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THE LATTICE OF SOLID σ-SUBGROUPS OF A RETRACTABLE GROUP

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1. Introduction. The concept of a retractable group was introduced in [2] and there it was shown that the class of lattice-ordered groups is a proper subclass of the class of retractable groups, which in turn is a proper subclass of the class of torsion free groups. In 1942 G. BIRKHOFF [1] proved that the collection of l-ideals of a lattice-ordered group is a complete sublattice of the lattice of subgroups and that this sublattice is Brouwerian. In 1962 this result was generalized by K. LORENZ [5]. Lorenz showed that the collection of convex l-subgroups of a lattice-ordered group is a complete sublattice of the lattice of subgroups and again this sublattice is Brouwerian. In [2, Theorem 4.2 (iv)] it was shown that the collection of ϱ-σ-subgroups of a retractable group is a complete sublattice of the lattice of subgroups. The dual assertion is true for λ-σ-subgroups, and hence, is true for solid σ-subgroups. The main result of this paper (Theorem 4.4) is that the collection of normal solid σ-subgroups is Brouwerian. This is a generalization of Birkhoff's result cited above. We note that the normal solid σ-subgroups are kernels of σ-τ-homomorphisms (see [2], Section 4).

In Section 2 we give the definitions and notation that will be used throughout the paper. In addition, we recall some results from [2] that will be frequently used. In Section 3 we give sufficient conditions for a ϱ-σ-subgroup to be a λ-σ-subgroup (Theorem 3.1 and Corollary 3.3) and sufficient conditions for the transitivity of ϱ-σ-subgroups (Theorems 3.7 and 3.8). In addition to the main result in Section 4, we show that if H and J are disjoint solid σ-subgroups, then they commute elementwise (Theorem 4.2). Finally, in Section 5 we give an example to illustrate the scope and limitations of our theory.

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2. Preliminaries. Throughout this paper, G will denote a group, written multiplicatively and with identity i, and $F(G)$ will denote the collection of all finite, nonempty subsets of G. Then $F(G)$ is a join monoid, that is, $F(G)$ is a join semilattice in which $A \vee B = A \cup B$, $F(G)$ is a monoid in which $AB = \{ab \mid a \in A \text{ and } b \in B\}$, $A(B \vee C) = AB \vee AC$, and $(A \vee B)\, C = AC \vee BC$. A homomorphism σ of $F(G)$ into G such that $\{g\}\, \sigma = g$ for every g in G, will be called a *retraction* of G. We will denote by Ret G the collection of all retractions of G. If Ret G is nonempty, then G is said to be a *retractable* group. If $\sigma \in$ Ret G then the *kernel* of σ is the set Ker $\sigma = \{A \mid A \in F(G) \text{ and } A\sigma = i\}$. If Ker σ is a convex subsemilattice of $F(G)$, then σ is said to be an *l-retraction* of G. There is a one-to-one correspondence between the lattice-orderings of G and the l-retractions of G [2, Corollary 3.3]; in this case $\bigvee A$ equals $A\sigma$ for all $A \in F(G)$.

If H is a subgroup of G and $\sigma \in$ Ret G, let

$$\varrho_{H,\sigma} = \{(A, B) \mid A, B \in F(G) \text{ and } H(A\sigma) = H(B\sigma)\},$$

$$\lambda_{H,\sigma} = \{(A, B) \mid A, B \in F(G) \text{ and } (A\sigma)\, H = (B\sigma)\, H\}.$$

It was shown in [2, Theorem 2.12] that the mapping given by $H \to \varrho_{H,\sigma}$ is a complete lattice isomorphism from the lattice of all subgroups of G into the lattice of all equivalence relations of $F(G)$. (Dually, so is the mapping $H \to \lambda_{H,\sigma}$.) It is easily seen that H is normal in G if and only if $\lambda_{H,\sigma} = \varrho_{H,\sigma}$ (or $\lambda_{H,\sigma} \supseteq \varrho_{H,\sigma}$). We call H a *σ-subgroup* of G provided that $A\sigma \in H$ for every $A \in F(H)$. If σ is an l-retraction, then σ-subgroups correspond to l-subgroups. If H is a σ-subgroup, then the restriction of σ to $F(H)$ is a retraction of H and we will denote the restriction by σ_H.

If θ is an equivalence relation on a set X and $x \in X$, then $[x]\, \theta$ will denote the equivalence class containing x.

Theorem 2.1. *If $\sigma \in$ Ret G and H is a subgroup of G, then the following are equivalent:*

(i) *H is a σ-subgroup of G;*
(ii) $F(H) \subseteq [\{i\}]\varrho_{H,\sigma}$;
(iii) *if $(A, B) \in \varrho_{H,\sigma}$ and $C \in F(H)$, then $(A, CB) \in \varrho_{H,\sigma}$;*
(iv) *if $A \in F(G)$, then $F(H)([A]\, \varrho_{H,\sigma}) \subseteq [A]\, \varrho_{H,\sigma}$;*
(v) $F(H)([\{i\}]\, \varrho_{H,\sigma}) \subseteq [\{i\}]\, \varrho_{H,\sigma}$.

