Czechoslovak Mathematical Journal

Ivan Chajda

Systems of equations and tolerance relations

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 2, 302–308

Persistent URL: http://dml.cz/dmlcz/101322

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

SYSTEMS OF EQUATIONS AND TOLERANCE RELATIONS

IVAN CHAJDA, Přerov

(Received May 29, 1974)

Systems of equations are very useful tools for various mathematical investigations. Some generalizations of equations and their systems for abstract algebras and quasi-algebras are introduced by J. SŁOMINSKI in [1]. A completely algebraic foundation of the theory of equations and systems of equations is given there from the abstract algebraic point of view. Other problems like approximation of solutions and their stability have not yet been algebraized. We can formulate these problems to advantage by means of the so called tolerance relations. The concept of tolerance relation was introduced by E. C. ZEEMAN, for automata by M. ARBIB and for algebraic systems by B. ZELINKA (see [2] and [3]). By tolerance relation on a set A we mean a reflexive and symmetric binary relation on A. This concept was derived from "classical tolerance T_ε" on the set of real numbers defined by the rule:

$$xT_\varepsilon y \quad \text{if and only if} \quad |x - y| < \varepsilon \quad \text{for fixed real} \quad \varepsilon > 0 .$$

This relation is reflexive and symmetric (but not transitive) and these properties are abstracted to define tolerance relations.

The purpose of this paper is to give the foundations of the theory of stability and approximate solutions of systems of equations from an abstract algebraic point of view. Some theorems on these concepts are proved here but many other problems still remain open.

1. TOLERANCE RELATIONS

By *tolerance relation* T on a set A we mean a reflexive and symmetric binary relation on A, i.e., for each $a, b \in A$ it holds

1) aTa,

2) if aTb, then bTa.

Let A be an algebra with the set of elements A and the set of fundamental operations Ω, in brief notation $\mathfrak{A} = (A, \Omega)$. A tolerance relation T on A is called *compatible on* $\mathfrak{A}$ if for each n-ary operation $\omega \in \dot{\Omega}$ and arbitrary elements $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ such that $a_i T b_i$ for $i = 1, \ldots, n$,

$$\omega(a_1, \ldots, a_n)\, T\, \omega(b_1, \ldots. b_n)$$

holds.

Theorem 1. *Let $T_1, \ldots, T_m$ be tolerance relations on A and let $T = T_1 . T_2 . \ldots . T_m$ be the product of relations (in the usual sense). Then $T_i \subseteq T$ for all $i = 1, \ldots, m$ and T is a reflexive relation.*

Proof. For every $a \in A$ it is aT_ia, from which aTa and T is reflexive. Let aT_ib for $a, b \in A$ and $i \in \{1, \ldots, m\}$, then aT_ja for each $j < i$ and bT_kb for each $k > i$; thus $aT_1a \ldots aT_ib \ldots bT_mb$, i.e., aTb. Accordingly, $T_i \subseteq T$.

Corollary 2. *Let $T_1, \ldots, T_m$ be tolerance relations on A, then $\bigcup_{i=1}^{m} T_i \subseteq T_1 . T_2 . \ldots$ $\ldots . T_m$.*

Theorem 3. *Let $T_1, \ldots, T_n$ be tolerance relations on A. The product $T_1 \ldots T_n$ is a tolerance relation on A if and only if $T_1 \ldots T_n = T_n \ldots T_1$. Further, if $T_1, \ldots, T_n$ are compatible tolerance relations on $\mathfrak{A} = (A, \Omega)$ and $T_1 \ldots T_n = T_n \ldots T_1$, then $T_1 \ldots T_n$ is a compatible tolerance relation on $\mathfrak{A}$.*

