(M), \mathbf{O}^m) \xrightarrow{T} H^0(\Omega(M), \mathbf{O}^m) \to H^1(\Omega(M), \mathbf{A}) \to H^1(\Omega(M), \mathbf{O}^m) \to \ldots .$$

Since $\Omega(M)$ is a Stein space, we have $H^1(\Omega(M), \mathbf{O}^m) = 0$. Hence we have

(9) $$H^1(\Omega(M), \mathbf{A}) = H^0(\Omega(M), \mathbf{O}^m)/TH^0(\Omega(M), \mathbf{O}^m) .$$

Let v be any element of $H^0(\Omega(M), \mathbf{O}^m)$. Since $\Omega(M)$ is an analytic set in a Stein manifold Ω, v can be extended to an element V of $H^0(\Omega, \mathbf{O}^m)$. Since $H^1(\Omega, \mathbf{A}) = 0$, we have $H^0(\Omega, \mathbf{O}^m) = TH^0(\Omega, \mathbf{O}^m)$ by (3). Hence there is an element U of $H^0(\Omega, \mathbf{O}^m)$ such that $TU = V$. Since the variable x plays only the role of a parameter in the differential operator T given in (1), the restriction $u \in H^0(\Omega(M), \mathbf{O}^m)$ of U to $\Omega(M)$ satisfies $Tu = v$. Thus we have proved $H^0(\Omega(M), \mathbf{O}^m) = TH^0(\Omega(M), \mathbf{O}^m)$. By (9) we have $H^1(\Omega(M), \mathbf{A}) = 0$.

Lemma 2. *If $H^1(\Omega, \mathbf{A}) = 0$, we have $H^1(\Omega(z, x), \mathbf{A}) = 0$ for any point (z, x) of Ω.*

Proof. Since $\{x\}$ is an analytic set in S, we have Lemma 2 by Lemma 1.

By Kajiwara [4] we have the following Lemma.

Lemma 3. *Let (z, x) be a point of Ω. The necessary and sufficient condition for $H^1(\Omega(z, x), \mathbf{A}) = 0$ is that either $\Omega(z, x)$ is simply connected or $\Omega(z, x)$ is a doubly connected domain with $H^0(\Omega(z, x), \mathbf{A}) = 0$.*

We shall induce an equivalence relation $\sim$ in Ω. We say that two points (z, x) and (z', x') of Ω satisfy $(z, x) \sim (z', x')$ if and only if $x = x'$ and $\Omega(z, x) = \Omega(z, x')$. Let $\tilde{\Omega}$ be the factor space of Ω by the equivalence relation $\sim$. Any point $\tilde{x}$ of $\tilde{\Omega}$ is an equivalence class in Ω. Let (z, x) be a representative of the class $\tilde{x}$. Then the subset $\Omega(z, x)$ of Ω coincides with the equivalence class $\tilde{x}$. Therefore we have

$$\tilde{\Omega} = \{\Omega(z, x); (z, x) \in \Omega\} . \tag{10}$$

We put

(11) $\tilde{\Omega}_1 =$ the set of all simply connected $\Omega(z, x)$ for points (z, x) of Ω

and

(12) $\tilde{\Omega}_2 =$ the set of all doubly connected $\Omega(z, x)$ for points (z, x) of Ω .

If $H^1(\Omega, \mathbf{A}) = 0$, we have $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$ and $\tilde{\Omega}_1 \cap \tilde{\Omega}_2 = \emptyset$ by Lemma 3.

Lemma 4. *If $H^1(\Omega, \mathbf{A}) = 0$, $\tilde{\Omega}_2$ is an open subset of $\tilde{\Omega}$.*

Proof. Let (z, x) be a point of Ω such that $\Omega(z, x)$ is doubly connected. $\mathbf{C} \times \{x\} - \Omega(z, x)$ has a compact connected component K. Let (z', x) be any point of $\Omega(z, x)$. There is a closed curve γ through z' in $\mathbf{C}$ such that $\gamma \times \{x\}$ is a closed curve in $\Omega(z, x)$ and K is contained in $\Gamma \times \{x\}$ where Γ is the domain in $\mathbf{C}$ surrounded by γ Since $\gamma \times \{x\} \subset \Omega$, there is an open connected neighborhood U of x such that $\gamma \times U \subset \Omega$. If $\Omega(z', x')$ were not doubly connected for a point x' of U, $\Omega(z', x')$ would be simply connected by Lemma 3. Then the domain $\Gamma \times \{x'\}$ in $\mathbf{C} \times \{x'\}$ surrounded by the closed curve $\gamma \times \{x'\}$ in $\Omega(z', x')$ is contained in $\Omega(z', x')$. There is an open connected neighbourhood U' of x' in S such that $\Gamma \times U' \subset \Omega$ and $U' \subset U$. By the theorem of continuity, any holomorphic function in $(\gamma \times U) \cup (\bar{\Gamma} \times U')$ is continued to a holomorphic function in $\bar{\Gamma} \times U$. Since Ω is a Stein manifold, we have $\bar{\Gamma} \times U \subset \Omega$. Especially we have $K \subset \Gamma \times \{x\} \subset \Omega$. But this is a contradiction. Thus we have proved that $\Omega(z', x')$ is doubly connected for any point x' of U. Since $\{\Omega(z', x') \in \tilde{\Omega}; x' \in U\}$ is a neighbourhood of $\Omega(z, x)$ in $\tilde{\Omega}$, $\tilde{\Omega}_2$ is an open subset of $\tilde{\Omega}$.