The equivalence of (i) and (ii) was given in [2, Corollary 2.13] and the equivalence of (i), (iii), (iv), and (v) is straightforward. Of course, (ii) through (v) may be replaced by the corresponding assertions involving $\lambda_{H,\sigma}$.

Again, let $\sigma \in$ Ret G and H be a subgroup of G. Then H is said to be a *ϱ-σ-subgroup* (resp., λ-σ-subgroup) if $A = \{a_1, \ldots, a_n\} \in F(G)$ and $h_1, \ldots, h_n \in H$ implies that $(A, \{h_1 a_1, \ldots, h_n a_n\}) \in \varrho_{H,\sigma}$ (resp., $(A, \{a_1 h_1, \ldots, a_n h_n\}) \in \lambda_{H,\sigma}$). We call H a *convex ϱ-σ-subgroup* (resp., *convex λ-σ-subgroup*) if $\varrho_{H,\sigma}$ (resp., $\lambda_{H,\sigma}$)

is a join congruence on $F(G)$. If H is both a ϱ-σ-subgroup and a λ-σ-subgroup, then H is said to be a *solid σ-subgroup.* (In [2] and [3], a ϱ-σ-subgroup was called a c-σ-subgroup and a convex ϱ-σ-subgroup was called a convex σ-subgroup.) Clearly, a normal ϱ-σ-subgroup is a solid σ-subgroup. It was proven in [2, Theorem 4.2 (ii) and (iii)] that a convex ϱ-σ-subgroup is a ϱ-σ-subgroup and a ϱ-σ-subgroup is a σ-subgroup. Moreover, the collection $\mathscr{R}_\sigma(G)$ of all ϱ-σ-subgroups is a complete sublattice of the lattice of all subgroups of G [2, Theorem 4.2 (iv)] and the collection of all convex ϱ-σ-subgroups is a dual ideal of $\mathscr{R}_\sigma(G)$ in which joins and meets of nonvoid subcollections agree with those in $\mathscr{R}_\sigma(G)$ [2, Theorem 4.1 and Corollary 4.8]. In particular, there is a smallest convex ϱ-σ-subgroup, which is necessarily normal in G. Also, the lattice of convex ϱ-σ-subgroups is a Brouwerian lattice [2, Corollary 4.6]. If σ is an l-retraction then $\{i\}$ is a convex ϱ-σ-subgroup, each ϱ-σ-subgroup is a convex ϱ-σ-subgroup, and the convex ϱ-σ-subgroups become convex l-subgroups in the lattice-ordering of G induced by σ.

If $\sigma \in \operatorname{Ret} G$, H is a normal solid σ-subgroup of G, and $X\sigma^* = H(\{a_1, \ldots, a_n\}\, \sigma)$, for every $X = \{Ha_1, \ldots, Ha_n\} \in F(G/H)$, then $\sigma^* \in \operatorname{Ret} G/H$ [2, Theorem 4.3 (i)] and there is a lattice isomorphism between the ϱ-σ-subgroups of G that contain H and the ϱ-σ^*-subgroups of G/H [2, Corollary 4.5].

In the sequel we will have occasion to use retractions constructed from a given retraction σ of G. If ϕ is an automorphism or an anti-automorphism of G, then $\phi\sigma\phi^{-1}$ (we do not distinguish in notation between the image of an element under a function and the image of a subset under the function) is a retraction of G [2, Theorem 5.1]. If ϕ is the anti-automorphism of G given by $g\phi = g^{-1}$, then $\sigma' = \phi\sigma\phi^{-1}$ is called the *dual* of σ. (If σ is an l-retraction, then σ' is an l-retraction and induces the dual lattice-ordering on G.) If $\sigma = \sigma'$, then σ is said to be *self dual.* If G is abelian, ϕ is an endomorphism of G, $A \in F(G)$, and $\sigma^\wedge$ is given by $A\sigma^\wedge = ((AA^{-1})\, \sigma\phi)\, (A\sigma)$, then $\sigma^\wedge$ is a retraction of G [2, Theorem 5.5].

If $X \subseteq G$, then $[X]$ will denote the subgroup of G generated by X. The rational numbers will be denoted by Q.

3. Subgroups. We begin this section by showing that if $\sigma \in \operatorname{Ret} G$, then the collection of convex ϱ-σ-subgroups is identical with the collection of convex λ-σ-subgroups.