Proof. Let $T_1 \ldots T_n = T_n \ldots T_1$ and $aT_1 \ldots T_nb$ for any $a, b \in A$. Then there exist $c_i \in A$ such that $c_0 = a$, $c_n = b$ and $c_{i-1}T_ic_i$ for each $i = 1, \ldots, n$. From the symmetry of $T_1, \ldots, T_n$ we obtain $c_iT_ic_{i-1}$ for $i = 1, \ldots, n$, i.e. $bT_n \ldots T_1a$, thus $T_1 \ldots T_n$ is symmetric. By Theorem 1, $T_1 \ldots T_n$ is also reflexive, thus $T_1 \ldots T_n$ is a tolerance relation on A. If $T_1, \ldots, T_n$ are compatible on $\mathfrak{A} = (A, \Omega)$, ω is an m-ary operation from Ω and $a_i, b_i \in A$ satisfy $a_iT_1 \ldots T_nb_i$ where $i = 1, \ldots, m$, then there exist $d_{ij} \in A$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$ such that $d_{i0} = a_i$, $d_{in} = b_i$ and $d_{i,j-1}T_jd_{ij}$. From the compatibility of $T_1 \ldots T_n$ we obtain

$$\omega(d_{1,j-1}, \ldots, d_{m,j-1})\, T_j\, \omega(d_{1j}, \ldots, d_{mj}),$$

thus $\omega(a_1, \ldots, a_m)\, T_1 \ldots T_n\, \omega(b_1, \ldots, b_m)$, i.e., $T_1 \ldots T_n$ is compatible on $\mathfrak{A}$.

Conversely, let $T_1, \ldots, T_n$ and $T_1 \ldots T_n$ be tolerance relations on A and $aT_1 \ldots T_nb$ for some $a, b \in A$. Then $bT_1 \ldots T_na$ and there exist $c_i \in A$ such that $c_0 = b$, $c_n = a$ and $c_{i-1}T_ic_i$ for $i = 1, \ldots, n$. Thus $c_iT_ic_{i-1}$. Hence we obtain $aT_n \ldots T_1b$, i.e., $T_1 \ldots T_n \subseteq$ $\subseteq T_n \ldots T_1$. The other inclusion $T_n \ldots T_1 \subseteq T_1 \ldots T_n$ follows in a similar manner, i.e., $T_1 \ldots T_n = T_n \ldots T_1$.

Corollary 4. *Let T_1, T_2 be two tolerance relations on A. The product $T_1 . T_2$ is a tolerance relation on A if and only if T_1, T_2 are commutative. Further, if T_1, T_2 are commutative and compatible tolerance relations on $\mathfrak{A} = (A, \Omega)$, then $T_1 . T_2$ is a compatible tolerance relation on $\mathfrak{A}$.*

It is clear that every tolerance relation is commutative with itself. Hence the following corollary of Theorems 3 and 1 is evident:

Corollary 5. *If T is a tolerance relation on A, then the same holds for T^2 and $T \subseteq T^2$. If T is a compatible tolerance relation on $\mathfrak{A} = (A, \Omega)$, then T^2 is a compatible tolerance relation on $\mathfrak{A}$. If n, k are positive integers and $k < n$, then T^n is a tolerance relation and $T^{n-k} \subseteq T^n$. If T is compatible on $\mathfrak{A}$, then T^n is also compatible on $\mathfrak{A}$.*

Remark. Let A be the set of real numbers. By the "classical tolerance" on A we mean the relation T_ε introduced by:

$$aT_\varepsilon b \quad \text{if and only if} \quad |a - b| < \varepsilon \quad \text{for fixed real} \quad \varepsilon > 0\,.$$

If T_ε is a classical tolerance on A, then it is easy to prove:

$$aT_\varepsilon^n b \quad \text{if and only if} \quad |a - b| < n\,.\,\varepsilon \quad \text{for positive integer} \quad n\,.$$

2. SYSTEMS OF EQUATIONS OVER ALGEBRAS

Let $\mathfrak{A} = (A, \Omega)$ be an algebra and X a set disjoint with A and Ω. Consider X to be a well-ordered set, i.e. $X = \{x_\mu, \mu < s\}$, where s is an ordinal number. By $\mathfrak{A}(X)$ we denote the set of all expressions generated by elements of A, Ω, X which become elements of A if all x_μ, $\mu < s$ are replaced by elements of A, i.e., the right number of elements from X and A being taken after each operation symbol; $\tau(x_\mu)$ denotes an element of $\mathfrak{A}(X)$. If each x_μ is replaced by a_μ in all places in $\tau(x_\mu)$, we obtain an element $\tau(a_\mu) \in A$.