Lemma 5. *Assume that $H^1(\Omega, \mathbf{A}) = 0$. Then either $\tilde{\Omega} = \tilde{\Omega}_1$ or $\tilde{\Omega} = \tilde{\Omega}_2$. In case that $\Omega = \tilde{\Omega}_2$, we have $H^0(\Omega(z, x), \mathbf{A}) = 0$ for any point (z, x) of Ω.*

Proof. By Lemma 3 it suffices to prove that either $\tilde{\Omega} = \tilde{\Omega}_1$ or $\tilde{\Omega} = \tilde{\Omega}_2$. Assume that there were two points (z_1, x_1) and (z_2, x_2) of Ω such that $\Omega(z_1, x_1) \in \tilde{\Omega}_1$ and $\Omega(z_2, x_2) \in \tilde{\Omega}_2$. There is an open connected neighborhood U of x_1 in S such that $\{z_1\} \times U \subset \Omega$. For any point x of U, either $\Omega(z_1, x)$ is simply connected or doubly connected by Lemma 3. Hence, there is a family $\{\Omega'(x); x \in U\}$ of subsets of Ω such that $\Omega'(x)$ is a simply connected subdomain containing (z_1, x) of $\Omega(z_1, x)$ for any $x \in U$, $\Omega' = \bigcup_{x \in U} \Omega'(x)$ is an open connected subset of Ω and $\Omega'(x_1) = \Omega(z_1, x_1)$. We put $\Omega'' = \Omega - \Omega(z_1, x_1)$. Then $\{\Omega', \Omega''\}$ is an open covering of Ω. For arbitrary but fixed point x of U, the initial value problem

$$(13) \qquad Tu = 0\,, \quad u = (1, 1, \dots, 1) \quad \text{at} \quad z = z_1$$

of a system of linear ordinary differential equations has a holomorphic solution $b(z, x)$ in the simply connected domain $\Omega'(x)$. Then $b(z, x)$ is holomorphic in Ω' as functions in z and the parameter x, that is, $b(z, x) \in H^0(\Omega', \mathbf{A})$.

(a) *In case that the dimension of S is one.* Since there holds $H^1(\Omega, \mathbf{A}) = 0$ in the Mayer-Vietoris sequence

$$(14) \qquad \dots \to H^0(\Omega', \mathbf{A}) + H^0(\Omega'', \mathbf{A}) \to H^0(\Omega' \cap \Omega'', \mathbf{A}) \to H^1(\Omega, \mathbf{A}) \to \dots\,,$$

for $(x - x_1)^{-1}\, b(z, x) \in H^0(\Omega' \cap \Omega'', \mathbf{A})$ there are $u_1 \in H^0(\Omega', \mathbf{A})$ and $u_2 \in H^0(\Omega'', \mathbf{A})$ such that

$$(15) \qquad \frac{b(z, x)}{x - x_1} = u_1(z, x) - u_2(z, x)$$

in $\Omega' \cap \Omega''$. By Lemma 3 $u_2(z, x) = 0$ for any point (z, x) of Ω'' such that $\Omega(z, x) \in \tilde{\Omega}_2$. By Lemma 4 such points (z, x) of Ω'' form an open subset of Ω'' containing (z_2, x_2). Hence $u_2(z, x)$ is identically zero in Ω'' by the theorem of identity. By (15) we have $(x - x_1)^{-1}\, b(z, x) \in H^0(\Omega', \mathbf{A})$. Since $b(z_1, x_1) = (1, 1, \dots, 1)$, this means that $(x - x_1)^{-1}$ is holomorphic in $x = x_1$. But this is a contradiction.

(b) *In case that the dimension of S is larger than one.* There is an analytic set M in S containing x_1 and x_2 such that M is regular and one dimensional in a neighborhood of x_1. By Lemma 1 we have $H^1(\Omega(M), \mathbf{A}) = 0$. By the argument given in (a), we also arrive at a contradiction.

Any way, we have proved under the assumption $H^1(\Omega, \mathbf{A}) = 0$ that either $\tilde{\Omega} = \tilde{\Omega}_1$ or $\tilde{\Omega} = \tilde{\Omega}_2$.

3. IF $H^1(\Omega, \mathbf{A}) = 0$ AND $\tilde{\Omega} = \tilde{\Omega}_2$, $\tilde{\Omega}$ IS A HAUSDORFF SPACE

We define mappings π of Ω onto $\tilde{\Omega}$ and φ of Ω in S by putting

$$(16) \qquad \pi(z, x) = \Omega(z, x)\,, \quad \varphi(\Omega(z, x)) = x \quad \text{for} \quad (z, x) \in \Omega$$

π and φ are continuous. Moreover φ is a locally homeomorphic mapping. For the projection p of $\mathbf{C} \times S$ onto S, the restriction $p \mid \Omega$ of p to Ω satisfies $p \mid \Omega = \varphi \circ \pi$. The fact that $\tilde{\Omega}$ is a Hausdorff space means that the singular number of connected components of $\Omega(x)$ never forks to the plural number of connected components of $\Omega(x)$ as x varies in S. If $\tilde{\Omega}$ is a Hausdorff space, we can induce a complex structure in $\tilde{\Omega}$ such that φ is a locally biholomorphic mapping of $\tilde{\Omega}$ in S. We call this complex structure in $\tilde{\Omega}$ the *natural complex structure* in $\tilde{\Omega}$. Then the pair $(\tilde{\Omega}, \varphi)$ of $\tilde{\Omega}$ and φ is an unramified domain over the Stein manifold S and Docquier-Grauert's theory [1] is applicable for this $(\tilde{\Omega}, \varphi)$.