Theorem 3.1. *If $\sigma \in \operatorname{Ret} G$ and H is a convex ϱ-σ-subgroup of G, then H is a convex λ-σ-subgroup. Hence, each convex ϱ-σ-subgroup is a solid σ-subgroup.*

Proof. Let J be the smallest convex ϱ-σ-subgroup of G. Then $H \supseteq J$ and G/J is a lattice-ordered group, where the join of $\{Ja_1, \ldots, Ja_n\}$ equals $\{Ja_1, \ldots, Ja_n\}\, \sigma^*$ for every $\{Ja_1, \ldots, Ja_n\} \in F(G/J)$ [2, Theorem 4.3 (i)]. Moreover, H/J is a convex l-subgroup of G/J [2, Corollary 4.6]. Let $(A, B) \in \lambda_{H,\sigma}$ and $C \in F(G)$, where $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_p\}$. Then $(\{Ja_1, \ldots, Ja_m\}, \{Jb_1, \ldots, Jb_n\}) \in \lambda_{H/J,\sigma^*}$. Since H/J is a convex l-subgroup of G/J, $\bigvee_{i=1}^{m} (Ja_iH/J) =$

$$= (\{Ja_1, \ldots, Ja_m\}\,\sigma^*)\,H/J = (\{Jb_1, \ldots, Jb_n\}\,\sigma^*)\,H/J = \bigvee_{i=1}^{n} (Jb_iH/J). \text{ Hence,}$$

$$(\{Ja_1, \ldots, Ja_m, Jc_1, \ldots, Jc_p\}\,\sigma^*)\,H/J = (\bigvee_{i=1}^{m} (Ja_iH/J)) \vee (\bigvee_{i=1}^{p} (Jc_iH/J)) =$$

$$= (\bigvee_{i=1}^{n} (Jb_iH/J)) \vee (\bigvee_{i=1}^{p} (Jc_iH/J)) = (\{Jb_1, \ldots, Jb_n, Jc_1, \ldots, Jc_p\}\,\sigma^*)\,H/J\,.$$

It follows that $(A \cup C, B \cup C) \in \lambda_{H,\sigma}$ and so H is a convex λ-σ-subgroup of G.

In view of Theorem 3.1, we will call a convex ϱ-σ-subgroup simply a convex σ-subgroup. We have not been able to determine if each ϱ-σ-subgroup is a λ-σ-subgroup.

The proofs of Theorem 3.2, Corollaries 3.3 and 3.4, and Theorem 3.5 are straightforward and will be omitted.

Theorem 3.2. *Let ϕ be an automorphism or an anti-automorphism of G, $\sigma \in$ Ret G, $\tau = \phi\sigma\phi^{-1}$, H be a subgroup of G, and $J = H\phi^{-1}$.*

(i) *If H is a σ-subgroup, then J is a τ-subgroup of G.*

(ii) *If ϕ is an automorphism and H is a ϱ-σ-subgroup, then J is a ϱ-τ-subgroup of G; if ϕ is an anti-automorphism and H is a ϱ-σ-subgroup, then J is a λ-τ-subgroup of G.*

(iii) *If H is a solid σ-subgroup, then J is a solid τ-subgroup of G.*

(iv) *If H is a convex σ-subgroup, then J is a convex τ-subgroup of G.*

Corollary 3.3. *If $\sigma \in$ Ret G and H is a ϱ-σ-subgroup of G, then the following are equivalent:*

(i) *H is a ϱ-σ'-subgroup;*

(ii) *H is a λ-σ-subgroup;*

(iii) *H is a solid σ-subgroup;*

(iv) *H is a solid σ'-subgroup.*

Corollary 3.4. *Let $\sigma \in$ Ret G and H be a subgroup of G.*

(i) *H is a σ-subgroup if and only if H is a σ'-subgroup.*

(ii) *H is a convex σ-subgroup if and only if H is a convex σ'-subgroup.*

Theorem 3.5. *Let G be an abelian group, ϕ be an endomorphism of G, and $\sigma \in$ Ret G. If H is a solid σ-subgroup and H is ϕ-invariant, then H is a solid $\sigma^\wedge$-subgroup.*

Example 3.6. Let $K = Q \times Q \times Q$, the direct product of three copies of the rationals, and define $\{(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)\}\,\sigma = (\bigvee a_i, \bigvee b_i, \bigvee c_i)$.Then $\sigma \in$ Ret K and $H = \{(0, 0, c) |\; c \in Q\}$ is a convex σ-subgroup of K. If ϕ is the endomorphism of K given by $(a, b, c)\,\phi = (c, -c, 0)$, then neither H, $H\phi$, nor $H + H\phi$ is a $\sigma^\wedge$-subgroup of K.

Let $\sigma \in \operatorname{Ret} G$ and H and J be normal solid σ-subgroups of G. We say that G is the *σ-product* of H and J, denoted $G = H \otimes J$, provided that G is the direct product of H and J and if $\{a_1, \ldots, a_n\} \in F(G)$, then $\{a_1, \ldots, a_n\} \sigma = (\{h_1, \ldots, h_n\} \sigma_H) \cdot (\{j_1, \ldots, j_n\} \sigma_J)$, where $h_i \in H$, $j_i \in J$, and $a_i = h_i j_i$. If σ is an l-retraction, H and J are normal convex σ-subgroups of G, and G is the σ-product of H and J, then G is the cardinal product of the convex l-subgroups H and J. The extension of the definition of a (restricted) σ-product to more than two factors is immediate.

A second problem which we have not been able to answer concerns the transitivity of ϱ-σ-subgroups. (Transitivity of σ-subgroups is trivial.) We show in Example 5.1 that the property of being a convex σ-subgroup need not be transitive.