Let us introduce an equivalence relation θ on $\mathfrak{A}(X)$ by the rule:

$$\tau(x_\mu)\,\theta\,\varrho(x_\mu) \quad \text{for} \quad \tau, \varrho \in \mathfrak{A}(X) \quad \text{if and only if} \quad \tau(a_\mu) = \varrho(a_\mu)$$

for each set $\{a_\mu, \mu < s\}$ of elements from A. It is clear that θ is a congruence on $(\mathfrak{A}(X), \Omega)$. By formal $\mathfrak{A}$-polynomial algebra denoted by $\mathrm{For}(\mathfrak{A}, X)$ we mean the factor algebra $(\mathfrak{A}(X), \Omega)/\theta$.

Definition 1. Let $\mathfrak{A} = (A, \Omega)$ be an algebra and $X = \{x_\mu, \mu < s\}$ be a set fulfilling $X \cap A = \emptyset = X \cap \Omega$. Let S be the set of all elements of $\mathrm{For}(\mathfrak{A}, X)$. Each subset E of the Cartesian product $S \times S$ is called system of equations over $\mathfrak{A}$. Each pair $\langle \tau, \vartheta \rangle \in E$ is called *$\mathfrak{A}$-equation* (or briefly *equation*). Elements of X generating τ, ϑ for $\langle \tau, \vartheta \rangle \in E$ are called *unknowns of E.*

The theory of systems of equations over algebras and quasi-algebras is introduced in [1]. The above formulated definition is taken from there.

Definition 2. Let $\mathfrak{A} = (A, \Omega)$ be an algebra, E a system of equations over $\mathfrak{A}$ and $X = \{x_\mu, \mu < s\}$ a set of unknowns of E. Let $\sim$ be a congruence relation on $\mathfrak{A}$. Any set $\{V_\mu, \mu < s\}$ of elements $V_\mu \in A$ such that $\tau(V_\mu) \sim \vartheta(V_\mu)$ for each $\langle \tau, \vartheta \rangle \in E$ is called *solution with the regularizer* $\sim$. If $\sim$ is the identity relation on A, the solution with the regularizer $\sim$ is called *proper*.

Remark. A proper solution is a solution in the sense of the classical definition. The above mentioned definition is, however, more general than the classical one.

3. APPROXIMATE SOLUTIONS

In numerical analysis the so called "approximate solutions" of systems of equations are studied. To the author's knowledge this theory is not yet algebraized. The purpose of this section is to give its algebraic foundation and to investigate the theory of stability from the abstract algebraic point of view.

Definition 3. Let $\mathfrak{A} = (A, \Omega)$ be an algebra and E a system of equations over A with the set of unknowns $X = \{x_\mu, \mu < s\}$. Any set $\{V_\mu, \mu < s\}$ of elements from A is called *approximate T-solution of E*, if T is a tolerance on A and

$$\tau(V_\mu)\ T\ \vartheta(V_\mu) \quad \text{for each equation} \quad \langle \tau, \vartheta \rangle \in E\,.$$

Remark. If T_ε is the "classical tolerance", then Definition 3 introduces an approximate solution in the sense of numerical methods. Further, each congruence is a tolerance relation, hence each solution of E with the regularizer $\sim$ is an approximate $\sim$-solution. If $\sim$ is an identity on E (i.e., any solution with the regularizer $\sim$ is proper), the approximate *identity*-solution is called *exact solution*.

Element $\vartheta \in \text{For}\,(\mathfrak{A}, X)$ generated only by A and Ω is called *constant*, because it does not depend on elements from X. This fact can be expressed by the symbolism ϑ_{const}. System E of equations over A is said to be *normalized*, if $\vartheta = \vartheta_{\text{const}}$ for each $\langle \tau, \vartheta \rangle \in E$. We can easy prove

Theorem 6. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, T a compatible tolerance relation on A, E a system of equations over A and $\{V_\mu, \mu < s\}$ an approximate T'-solution of E, where T' is a tolerance relation on A. Let $T_0 = T \,.\, T' \,.\, T$. Then every set $\{W_\mu, \mu < s\}$ of elements $W_\mu \in A$ such that $W_\mu T V_\mu$ for $\mu < s$ is an approximate T_0-solution of E. If E is a normalized system and $T \,.\, T' = T' \,.\, T$, then $\{W_\mu, \mu < s\}$ is an approximate $T \,.\, T'$-solution of E.*