Lemma 6. *If* $\tilde{\Omega} = \tilde{\Omega}_2$, $\tilde{\Omega}$ *is a Hausdorff space.*

Proof. If $\tilde{\Omega}$ were not a Hausdorff space, there would be two points $\Omega(z_1, x_1)$ and $\Omega(z_2, x_2)$ of $\tilde{\Omega}$ such that the set $\mathbf{B}$ of all intersections of neighborhoods of $\Omega(z_1, x_1)$ and $\Omega(z_2, x_2)$ in $\tilde{\Omega}$ forms a base of a filter in $\tilde{\Omega}$. Then $\mathbf{B}$ converges to points $\Omega(z_1, x_1)$ and $\Omega(z_2, x_2)$ in $\tilde{\Omega}$. Since φ is continuous, $\varphi(\mathbf{B})$ converges also to $x_1 = \varphi(\Omega(z_1, x_1))$ and $x_2 = \varphi(\Omega(z_2, x_2))$ in the Hasudorff space S. Hence we have $x_1 = x_2$. In other words, $\Omega(z_1, x_1)$ and $\Omega(z_2, x_2)$ are two distinct connected components of the same open set $\Omega(x_1)$ in $\boldsymbol{C} \times \{x_1\}$ and in any neighborhood of x_1 there is a point x' such that $\Omega(z_1, x')$ and $\Omega(z_2, x')$ are the same connected component of $\Omega(x')$ in $\boldsymbol{C} \times \{x'\}$. Since each $\Omega(z_i, x_1)$ is a doubly connected domain in $\boldsymbol{C} \times \{x_1\}$, $\boldsymbol{C} \times \{x_1\} - \Omega(z_i, x_1)$ has a compact connected components K_i for $i = 1, 2$. There is a closed curve γ_i in the complex plane $\boldsymbol{C}$ for $i = 1, 2$ such that $\gamma_i \times \{x_1\}$ is a closed curve in $\Omega(z_i, x_1)$ and K_i is contained in $\Gamma_i \times \{x_i\}$ where Γ_i is the domain in $\boldsymbol{C}$ surrounded by γ_i for $i = 1, 2$. Since $\gamma_i \times \{x_1\} \subset \Omega$, there is an open connected neighbourhood U of x_1 such that $\gamma_i \times U \subset \Omega$ for $i = 1, 2$. Since $\mathbf{B}$ is a base of a filter in $\tilde{\Omega}$, there is a point x' of U such that $\gamma_1 \times \{x'\}$ and $\gamma_2 \times \{x'\}$ are contained in the same connected component $\Omega(z_1, x') = \Omega(z_2, x')$ of $\Omega(x')$ which is a doubly connected domain in $\boldsymbol{C} \times$ $\times \{x'\}$. Hence one of $\gamma_1 \times \{x'\}$ and $\gamma_2 \times \{x'\}$ is contained in $\Omega(z_1, x') = \Omega(z_2, x')$. We may assume that $\gamma_1 \times \{x'\} \subset \Omega(z_1, x') = \Omega(z_2, x')$. Any holomorphic function in $(\gamma_1 \times U) \cup (\bar{\Gamma}_1 \times \{x'\})$ is continued to a holomorphic function in $\bar{\Gamma}_1 \times U$ by the theorem of identity. Since Ω is a Stein manifold, we have $\bar{\Gamma}_1 \times U \subset \Omega$. This means that $K_1 \subset \bar{\Gamma}_1 \times \{x_1\} \subset \Omega$. But this is a contradiction. Thus we have proved that $\tilde{\Omega}$ is a Hausdorff space.

4. SUFFICIENCY IN CASE THAT $\tilde{\Omega} = \tilde{\Omega}_2$

Lemma 7. *If* $\tilde{\Omega} = \tilde{\Omega}_2$, $H^0(\Omega(x), \mathbf{A}) = 0$ *for any point* x *of* S *and* $\tilde{\Omega}$ *is a Hausdorff space, then* $H^1(\Omega, \mathbf{A}) = 0$.