Theorem 3.7. *Let $\sigma \in \operatorname{Ret} G$ and H and J be normal solid σ-subgroups of G such that $G = H \otimes J$.*

(i) *If K is a ϱ-σ_H-subgroup of H, then K is a ϱ-σ-subgroup of G.*

(ii) *If K is a solid σ_H-subgroup of H, then K is a solid σ-subgroup of G.*

(iii) *If K is a convex σ_H-subgroup of H and H is a convex σ-subgroup of G, then K is a convex σ-subgroup of G.*

Proof. The verification of (i) and (ii) is routine. We prove only (iii). Let $(A, B) \in \varrho_{K,\sigma}$ and $C \in F(G)$, where $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_p\}$. Let $a_i = h_i j_i$, $b_i = s_i t_i$, and $c_i = x_i y_i$, where $h_i, s_i, x_i \in H$ and $j_i, t_i, y_i \in J$. Then $(A, B) \in \varrho_{H,\sigma}$ and so $(A \cup C, B \cup C) \in \varrho_{H,\sigma}$. Thus, $H(\{h_1, \ldots, h_m, x_1, \ldots, x_p\} \sigma_H)(\{j_1, \ldots, j_m, y_1, \ldots, y_p\} \sigma_J) = H(A \cup C) \sigma = H(B \cup C) \sigma = H(\{s_1, \ldots, s_n, x_1, \ldots, x_p\} \sigma_H)(\{t_1, \ldots, t_n, y_1, \ldots, y_p\} \sigma_J)$. Since $H \cap J = \{i\}$, it follows that $\{j_1, \ldots, j_m, y_1, \ldots, y_p\} \sigma = \{t_1, \ldots, t_n, y_1, \ldots, y_p\} \sigma$. Similarly, $(A, B) \in \varrho_{H,\sigma}$ implies that $\{j_1, \ldots, j_m\} \sigma = \{t_1, \ldots, t_n\} \sigma$. Therefore, $(A, B) \in \varrho_{K,\sigma}$ implies that $K(\{h_1, \ldots, h_m\} \sigma_H) = K(\{s_1, \ldots, s_n\} \sigma_H)$ and since $\{x_1, \ldots, x_p\} \in F(H)$ and K is a convex σ_H-subgroup of H, $K(\{h_1, \ldots, h_m, x_1, \ldots, x_p\} \sigma_H) = K(\{s_1, \ldots, s_n, x_1, \ldots, x_p\} \sigma_H)$. Consequently, $K(A \cup C) \sigma = K(B \cup C) \sigma$ and so K is a convex σ-subgroup of G.

As a second instance of transitivity we have

Theorem 3.8. *If H is a normal subgroup of G, $\tau \in \operatorname{Ret} H$ such that for every $g \in G$ and every $A \in F(H)$, $(g^{-1}Ag) \tau = g^{-1}(A\tau) g$, $\leqq$ is a linear ordering of G/H such that $(G/H, \leqq)$ is a linearly ordered group, then there is an extension σ of τ to a retraction of G such that $\{a_1, \ldots, a_n\} \sigma = (\{a_m a_n^{-1}, \ldots, a_n a_n^{-1}\} \tau) a_n$, where $\{a_1, \ldots, a_n\} \in F(G)$ and $Ha_1 \leqq \ldots \leqq Ha_{m-1} < Ha_m = \ldots = Ha_n$, and H is a convex σ-subgroup of G. Moreover,*

(i) *if J is a ϱ-τ-subgroup of H, then J is a ϱ-σ-subgroup of G;*

(ii) *if J is a solid τ-subgroup of H, then J is a solid σ-subgroup of G;*

(iii) *if J is a convex τ-subgroup of H, then J is a convex σ-subgroup of G.*

Proof. First we note that in [3, Theorem 3.18] that the existence of σ was established and it was shown that H is a convex σ-subgroup of G.

(i) Let $\{a_1, \ldots, a_n\} \in F(G)$, where $Ha_1 \leqq \ldots \leqq Ha_{m-1} < Ha_m = \ldots = Ha_n$, and $j_1, \ldots, j_n \in J$. Then $\{a_m a_n^{-1}, \ldots, a_n a_n^{-1}\} \in F(H)$ and so $\{a_m a_n^{-1}, \ldots, a_n a_n^{-1}\} \tau = j(\{j_m a_m a_n^{-1}, \ldots, j_n a_n a_n^{-1}\} \tau)$, for some $j \in J$. Now, $Hj_1 a_1 \leqq \ldots \leqq Hj_{m-1} a_{m-1} < Hj_m a_m = \ldots = Hj_n a_n$ and hence, $J(\{j_1 a_1, \ldots, j_n a_n\} \sigma) = J(\{j_m a_m a_n^{-1} j_n^{-1}, \ldots \ldots, j_n a_n a_n^{-1} j_n^{-1}\} \tau) j_n a_n = J(\{j_m a_m a_n^{-1}, \ldots, j_n a_n a_n^{-1}\} \tau) a_n = J(j^{-1}(\{a_m a_n^{-1}, \ldots \ldots, a_n a_n^{-1}\} \tau) a_n = J(\{a_1, \ldots, a_n\} \sigma)$. Therefore, J is a ϱ-σ-subgroup of G.