Proof. If T is compatible, then $W_\mu T V_\mu$ implies $\tau(W_\mu)\, T\, \tau(V_\mu)$ and $\vartheta(V_\mu)\, T\, \vartheta(W_\mu)$. If $\{V_\mu, \mu < s\}$ is an approximate T'-solution, then $\tau(V_\mu)\, T'\, \vartheta(V_\mu)$ for each $\langle \tau, \vartheta \rangle \in E$. However,

$$\tau(W_\tau)\, T\, \tau(V_\mu)\, T'\, \vartheta(V_\mu)\, T\, \vartheta(W_\mu) \Rightarrow \tau(W_\mu)\, T\, .\, T'\, .\, T\, \vartheta(W_\mu)\, .$$

By Theorem 3 the relation $T\, .\, T'\, .\, T$ is a tolerance relation on A and $\{W_\mu, \mu < s\}$ is an approximate T_0-solution, where $T_0 = T\, .\, T'\, .\, T$. If E is normalized, then

$\vartheta = \vartheta_{\text{const}}$ for each $\langle \tau, \vartheta \rangle \in E$. Then $\tau(W_\mu)\, T\, \tau(V_\mu)\, T'\, \vartheta(V_\mu) = \vartheta_{\text{const}} = \vartheta(W_\mu)$, i.e., $\{W_\mu, \mu < s\}$ is an approximate $T\, .\, T'$-solution, if $T\, .\, T'$ is a tolerance relation on A. This is true if and only if $T\, .\, T' = T'\, .\, T$ as follows from Theorem 3.

Corollary 7. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, T a compatible tolerance relation on $\mathfrak{A}$, E a system of equations over $\mathfrak{A}$ and $\{V_\mu, \mu < s\}$ an exact (i.e. proper) solution of E. Every set $\{W_\mu, \mu < s\}$ of elements $W_\mu \in A$ such that $W_\mu T V_\mu$ is an approximate T^2-solution of E.*

Proof. This follows directly from Theorem 6 for T' equal to the identity relation on $\mathfrak{A}$. By Theorem 3, T^2 is a tolerance relation on A.

Corollary 8. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, T a compatible tolerance relation on $\mathfrak{A}$, E a normalized system of equations over $\mathfrak{A}$ and $\{V_\mu, \mu < s\}$ an exact (i.e. proper) solution of E. Then every set $\{W_\mu, \mu < s\}$ of elements $W_\mu \in A$ such that $W_\mu T V_\mu$ is an approximate T-solution of E.*

Proof. If $\{V_\mu, \mu < s\}$ is an exact solution of E, $\{V_\mu, \mu < s\}$ is T'-approximate, where T' is the identity relation I on A. However, $I\, .\, T = T\, .\, I = T$ for every tolerance relation T on A, thus Theorem 6 implies the assertion.

Remark. These corollaries give a method for finiding approximate T-solutions of E if at least one exact solution of E is known.

4. STABILITY OF SOLUTIONS

Now, we can introduce a general concept of "stability" of solutions.

Definition 4. Let $\mathfrak{A} = (A, \Omega)$ be an algebra, $\sim$ a congruence relation on $\mathfrak{A}$, E a system of equations over $\mathfrak{A}$ with the set of unknowns $X = \{x_\mu, \mu < s\}$ and let T_1, T_2 be tolerance relations on A. Let $\{V_\mu, \mu < s\}$ be a solution of E with the regularizer $\sim$ (or approximate T_1-solution, or exact solution). If every set $\{W_\mu, \mu < s\}$ of elements $W_\mu \in A$ such that $W_\mu T_2 V_\mu$ is an approximate T_1-solution of E, then $\{V_\mu, \mu < s\}$ is called *$T_1 - T_2$-stable solution* (or *$T_1 - T_2$-stable approximate solution*, or *$T_1 - T_2$-stable exact solution*, respectively).