Proof. Let (z_0, x_0) be any point of Ω. Since $\Omega(z_0, x_0)$ is doubly connected, $\boldsymbol{C} \times$ $\times \{x_0\} - \Omega(z_0, x_0)$ has a compact connected component K. There is a closed curve γ in $\boldsymbol{C}$ such that $\gamma \times \{x_0\}$ is a closed curve in $\Omega(z_0, x_0)$ and $\gamma \times \{x_0\}$ goes round K just once. There are an open simply connected neighbourhood V of z_0 and an open connected neighborhood U of x_0 such that $\gamma \times U \subset \Omega$, $V \times U \subset \Omega$. Let f^i be the holomorphic solution in $V \times U$ of the initial value problem

(17) $$T(u_1, u_2, \ldots, u_m) = 0\,, \quad u_j = \delta_{ij} \quad \text{at} \quad z = z_0 \quad (j = 1, 2, \ldots, m)$$

for $i = 1, 2, \ldots, m$. For arbitrary but fixed point x of U, each $f^i(z, x)$ is analytically continued along the closed curve $\gamma \times \{x\}$. Let $g^i(z, x) = (g^i_1, g^i_2, \ldots, g^i_m)$ be the element of $H^0(V \times \{x\}, \mathbf{A})$ obtained after one round of analytic continuation of $f^i(z, x)$ along $\gamma \times \{x\}$ for $i = 1, 2, \ldots, m$. Since $f^1(z, x), f^2(z, x), \ldots, f^m(z, x)$ form a base of the vector space $H^0(V \times \{x\}, \mathbf{A})$ for $x \in U$, $g^1(z, x), g^2(z, x), \ldots, g^m(z, x)$ are represented as their linear combinations

(18) $$g^i(z, x) = \sum_{j=1}^{m} c_{ij}(x) f^j(z, x)$$

in V for $x \in U$. Substituting $z = z_0$ in (18), we have

(19) $$c_{ij}(x) = g^i_j(z_0, x)\,.$$

Since each $g^i_j(z, x)$ is a holomorphic function in $V \times U$, each $c_{ij}(x)$ is a holomorphic function in U. Since $H^0(\Omega(x), \mathbf{A}) = 0$ for any point x of U, we have

(20) $$\det\,(c_{ij}(x) - \delta_{ij}) \neq 0$$

for any point x of U as we see it at p.102 of Kajiwara [4]. Let Δ be the connected component of $\pi^{-1}(U)$ containing (z_0, x_0).

Now let $v = (v_1, v_2, \ldots, v_m)$ be any element of $H^0(\Omega, \mathbf{O}^m)$. Since V is simply connected, $Tu = v$ has a holomorphic solution $u = (u_1, u_2, \ldots, u_m)$ in $V \times U$. u is analytically continued along the closed curve $\gamma \times \{x\}$ for arbitrary but fixed point x of U. Let $u' = (u'_1, u'_2, \ldots, u'_m)$ be the element of $H^0(V \times \{x\}, \mathbf{O}^m)$ obtained after one round of analytic continuation of $u(z, x)$ along $\gamma \times \{x\}$. $u'(z, x)$ is holomorphic in $V \times U$ as functions in z and the parameter x. u' satisfies $Tu' = v$ by the theorem of identity. $u - u'$ is an element of $H^0(V \times \{x\}, \mathbf{A})$ for any $x \in U$. By the above argument there are $a_1(x), a_2(x), \ldots, a_m(x) \in H^0(U, \mathbf{O})$ such that

(21) $$u(z, x) - u'(z, x) = \sum_{i=1}^{m} a_i(x) f^i(z, x)$$

in $V \times U$. By (20) there are $b_1(x), b_2(x), \ldots, b_m(x) \in H^0(U, \mathbf{O})$ such that

(22) $$b_j(x) - \sum_{k=1}^{m} b_k(x)\, c_{kj}(x) = a_j(x) \quad (j = 1, 2, \ldots, m)$$

in U. We put

(23) $$w(z, x) = u(z, x) - \sum_{j=1}^{m} b_j(x) f^j(z, x)$$

in $V \times U$. Then for arbitrary but fixed point x of U, $w(z, x)$ is continued to

$$(24)\quad u'(z,x)-\sum_{j=1}^{m} b_j(x)\sum_{k=1}^{m} c_{jk}(x) f^k(z,x) = u'(z,x)-\sum_{k=1}^{m}(b_k(x)-a_k(x)) f^k(z,x) = \\ = u(z,x)-\sum_{k=1}^{m} b_k(x) f^k(z,x) = w(z,x)$$

by (22) and (23) after one round of analytic continuation along $\gamma \times \{x\}$. This means that $w(z, x)$ is single-valued along $\gamma \times \{x\}$ for any $x \in U$. Hence $w(z, x) \in$ $\in H^0(V \times U, \mathbf{O}^m)$ is continued to an element of $H^0(\Delta, \mathbf{O}^m)$ which is the holomorphic solution in Δ of $Tw = v$.

Now let $\{\Delta_j\}$ be an open covering of Ω consisting of open sets given in the above argument and let u^j be a solution of $Tu^j = v$ in Δ_j. If $\Delta_j \cap \Delta_k \neq \emptyset$, $u^j - u^k$ satisfies $T(u^j - u^k) = 0$ in $\Delta_j \cap \Delta_k$. Since $H^0(\Omega(x), \mathbf{A}) = 0$ for any point x of S, we have $u^j = u^k$ in $\Delta_j \cap \Delta_k$. If we put $u = u^j$ in Δ_j, we obtain a well-defined element u of $H^0(\Omega, \mathbf{O}^m)$ which satisfies $Tu = v$. We have $H^1(\Omega, \mathbf{A}) = 0$ by (4).