(ii) is immediate from (i) and the dual assertion for λ-σ-subgroups of G.

(iii) Let $(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\}) \in \varrho_{J,\sigma}$ and $\{c_1, \ldots, c_p\} \in F(G)$, where $Ha_1 \leqq \ldots \ldots \leqq Ha_{r-1} < Ha_r = \ldots = Ha_m$, $Hb_1 \leqq \ldots \leqq Hb_{s-1} < Hb_s = \ldots = Hb_n$, and $Hc_1 \leqq \ldots \leqq Hc_{t-1} < Hc_t = \ldots = Hc_p$. Then $(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\}) \in \varrho_{H,\sigma}$ and hence, $Ha_m = Hb_n$.

Case 1. $Hc_p < Ha_m$. Then $\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \sigma = (\{a_r a_m^{-1}, \ldots, a_m a_m^{-1}\} \tau) a_m$ and $\{b_1, \ldots, b_n, c_1, \ldots, c_p\} \sigma = (\{b_s b_n^{-1}, \ldots, b_n b_n^{-1}\} \tau) b_n$. Since $(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\}) \in \varrho_{J,\sigma}$, we have that $J(\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \sigma) = J(\{b_1, \ldots, b_n, c_1, \ldots, c_p\} \sigma)$.

Case 2. $Hc_p = Ha_m$. Then, since $a_m b_n^{-1} \in H$ and $(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\}) \in \varrho_{J,\sigma}$, $J(\{a_r b_n^{-1}, \ldots, a_m b_n^{-1}\} \tau) = J(\{a_r a_m^{-1} a_m b_n^{-1}, \ldots, a_m a_m^{-1} a_m b_n^{-1}\} \tau) = J(\{a_r a_m^{-1}, \ldots, a_m a_m^{-1}\} \tau) a_m b_n^{-1} = J(\{a_1, \ldots, a_m\} \sigma) b_n^{-1} = J(\{b_1, \ldots, b_n\} \sigma) b_n^{-1} = J(\{b_s b_n^{-1}, \ldots, b_n b_n^{-1}\} \tau b_n) b_n^{-1} = J(\{b_s b_n^{-1}, \ldots, b_n b_n^{-1}\} \tau)$. Also, $c_p b_n^{-1} \in H$ and $\{c_t c_p^{-1}, \ldots, c_p c_p^{-1}\} \in F(H)$. Thus, $\{c_t b_n^{-1}, \ldots, c_p b_n^{-1}\} \in F(H)$. Since J is a convex τ-subgroup of H and $(\{a_r b_n^{-1}, \ldots, a_m b_n^{-1}\}, \{b_s b_n^{-1}, \ldots, b_n b_n^{-1}\}) \in \varrho_{J,\tau}$, $(\{a_r b_n^{-1}, \ldots, a_m b_n^{-1}, c_t b_n^{-1}, \ldots, c_p b_n^{-1}\}, \{b_s b_n^{-1}, \ldots, b_n b_n^{-1}, c_t b_n^{-1}, \ldots, c_p b_n^{-1}\}) \in \varrho_{J,\tau}$. Therefore, $J(\{a_r a_m^{-1}, \ldots, a_m a_m^{-1}, c_t a_m^{-1}, \ldots, c_p a_m^{-1}\} \tau) a_m b_n^{-1} = J(\{a_r b_n^{-1}, \ldots, a_m b_n^{-1}, c_t b_n^{-1}, \ldots, c_p b_n^{-1}\} \tau) = J(\{b_s b_n^{-1}, \ldots, b_n b_n^{-1}, c_t b_n^{-1}, \ldots, c_p b_n^{-1}\} \tau)$. It follows that $J(\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \sigma) = J(\{b_1, \ldots, b_n, c_1, \ldots, c_p\} \sigma)$.

Case 3. $Hc_p > Ha_m$. Then $\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \sigma = (\{c_t c_p^{-1}, \ldots, c_p c_p^{-1}\} \tau) c_p = \{b_1, \ldots, b_n, c_1, \ldots, c_p\} \sigma$.

Consequently, J is a convex σ-subgroup of G.

Theorem 3.9. *If σ is a self dual retraction of G and H and J are ϱ-σ-subgroups of G, then $HJ = JH$.*

Proof. Let $h \in H$ and $j \in J$. It was shown in the proof of [2, Corollary 5.3] that if $\{i, h\} \sigma = a$, then $a^2 = h$. Hence, $h = a^2 = (\{i, h\} \sigma)(\{i, h\} \sigma) = \{i, h, h^2\} \sigma$. Thus, if $A = \{i, h, h^2\}$, we have $h = A\sigma$ and $i \in A \in F(H)$. Similarly, $j^{-1} = B\sigma$, where $B = \{i, j^{-1}, j^{-2}\} \in F(J)$. Since H is a ϱ-σ-subgroup of G, $A \cup B \in F(G)$, $i \in A \cap B$, and $i, h^{-1}, h^{-2} \in H$, $H(A \cup B) \sigma = H(B\sigma)$. Similarly, $J(A \cup B) \sigma = J(A\sigma)$. Therefore, $(A \cup B) \sigma = h_1 j^{-1} = j_1 h$, for some $h_1 \in H$ and $j_1 \in J$. Hence, $hj = j_1^{-1} h_1 \in JH$. It follows that $HJ = JH$.

