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LINEAR OPERATORS ON $C_X(\Omega)$ FOR Ω DISPERSED

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In [1], we gave a characterization of unconditionally converging and Dunford-Pettis operators on c_X, the B-space of X-valued convergent sequences when X is a B-space, by using some results of BATT ([2]) on unconditionally converging series of bounded linear operators. In this note we point out that the methods of [1] can be employed to give characterizations of unconditionally converging, weakly compact, compact and Dunford-Pettis operators on $C_X(\Omega)$, the B-space of X-valued continuous functions on a dispersed, compact space Ω. The essential tools used are the results of Batt ([2]) and the characterization of the dual of $C(\Omega)$ given in [10].

Let S be a compact Hausdorff space with $\mathscr{B}$ the Borel sets of S and let X, Y be B-spaces with $L(X, Y)$ the space of bounded linear operators from X into Y. First we recall the following representation of bounded linear operators $T: C_X(S) \to Y$, where $C_X(S)$ is the B-space of all continuous functions $f : S \to X$ with $\|f\| = \sup \{\|f(t)\| : t \in S\}$. For each such T there is a unique finitely additive set function $m : \mathscr{B} \to L(X, Y'')$ such that $Tf = \int_S f \, dm$ ([4], p. 217; m has other properties which we do not list). The set function m is called the representing measure of T, and m is said to be strongly bounded if there is a positive regular Borel measure λ on $\mathscr{B}$, called a control measure for m, such that $\lim_{\lambda(E)\to 0}$ semi-var $(m)(E) = 0$ (see [6], Lemma 2 for another formulation).

Before stating the main theorem, we first establish a preliminary result concerning Dunford-Pettis operators which we will need later. A bounded lineat operator T between B-spaces is said to be a Dunford-Pettis (DP) operator if T carries weak Cauchy sequences into norm-convergent sequences ([1]; [6], §6), and T is said to be an unconditionally converging (u.c.) operator if T carries weakly unconditionally Cauchy series (w.u.c. series) into unconditionally converging series ([9]).

Proposition 1. *If $T : C_X(S) \to Y$ is a DP operator, then its representing measure m is strongly bounded and $m(E) : X \to Y$ is a DP operator for each $E \in \mathscr{B}$.*

Proof. Since a DP operator is u.c., m is strongly bounded ([6], Theorem 3). For each $\phi \in C(S)$ let $T_\phi : X \to Y$ be the bounded linear operator defined by $T_\phi(x) = T(\phi x)$. Since the linear map $\hat{\phi} : X \to C_X(S)$, $\hat{\phi}(x) = \phi x$, is norm-continuous, it is weak-continuous ([7], V.3.15), and if $\{x_i\}$ is weak-Cauchy in X, $\{\phi x_i\}$ is weak-Cauchy in $C_X(S)$, and $\{T_\phi(x_i)\}$ is norm-convergent. That is, T_ϕ is a DP operator.

Let $K \subseteq S$ be compact. For G open and $G \supseteq K$ choose $\phi_G \in C(S)$ such that $0 \leqq \phi_G \leqq 1$, $\phi_G(t) = 1$ for $t \in K$, and support $(\phi_G) \subseteq G$. For $\|x\| \leqq 1$,

$$(1) \qquad \|T_{\phi_G} x - m(K)\, x\| = \|\textstyle\int_S (\phi_G x - C_K x)\, dm\| \leqq \text{semi-var}\,(m)\,(G \setminus K)$$

so that $m(K)$ is a DP operator from (1), the fact that m is strongly bounded and Proposition 3 of [1]. If $E \in \mathscr{B}$, then $m(E)$ is a DP operator since $m(K)$ is a DP operator for each compact K and m is regular ([6], Lemma 2).

We establish a partial converse for Proposition 1 in Theorem 2.

For the remainder of this paper Ω will denote a compact, Hausdorff, dispersed space, i.e., Ω has no non-void perfect sets ([10]). Suppose $T: C_X(\Omega) \to Y$ is a bounded linear operator whose representing measure m is strongly bounded with control measure λ. Now λ has the form, $\lambda = \sum_{i=1}^{\infty} a_i \delta_{t_i}$, where $\{a_i\} \in l^1$, $t_i \in \Omega$, and δ_t is the Dirac measure concentrated at $t \in \Omega$ ([10]). Since m is absolutely continuous with respect to λ, m has the form $m = \sum_{i=1}^{\infty} T_i \varepsilon_{t_i}$, where $T_i = m(\{t_i\})$ and ε_t is the (vector) "Dirac measure" concentrated at $t \in \Omega$, that is, $\varepsilon_t : C_X(\Omega) \to X$ is defined by $\varepsilon_t(f) = f(t)$. If we set $S_i = T_i \varepsilon_{t_i} \in L(C_X(\Omega), Y)$, then $T = \sum_{i=1}^{\infty} S_i$. Since m is strongly bounded, the series $\sum_{i=1}^{\infty} T_i$ is bounded-evaluation convergent, i.e., $\sum_{i=1}^{\infty} T_i x_i$ converges unconditionally for each bounded sequence $\{x_i\}$ in X ([12], §3; [2], Theorem 6). Thus the series $\sum_{i=1}^{\infty} S_i$ converges in the uniform operator topology of $L(C_X(\Omega), Y)$ ([2], Theorem 2). From these observations, we obtain

Theorem 2. *Let $T: C_X(\Omega) \to Y$ be a bounded linear operator whose representing measure m is strongly bounded. Then*

(i) *T is compact (weakly compact) iff $m(E)$ is compact (weakly compact) for each $E \in \mathscr{B}$.*

(ii) *T is u.c. iff $m(E)$ is u.c. for each $E \in \mathscr{B}$.*

(iii) *T is DP iff $m(E)$ is DP for each $E \in \mathscr{B}$.*

Proof. The necessity of (i) is given in [4], Theorem 6; that of (ii) in [6], Theorem 3; and that of (iii) in Proposition 1. The sufficiency of each condition follows from the convergence of the series ΣS_i in the uniform operator topology and the fact that each class of operators in (i), (ii), and (iii) is closed in the uniform operator topology ([7], VI. 5.3 and VI. 4.4; [8], Proposition 1.2; [1], Proposition 3).

Remark 3. Part (ii) establishes a conjecture of DOBRAKOV ([5], [6]) for dispersed spaces. Also, see [11], Theorem 6, relative to this conjecture. It is still an open question as to whether (ii) is valid for arbitrary compact spaces. Part (i) is not valid for arbitrary compact spaces as is shown by Batt in [3], Example 2. It is also an open question as to whether (iii) holds for arbitrary compact spaces; see [1], Theorem 4, for the case when Ω is the one-point compactification of the positive integers.

Recall that a B-space X has property V if for each B-space Y a bounded linear operator $T: X \to Y$ is u.c. iff T is weakly compact ([9]). From Theorem 2, we have

Corollary 4. *$C_X(\Omega)$ has property V iff X has property V.*

Remark 5. This establishes a conjecture of PELCZYNSKI for dispersed spaces ([9]). See also [11], Theorem 14, concerning this problem.

Recall a B-space X has the Dunford-Pettis (DP) property if for each B-space Y any weakly compact operator $T: X \to Y$ is DP ([6], §6). From Theorem 2, we have

Corollary 6. *$C_X(\Omega)$ has the DP property iff X has the DP property.*

Remark 7. This result improves Theorem 13a of [6] and also Corollary 5 of [1].

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