5. SUFFICIENT CONDITION IN CASE THAT $\tilde{\Omega} = \tilde{\Omega}_1$

Lemma 8. *If $\tilde{\Omega} = \tilde{\Omega}_1$, $\tilde{\Omega}$ is a Hausdorff space and $\tilde{\Omega}$ is a Stein manifold for the natural complex structure in $\tilde{\Omega}$, then $H^1(\Omega, \mathbf{A}) = 0$.*

Proof. Let (z_0, x_0) be any point of Ω. Let V and U be, respectively, an open simply connected neighborhood of z_0 and a connected Stein neighborhood of x_0 as in the proof of Lemma 7. Let Δ be the connected component of $p^{-1}(U)$ containing (z_0, x_0). We put $\tilde{\Delta} = \pi(\Delta)$. Moreover, let $f_1^\Delta, f_2^\Delta, \ldots, f_m^\Delta$ be elements of $H^0(V \times U, \mathbf{A})$ obtained in the proof of Lemma 7. Since $\Omega(z_0, x)$ is simply connected for any point x of U, $f_1^\Delta(z, x), f_2^\Delta(z, x), \ldots, f_m^\Delta(z, x)$ are continued to elements of $H^0(\Delta, \mathbf{A})$ which are denoted by the same symbols. Since $\Omega(z_0, x)$ is simply connected for any point x of U, we have $H^0(\Delta, \mathbf{O}^m) = TH^0(\Delta, \mathbf{O}^m)$. Hence we have $H^1(\Delta, \mathbf{A}) = 0$ by (4) as Δ is a Stein manifold. Let V', U', Δ' be other such open sets and $f_1^{\Delta'}, f_2^{\Delta'}, \ldots, f_m^{\Delta'}$ be the corresponding elements of $H^0(\Delta', \mathbf{A})$. Let $\tilde{x} = \Omega(z, x)$ be any point of $\tilde{\Delta} \cap \tilde{\Delta}'$. For the fixed point x of $U \cap U'$, $f_1^\Delta(z', x), f_2^\Delta(z', x), \ldots, f_m^\Delta(z', x)$ and $f_1^{\Delta'}(z', x), f_2^{\Delta'}(z', x), \ldots$ $\ldots, f_m^{\Delta'}(z', x)$ are bases of the vector space $H^0(\Omega(z, x), \mathbf{A})$ as functions in z'. There are complex numbers $c_{ij}^{\Delta,\Delta'}(\tilde{x})$ $(i, j = 1, 2, \ldots, m)$ such that

$$(25)\qquad f_i^\Delta(z', x) = \sum_{j=1}^{m} c_{ij}^{\Delta,\Delta'}(\tilde{x}) f_j^{\Delta'}(z', x)$$

for any $(z', x) \in \Omega(z, x)$. Then $c^{\Delta,\Delta'}(\tilde{x}) = (c_{ij}^{\Delta,\Delta'}(\tilde{x}))$ is a regular matrix and $c^{\Delta,\Delta'}$ is a holomorphic mapping of $\tilde{\Delta} \cap \tilde{\Delta}'$ in the space $\mathrm{GL}(m, \mathbf{C})$ of regular $m \times m$ matrices by the arguments as in the proof of Lemma 7.

Let $\mathbf{U} = \{\Delta\}$ be an open covering of Ω consisting of such Δ. Then $\{\tilde{\Delta}\}$ is an open covering of $\tilde{\Omega}$ and $c^{\Delta,\Delta'}$ is a holomorphic mapping of $\tilde{\Delta} \cap \tilde{\Delta}'$ in $\mathrm{GL}(m, \mathbf{C})$ for any Δ and Δ' of $\mathbf{U}$. In the sum space $\bigcup_{\Delta \in \mathbf{U}} \tilde{\Delta} \times \mathbf{C}^m$ we shall induce an equivalence relation $\sim$.

Let $(\tilde{x}, w)$ and $(\tilde{x}', w')$ be, respectively, points of $\tilde{\Delta} \times C^m$ and $\tilde{\Delta}' \times C^m$. We say that $(\tilde{x}, w) \sim (\tilde{x}', w')$ if and only if $\tilde{x} = \tilde{x}'$ and

$$w'_j = \sum_{i=1}^{m} c_{ij}^{\Delta,\Delta'}(\tilde{x})\, w_i \,. \tag{26}$$

Then the factor space $\mathbf{B}$ of $\bigcup \tilde{\Delta} \times C^m$ by the equivalence relation $\sim$ is regarded as a complex analytic vector bundle over $\tilde{\Omega}$. Let π_Δ be the canonical mapping of $\tilde{\Delta} \times C^m$ in $\mathbf{B}$ and q_i be the projection of $\tilde{\Delta} \times C^m$ onto C defined by $q_i(\tilde{x}, w_1, w_2, \ldots, w_m) = w_i$ for $(\tilde{x}, w_1, w_2, \ldots, w_m) \in C^m$ $(i = 1, 2, \ldots, m)$. Let $\tilde{\mathbf{A}}$ be the sheaf of germs of holomorphic sections of $\mathbf{B}$ over $\tilde{\Omega}$. Since $\{\Delta\}$ is a Leray covering of Ω with respect to the sheaf $\mathbf{A}$, we have