We conclude this section by giving a description of the σ-subgroup generated by a subset of G.

Theorem 3.10. *Let* $\sigma \in \operatorname{Ret} G$, $X \subseteq G$, $H_1 = [X]$, $H_n = [\{A\sigma | A \in F(H_{n-1})\}]$ *for* $n > 1$, *and* $H = \bigcup H_n$.

(i) *H is the σ-subgroup of G generated by X.*

(ii) *If $xy = yx$ for all $x, y \in X$, then $H_n = \{(A\sigma)(B\sigma)^{-1} | A, B \in F(H_{n-1})\}$ for $n > 1$ and H is abelian. If, in addition, σ is an l-retraction, then $H_2 = H$.*

(iii) *If σ is self dual, then $H_n = \{A\sigma | A \in F(H_{n-1})\}$ for $n > 1$.*

Proof. (i) For each n, $H_n \subseteq H_{n+1}$. Hence, H is a subgroup of G. If $\{h_1, \ldots, h_m\} \in F(H)$, then $\{h_1, \ldots, h_m\} \in F(H_n)$ for some n, and so $\{h_1, \ldots, h_m\} \sigma \in H_{n+1} \subseteq H$. Therefore, H is a σ-subgroup of G. If J is any σ-subgroup of G containing X, then $J \supseteq H_1 = [X]$. If $J \supseteq H_n$, then $A\sigma \in J$ for every $A \in F(H_n)$. Thus, $H_{n+1} \subseteq J$ and so $H \subseteq J$.

(ii) First we show that H is abelian. Clearly, H_1 is abelian. If H_n is abelian, then $H_n \subseteq C(H_n)$, the centralizer of H_n. By [2, Theorem 2.14], the centralizer of any subset is a σ-subgroup. Hence, $C(H_n)$ is a σ-subgroup and the center of $C(H_n)$ is a σ-subgroup that contains H_n. Therefore, H is abelian.

For $n > 1$, let $T = \{(A\sigma)(B\sigma)^{-1} | A, B \in F(H_{n-1})\}$. Then $T \subseteq H_n$. If $x, y \in T$, then $x = (A\sigma)(B\sigma)^{-1}$ and $y = (C\sigma)(D\sigma)^{-1}$ for some $A, B, C, D \in F(H_{n-1})$. Thus, $xy^{-1} = (A\sigma)(B\sigma)^{-1}(D\sigma)(C\sigma)^{-1} = (AD)\sigma((CB)\sigma)^{-1} \in T$ and so $T = H_n$.

Next suppose that σ is an l-retraction. Then [2, Theorem 3.2 (v)] there is a lattice-ordering of G so that the join of A equals $A\sigma$ for every $A \in F(G)$. Hence, to show that H_2 is a σ-subgroup, it suffices to show that $\{i, h\} \sigma \in H_2$ for every $h \in H_2$. Now, $h \in H_2$ implies that $h = (A\sigma)(B\sigma)^{-1}$ for some $A, B \in H_1$. Then $\{i, h\} \sigma = \{i, (A\sigma)(B\sigma)^{-1}\} \sigma = \{B\sigma, A\sigma\} \sigma (B\sigma)^{-1} = ((A \cup B)\sigma)(B\sigma)^{-1} \in H_2$.

(iii) For $n > 1$, let $T = \{A\sigma | A \in F(H_{n-1})\}$. If $A, B \in F(H_{n-1})$ then $AB \in F(H_{n-1})$ and so $(A\sigma)(B\sigma) = (AB)\sigma \in T$. Moreover, $A^{-1} \in F(H_{n-1})$ and $(A\sigma)^{-1} = (A^{-1})\sigma \in T$ since σ is self dual. Hence, $T = H_n$.

4. Distributivity. In this section we prove our main result. First we establish a theorem which appears very similar to the definition of σ-product.

Theorem 4.1. *Let $\sigma \in \operatorname{Ret} G$, H be a ϱ-σ-subgroup of G and J be a λ-σ-subgroup of G such that $H \cap J = \{i\}$. If $h_1, \ldots, h_n \in H$ and $j_1, \ldots, j_n \in J$, then $\{h_1 j_1, \ldots, h_n j_n\} \sigma = (\{h_1, \ldots, h_n\} \sigma_H)(\{j_1, \ldots, j_n\} \sigma_J)$.*