Remark. If A is a set of real numbers and T_ε, T_δ are two "classical tolerances", i.e.,

$$xT_\varepsilon y \text{ iff } |x - y| < \varepsilon \text{ and } fT_\delta g \text{ iff } \max_x |f(x) - g(x)| < \delta,$$

then the $T_\delta - T_\varepsilon$-*stable exact solution* of the system of equations over the field of real numbers is a stable solution of the system of equations by the classical definition in the theory of stability. Definition 4 is, however, more general.

Now, we can state some assertions concerning stable solutions. From Corollary 8 we obtain immediately

Corollary 9. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, T a compatible tolerance relation on $\mathfrak{A}$ and E a normalized system of equations over $\mathfrak{A}$. Then every exact (i.e. proper) solution of E is a $T - T$-stable exact solution of E.*

We can generalize this result for non-normalized systems by the following

Theorem 10. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, T_1 a tolerance relation on A, T_2 a compatible tolerance relation on $\mathfrak{A}$, E a system of equations over $\mathfrak{A}$ and $\{V_\mu, \mu < s\}$ an approximate T_1-solution of E. Then $\{V_\mu, \mu < s\}$ is a $T_0 - T_2$-stable approximate solution of E, where $T_0 = T_2 \, . \, T_1 \, . \, T_2$.*

Proof. By Theorem 3, T_0 is a tolerance relation on A and by Theorem 6 every set $\{W_\mu, \mu < s\}$ where $W_\mu T_2 V_\mu$ is an approximate T_0-*solution of* E. Thus, by Definition 4, $\{V_\mu, \mu < s\}$ is a $T_0 - T_1$-stable approximate solution of E.

Corollary 11. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra, E a system of equations over $\mathfrak{A}$ and T a compatible tolerance relation on $\mathfrak{A}$. Then every exact (i.e. proper) solution of E is a $T^2 - T$-stable exact solution of E.*

Proof. If T_1 is an identity relation on A, then $T_0 = T \, . \, T_1 \, . \, T = T^2$. By Theorem 3, T_0 is a compatible tolerance relation on A and by Corollary 7 we obtain the result.

Corollary 12. *Let $\mathfrak{A} = (A, \Omega)$ be an algebra and let $\sim$ be a congruence relation on $\mathfrak{A}$. Let be a system of equations over $\mathfrak{A}$ and T a compatible tolerance relation on $\mathfrak{A}$. Every solution $\{V_\mu, \mu < s\}$ of E with the regularizer $\sim$ is a $T_0 - T$-stable solution of E, where $T_0 = T \, . \sim . \, T$ and every set $\{W_\mu, \mu < s\}$ of elements $W_\mu \in A$ such that $W_\mu T V_\mu$ is an approximate T_0-solution of E.*

Proof. By Theorem 3, T_0 is a tolerance relation on A. The first assertion follows directly from Theorem 10 and the other from Theorem 6.

Remark. This corollary characterizes the mutual relation between the solutions in the generalized sense of Definition 2 and the approximate and stable solutions with respect to compatible tolerance relations. For tolerance relations which are not compatible the problems of stability and approximate solutions remain still open.

Concluding remark. Comparing Corollaries 7 and 8 or Corollaries 9 and 11 we obtain new general results (which have not yet been formulated in the theories of approximation or stability, respectively). The ability of systems of equations "to be normalized" has the principal influence (given by the inclusion $T \subseteq T^2$) on the "*accuracy*" or "*stability*" of solutions. Further, Theorem 10 yields a general relation between "accuracy" and "stability" of a given system of equations.

References

[1] *Slominski J.:* On the solving of systems of equations over quasi-algebras and algebras (Bull. de l'Acad. Pol. des Sci., série des sci. math., astr. et phys., Vol. *X*, (1962), p. 627—635.)

[2] *Zelinka B.:* Tolerance in algebraic structures (Czech. Math. J. V, *20* (95), Praha 1970, p. 179—183).

[3] *Zelinka B.:* Tolerance in algebraic structures II (Czech. Math. J., to appear).

Author's address: Třída Lidových milicí 290, 750 00 Přerov, ČSSR.