$$H^1(\Omega, \mathbf{A}) = H^1(\{\Delta\}, \mathbf{A}) \,. \tag{27}$$

Let $\{g_{\Delta,\Delta'}\}$ be any element of $Z^1(\{\Delta\}, \mathbf{A})$. Let $\tilde{x} = \Omega(z, x)$ be any point of $\tilde{\Delta} \cap \tilde{\Delta}'$. There are complex numbers $a_1^{\Delta,\Delta'}(\tilde{x}), a_2^{\Delta,\Delta'}(\tilde{x}), \ldots, a_m^{\Delta,\Delta'}(\tilde{x})$ such that there holds

$$g_{\Delta,\Delta'}(z', x) = \sum_{i=1}^{m} a_i^{\Delta,\Delta'}(\tilde{x}) f_i^{\Delta}(z', x) \tag{28}$$

for any point z' of $\Omega(z, x)$. We define a mapping $s^{\Delta,\Delta'}$ of $\tilde{\Delta} \cap \tilde{\Delta}'$ in $\mathbf{B}$ by putting

$$s^{\Delta,\Delta'}(\tilde{x}) = \pi_\Delta(\tilde{x}, a_1^{\Delta,\Delta'}(\tilde{x}), a_2^{\Delta,\Delta'}(\tilde{x}), \ldots, a_m^{\Delta,\Delta'}(\tilde{x})) \tag{29}$$

for any $\tilde{x}$ in $\tilde{\Delta} \cap \tilde{\Delta}'$. Then $s^{\Delta,\Delta'} \in H^0(\tilde{\Delta} \cap \tilde{\Delta}', \tilde{\mathbf{A}})$. Since $\tilde{\mathbf{A}}$ is an analytic coherent sheaf over the Stein manifold $\tilde{\Omega}$, we have

$$\{s^{\Delta,\Delta'}\} \in Z^1(\{\tilde{\Delta}\}, \tilde{\mathbf{A}}) = B^1(\{\tilde{\Delta}\}, \tilde{\mathbf{A}}) \,. \tag{30}$$

There is an element $\{s^\Delta\}$ of $C^0(\{\tilde{\Delta}\}, \tilde{\mathbf{A}})$ such that

$$s^{\Delta,\Delta'} = s^{\Delta'} - s^{\Delta} \tag{31}$$

in $\tilde{\Delta} \cap \tilde{\Delta}'$. We put

$$g_\Delta(z, x) = \sum_{i=1}^{m} (q_i \circ (\pi_\Delta^{-1}) \circ s^\Delta)\,(\Omega(z, x))\, f_i^{\Delta}(z, x) \tag{32}$$

for $(z, x) \in \Delta$. Then $\{g_{\Delta,\Delta'}\}$ is the coboundary of $\{g_\Delta\} \in C^0(\{\Delta\}, \mathbf{A})$. Hence we have $H^1(\{\Delta\}, \mathbf{A}) = 0$. By (27) we have $H^1(\Omega, \mathbf{A}) = 0$.

6. NECESSITY IN CASE THAT $\tilde{\Omega} = \tilde{\Omega}_1$

Lastly we want to prove that $\tilde{\Omega}$ is a Hausdorff space and $\tilde{\Omega}$ is a Stein manifold for the natural complex structure in $\tilde{\Omega}$ in case that $\tilde{\Omega} = \tilde{\Omega}_1$. We shall do it under the following incidental assumption (A) concerning Ω and T.

(A) (1) There is an element $b^1(z, x)$ of $H^0(\Omega, \mathbf{A})$ such that the restriction $b^1 \mid \Omega(z, x)$ of b^1 to $\Omega(z, x)$ is not a zero vector in the vector space $H^0(\Omega(z, x), \mathbf{A})$ for any point (z, x) of Ω.

(2) For any point (z_0, x_0) of $\partial\Omega$ in $\boldsymbol{C} \times S$, there is an open neighborhood $U(x_0)$ of x_0 as following: For any connected component Δ of $p^{-1}(U(x_0))$ there are elements $b^2(z, x), b^3(z, x), \ldots, b^m(z, x)$ of $H^0(\Delta, \mathbf{A})$ such that $b^1 \mid \Omega(x) \cap \Delta$, $b^2 \mid \Omega(x) \cap \Delta, \ldots, b^m \mid \Omega(x) \cap \Delta$ form a base in the vector space $H^0(\Omega(x) \cap \Delta, \mathbf{A})$ for any point x of $U(x_0)$.

Lemma 9. *If $H^1(\Omega, \mathbf{A}) = 0$ and $\tilde{\Omega} = \tilde{\Omega}_1$, then under the assumption* (A) $\tilde{\Delta} = \varphi(\Delta)$ *is a Hausdorff space and $\tilde{\Delta}$ is a Stein manifold for the natural complex structure in $\tilde{\Delta}$.*