Proof. Since H is a ϱ-σ-subgroup and $h_1^{-1}, \ldots, h_n^{-1} \in H$, $H(\{h_1 j_1, \ldots, h_n j_n\} \sigma) = H(\{j_1, \ldots, j_n\} \sigma)$, and since J is a λ-σ-subgroup and $j_1^{-1}, \ldots, j_n^{-1} \in J$, $(\{h_1 j_1, \ldots, h_n j_n\} \sigma) J = (\{h_1, \ldots, h_n\} \sigma) J$. Thus, there exist $h \in H$ and $j \in J$ such that $h(\{j_1, \ldots, j_n\} \sigma) = \{h_1 j_1, \ldots, h_n j_n\} \sigma = (\{h_1, \ldots, h_n\} \sigma) j$. Therefore,
$(\{h_1, \ldots, h_n\} \sigma)^{-1} h = j(\{j_1, \ldots, j_n\} \sigma)^{-1} \in H \cap J = \{i\}$. Hence, $(\{h_1, \ldots, h_n\} \sigma_H) \cdot (\{j_1, \ldots, j_n\} \sigma_J) = (\{h_1, \ldots, h_n\} \sigma)(\{j_1, \ldots, j_n\} \sigma) = \{h_1 j_1, \ldots, h_n j_n\} \sigma$.

It is well known that if G is a lattice-ordered group and H and J are convex l-subgroups of G such that $H \cap J = \{i\}$, then H and J commute elementwise. Since each convex σ-subgroup is a solid σ-subgroup, our next theorem is a generalization of this result.

Theorem 4.2. *If $\sigma \in \text{Ret } G$, H and J are solid σ-subgroups of G such that $H \cap J = \{i\}$, $h \in H$ and $j \in J$, then $hj = jh$.*

Proof. The proof is divided into steps.

$$(1) \qquad (\{i, h\} \sigma)(\{i, j\} \sigma) = (\{i, j\} \sigma)(\{i, h\} \sigma).$$

Since H is a ϱ-σ-subgroup and J is a λ-σ-subgroup, we have by Theorem 4.1 that $\{h, j\} \sigma = \{hi, ij\} \sigma = (\{h, i\} \sigma)(\{i, j\} \sigma)$. Dually, $\{h, j\} \sigma = (\{i, j\} \sigma)(\{i, h\} \sigma)$.

$$(2) \qquad \{hj, i\} \sigma = \{jh, i\} \sigma.$$

By (1) and Theorem 4.1, $\{hj, i\} \sigma = (\{h, i\} \sigma)(\{j, i\} \sigma) = (\{j, i\} \sigma)(\{h, i\} \sigma) = \{jh, i\} \sigma$.

$$(3) \qquad (\{h^{-1}, j^{-1}\} \sigma) h = h(\{h^{-1}, j^{-1}\} \sigma) \quad \text{and} \quad (\{h^{-1}, j^{-1}\} \sigma) j = j(\{h^{-1}, j^{-1}\} \sigma).$$

Since h and j are arbitrary, we have by (2) that $(\{h^{-1}, j^{-1}\} \sigma) h = \{i, j^{-1}h\} \sigma = \{i, hj^{-1}\} \sigma = h(\{h^{-1}, j^{-1}\} \sigma)$. Similarly, we obtain the other equality.

$$(4) \qquad hj = jh.$$

By (3), $hj(\{h^{-1}, j^{-1}\} \sigma) = h(\{h^{-1}, j^{-1}\} \sigma) j = \{h, j\} \sigma = j(\{h^{-1}, j^{-1}\} \sigma) h = jh(\{h^{-1}, j^{-1}\} \sigma)$. Hence, $hj = jh$.

Corollary 4.3. *If $\sigma \in \text{Ret } G$ and H and J are solid σ-subgroups of G such that $H \cap J = \{i\}$, then $[H \cup J] = HJ$ and HJ is the σ-product of H and J.*

Before proving our main result, we recall some properties of the lattices mentioned in Section 2. If $\sigma \in \text{Ret } G$, then the collection $\mathscr{R}_\sigma(G)$ of ϱ-σ-subgroups is a complete sublattice of the lattice of all subgroups of G. Dually, the collection $\mathscr{L}_\sigma(G)$ of λ-σ-subgroups is a complete sublattice of the lattice of subgroups of G. The intersection $\mathscr{S}_\sigma(G)$ of these two collections is the collection of all solid σ-subgroups, contains all normal ϱ-σ-subgroups, and is also a complete sublattice of the lattice of all subgroups of G. By Theorem 3.1 the collection of convex σ-subgroups is a subset of $\mathscr{S}_\sigma(G)$ and, consequently, is a dual ideal of $\mathscr{S}_\sigma(G)$ in which joins and intersections of nonvoid subcollections agree with those in $\mathscr{S}_\sigma(G)$. Since $\mathscr{S}_\sigma(G)$ is a complete sublattice of the lattice of subgroups of G, the collection of normal solid σ-subgroups is a complete sublattice of the lattice of subgroups of G.