Proof. Although we have not proved that $\tilde{\Delta}$ is a Hausdorff space, we can speak of the sheaf $\mathbf{O}$ over $\tilde{\Delta}$. A continuous function g in an open set $\tilde{U}$ of $\tilde{\Delta}$ is said to be *holomorphic* in $\tilde{U}$ if and only if $g(\pi(z, x))$ is a holomorphic function in $\pi^{-1}(\tilde{U})$. For any open covering $\tilde{\mathbf{U}} = \{\tilde{U}_j\}$ of $\tilde{\Delta}$ and for any element $\{g_{jk}\}$ of $Z^1(\tilde{\mathbf{U}}, \mathbf{O})$, $\{g_{jk}(\pi(z, x))\, b^1(z, x)\}$ is an element of $Z^1(\pi^{-1}(\tilde{\mathbf{U}}), \mathbf{A})$. Since $H^1(\Omega, \mathbf{A}) = 0$, we have $Z(\pi^{-1}(\tilde{\mathbf{U}}), \mathbf{A}) = B(\pi^{-1}(\tilde{\mathbf{U}}), \mathbf{A})$. Hence there is an element $\{u_j(z, x)\}$ of $C^0(\pi^{-1}(\tilde{\mathbf{U}}), \mathbf{A})$ such that $\{g_{jk}(\pi(z, x))\, b^1(x, z)\}$ is a coboundary of $\{u_j(z, x)\}$. Let $\tilde{x} = \Omega(z, x)$ be a point of $\tilde{U}_j$. Since $b^1 \mid \Omega(z, x) \cap \Delta$, $b^2 \mid \Omega(z, x) \cap \Delta, \ldots, b^m \mid \Omega(z, x) \cap \Delta$ form a base of the vector space $H^0(\Omega(z, x) \cap \Delta, \mathbf{A})$, each $u_j \mid \Omega(z, x) \cap \pi^{-1}(\tilde{U}_j)$ of $H^0(\Omega(z, x) \cap \pi^{-1}(\tilde{U}_j), \mathbf{A})$ is represented as a linear combination

$$u_j(z', x) = g_j(\tilde{x})\, b^1(z', x) + \ldots \tag{33}$$

of $b^1(z', x), b^2(z', x), \ldots, b^m(z', x)$ for $z' \in \Omega(z, x) \cap \pi^{-1}(\tilde{U}_j)$. Each g_j is a holomorphic function in $\tilde{U}_j \cap \tilde{\Delta}$. We put $\tilde{\mathbf{U}} \cap \tilde{\Delta} = \{\tilde{U}_j \cap \tilde{\Delta}\}$. Then, for the restriction $g_{jk} \mid \tilde{\Delta}$ of g_{jk} to $\tilde{U}_j \cap \tilde{U}_k \cap \tilde{\Delta}$, the restriction $\{g_{jk} \mid \tilde{\Delta}\} \in Z^1(\tilde{\mathbf{U}} \cap \tilde{\Delta}, \mathbf{O})$ of $\{g_{jk}\} \in$ $\in Z^1(\tilde{\mathbf{U}}, \mathbf{O})$ to $\tilde{\Delta}$ is a coboundary of the element $\{g_j\}$ of $C^0(\mathbf{U} \cap \tilde{\Delta}, \mathbf{O})$.

If $\tilde{\Delta}$ were not a Hausdorff space, there would be a point x_0 of S and two distinct points $\Omega(z_1, x_0)$ and $\Omega(z_2, x_0)$ of $\tilde{\Omega}$ such that the set $\mathbf{B}$ of all intersections of neighborhoods of $\Omega(z_1, x_0)$ and $\Omega(z_2, x_0)$ forms a base in $\tilde{\Omega}$ as we have seen it in the proof of Lemma 6.

(a) *In case that the dimension of S is one.* There is an open connected neighbourhood $\tilde{\Omega}_1$ of $\Omega(z_1, x_0)$ such that $\Omega(z_2, x_0) \notin \tilde{\Omega}_1$. We put $\tilde{\Omega}_2 = \tilde{\Omega} - \{\Omega(z_1, x_0)\}$. Then $\{\tilde{\Omega}_1, \tilde{\Omega}_2\}$ is an open covering of $\tilde{\Omega}$. By the above argument, for $(\varphi(\Omega(z, x)) -$ $- x_0)^{-1} \in H^0(\tilde{\Omega}_1 \cap \tilde{\Omega}_2, \mathbf{O})$, there are $f_1(\tilde{x}) \in H^0(\tilde{\Omega}_1, \mathbf{O})$ and $f_2(\tilde{x}) \in H^0(\tilde{\Omega}_2, \mathbf{O})$ such that there holds

$$\frac{1}{\varphi(\Omega(z, x)) - x_0} = f_1(\tilde{x}) - f_2(\tilde{x}) \tag{34}$$

for $\tilde{x} = \Omega(z, x)$ of $\tilde{\Omega}_1 \cap \tilde{\Omega}_2 = \tilde{\Omega}_1 - \{\Omega(z_1, x_0)\}$. Since the right-hand side of (34) has a limit with respect to the trace $\mathbf{B} \cap (\tilde{\Omega}_1 \cap \tilde{\Omega}_2)$ of the filter $\mathbf{B}$ to $\tilde{\Omega}_1 \cap \tilde{\Omega}_2$ and since the left-hand side $(x - x_0)^{-1}$ has not it, this is a contradiction.

(b) *In case that the dimension of S is larger than one.* There is an analytic set M in S such that M is regular and one dimensional in a neighborhood of x_0 By Lemma 1 we have $H^1(\Omega(M), \mathbf{A}) = 0$. By the argument in (a) we arrive at a contradiction.

Any way, we have proved that $\tilde{\Omega}$ is a Hasudorff space. Next we shall prove that $\tilde{\Omega}$ is a Stein manifold for the natural complex structure in $\tilde{\Omega}$.

(c) *In case that the dimension of S is two.* Since the restriction to $\tilde{\Delta}$ of any element of $Z^1(\tilde{\mathbf{U}}, \mathbf{O})$ is a coboundary of an element of $C^0(\tilde{\mathbf{U}} \cap \tilde{\Delta}, \mathbf{O})$, $\tilde{\Delta}$ is a Stein manifold for the natural complex structure in $\tilde{\Delta}$ by the proof of (3) of Lemma 11 of Kajiwara-Kazama [5]. $\tilde{\Omega}$ is a Stein manifold by Docquier-Grauert [1].

(d) *In case that the dimension of S is larger than two.* If $\tilde{\Delta}$ were not p_7*-convex in the sense of Docquier-Grauert [1] there are a Stein subdomain U of S and an analytic set M in S such that $\tilde{\Delta} \cap \varphi^{-1}(U \cap M)$ is not p_7-convex in the sense of Docquier-Grauert and M is regular and two dimensional in U. By Lemma 1 we have $H^1(\Omega(M), \mathbf{A}) = 0$. By the argument in (c) $\tilde{\Delta} \cap \varphi^{-1}(U \cap M)$ is a Stein manifold. But this is a contradiction. $\tilde{\Omega}$ is a Stein manifold by Docquier-Grauert [1].

Any way, we have proved that $\tilde{\Omega}$ is a Stein manifold.

7. STATEMENT OF THE THEOREM

Theorem. *Let S be a Stein manifold and Ω be a Stein subdomain of $C \times S$. For any point x of S we put $\Omega(x) = \Omega \cap (C \times \{x\})$. For any point (z, x) of Ω let $\Omega(z, x)$ be the connected component of $\Omega(x)$ containing (z, x). Let $\tilde{\Omega}$ be the set of all $\Omega(z, x)$ for $(z, x) \in \Omega$. In the factor set $\tilde{\Omega}$ of Ω we induce a factor topology of Ω. Let $\tilde{\Omega}_1$ be the set of all simply connected $\Omega(z, x)$ for $(z, x) \in \Omega$ and $\tilde{\Omega}_2$ be the set of all doubly connected $\Omega(z, x)$ for $(z, x) \in \Omega$. Let $\mathbf{O}$ be the sheaf of all germs of holomorphic functions in Ω, m be a positive integer and $a_{jk}(z, x)$ be holomorphic functions in Ω for $j, k = 1, 2, \ldots, m$. Let $\mathbf{A}$ be the kernel of the homomorphism of $\mathbf{O}^m$ in $\mathbf{O}^m$ defined by*

$$T(u_1, u_2, \ldots, u_m) =$$

$$= \left(\frac{\partial u_1}{\partial z} + \sum_{k=1}^{m} a_{1k}(z, x)\, u_k,\ \frac{\partial u_2}{\partial z} + \sum_{k=1}^{m} a_{2k}(z, x)\, u_k, \ldots, \frac{\partial u_m}{\partial z} + \sum_{k=1}^{m} a_{mk}(z, x)\, u_k\right)$$

for $(u_1, u_2, \ldots, u_m) \in \mathbf{O}^m$.

If $H^1(\Omega, \mathbf{A}) = 0$, either $\tilde{\Omega} = \tilde{\Omega}_1$ or $\tilde{\Omega} = \tilde{\Omega}_2$. In case that $\tilde{\Omega} = \tilde{\Omega}_2$, the necessary and sufficient condition that $H^1(\Omega, \mathbf{A}) = 0$ is that $\tilde{\Omega}$ is a Hausdorff space and $H^0(\Omega(x), \mathbf{A}) = 0$ for any point x of S. In case that $\tilde{\Omega} = \tilde{\Omega}_1$, if $\tilde{\Omega}$ is a Hausdorff *space and* $\tilde{\Omega}$ *is a Stein manifold for the natural complex structure in* $\tilde{\Omega}$, *we have* $H^1(\Omega, \mathbf{A}) = 0$. *In case that* $\tilde{\Omega} = \tilde{\Omega}_1$, *under the assumption* (A) *if* $H^1(\Omega, \mathbf{A}) = 0$, $\tilde{\Omega}$ *is a Hausdorff space and* $\tilde{\Omega}$ *is a Stein manifold for the natural complex structure in* $\tilde{\Omega}$.

References

[1] *F. Docquier* und *H. Grauert,* Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeit, Math. Ann. *140* (1960), 94—123.

[2] *L. Ehrenpreis,* Sheaves and differential equations, Proc. Amer. Math. Soc. *7* (1956), 1131 to 1139.

[3] *S. Hitotumatu,* On some conjectures concerning pseudoconvex domains, J. Math. Soc. Jap. *6* (1954), 177—195.

[4] *J. Kajiwara,* On an application of L. Ehrenpreis' method to ordinary differential equations, Kōdai Math. Sem. Rep. *15* (1963), 94—105.

[5] *J. Kajiwara* and *H. Kazama,* Two dimensional complex manifold with vanishing cohomology set, Math. Ann. (1973) (in press).

[6] *H. Suzuki,* On the global existence of holomorphic solutions of the equation $\partial u/\partial x_1 = f$ (to appear).

Authors' adresses: J. Kajiwara, Department of Mathematics, Faculty of Science, Kyushu University, Fukuoka, 812, Japan, Y. Mori, Department of Mathematics, Faculty of Science and Technology, University of the Ryukyus, Naha, 903, Japan.