Theorem 4.4. *If $\sigma \in \text{Ret } G$ and $\mathscr{N}_\sigma(G) = \{H \mid H$ is normal solid σ-subgroup of $G\}$, then $\mathscr{N}_\sigma(G)$ is a Brouwerian lattice.*

Proof. Since $\mathcal{N}_\sigma(G)$ is compactly generated, it suffices to show that $\mathcal{N}_\sigma(G)$ is distributive. Suppose (by way of contradiction) that $\mathcal{N}_\sigma(G)$ is not distributive. Since $\mathcal{N}_\sigma(G)$ is modular, there exist $H, J, K \in \mathcal{N}_\sigma(G)$ such that $HJ = HK = JK$, $H \cap J = H \cap K = J \cap K$, and H, J, and K are pairwise incomparable. If τ is the retraction of $HJ/H \cap J$ induced by σ, then the sublattice $\{H \cap J, H, J, K, HJ\}$ is isomorphic to a sublattice of $\mathcal{N}_\tau(HJ/H \cap J)$. Hence, we may further assume that $G = HJ$ and $H \cap J = \{i\}$.

Next we show that σ must be self dual. Let $\{h_1, \ldots, h_n\} \in \operatorname{Ker} \sigma_H$. Then for every $1 \leqq i \leqq n$, $h_i = j_i k_i$ for some $j_i \in J$ and $k_i \in K$. Then by Theorem 4.1, $i = \{h_1, \ldots, h_n\} \sigma_H = \{j_1 k_1, \ldots, j_n k_n\} \sigma = (\{j_1, \ldots, j_n\} \sigma_J)(\{k_1, \ldots, k_n\} \sigma_K)$. Since $J \cap K = \{i\}$, $\{j_1, \ldots, j_n\} \sigma = i = \{k_1, \ldots, k_n\} \sigma$. Now $k_i = j_i^{-1} h_i$ and, as above, $i = \{k_1, \ldots, k_n\} \sigma = (\{j_1^{-1}, \ldots, j_n^{-1}\} \sigma)(\{h_1, \ldots, h_n\} \sigma)$. Hence, $\{j_1^{-1}, \ldots, j_n^{-1}\} \sigma = i$. Thus, $\operatorname{Ker} \sigma_H \subseteq (\operatorname{Ker} \sigma_H)^{-1}$, where $(\operatorname{Ker} \sigma_H)^{-1} = \{A^{-1} \mid A \in \operatorname{Ker} \sigma_H\}$. By [2, Corollary 5.2 (i)], $\operatorname{Ker} \sigma'_H = (\operatorname{Ker} \sigma_H)^{-1}$ and by [2, Theorem 2.9 (ii)], $\sigma_H = \sigma'_H$. Therefore, σ_H is self dual. Similarly, σ_J is self dual and since G is the σ-product of H and J, σ is self dual.

If $g \in G$ and $\{i, g\} \sigma = a$, then, since σ is self dual, $a^2 = g$. (This was observed in the proof of Theorem 3.9.) Also, by [2, Theorem 2.4 (ii)], if $r_1, \ldots, r_n$ are integers with $r_1 < \ldots < r_n$, then $\{g^{r_1}, \ldots, g^{r_n}\} \sigma = a^{r_n - r_1} g^{r_1}$. Let $i \neq k \in K$. Then $k = hj$ for some $i \neq h \in H$ and $i \neq j \in J$. Let $h_1 = \{i, h\} \sigma$ and $j_1 = \{i, j\} \sigma$. Then, by Theorem 4.1 and the above, $\{h^{-1}j^2, h^{-1}j, h\} \sigma = \{h^{-1}j^2, h^{-1}j, hi\} \sigma = (\{h^{-1}, h\} \sigma) . (\{j^2, j, i\} \sigma) = (h_1^2 h^{-1})(j_1^2) = j$. Since K is a solid σ-subgroup, $Kj = K(\{h^{-1}j^2, h^{-1}j, h\} \sigma) = K(\{ih^{-1}j^2, ih^{-1}j, k^2 h\} \sigma) = K(\{h^{-1}j^2, h^{-1}j, h^3 j^2\} \sigma) = K(\{h^{-1}, h^3\} \sigma)(\{j, j^2\} \sigma) = K(h_1^4 h^{-1})(j_1 j) = Khjj_1 = Kkj_1 = Kj_1$. Therefore, $j_1 = j_1^2 j_1^{-1} = jj_1^{-1} \in K$ and hence, $j = j_1^2 \in K$, a contradiction, since $J \cap K = \{i\}$. Thus, $\mathcal{N}_\sigma(G)$ is a distributive lattice.

Corollary 4.5. *If G is an abelian group of finite rank and $\sigma \in$ Ret G, then G has only finitely many solid σ-subgroups. Hence, $\mathcal{R}_\sigma(G)$ is a finite distributive lattice.*

Proof. If D is a divisible closure of G, then σ can be uniquely extended to a retraction τ of D [3, Theorem 3.7]. Moreover, there is a one-to-one correspondence between the solid σ-subgroups of G and the solid τ-subgroups of D [3, Theorem 3.9 (ii)]. Thus, we may assume that G is divisible. Now each solid σ-subgroup of G is divisible [2, Corollary 4.10]. Since G has finite dimension as a rational vector space, the length of each chain in $\mathcal{R}_\sigma(G)$ is bounded by the dimension of G. Since $\mathcal{R}_\sigma(G)$ is distributive, it follows that $\mathcal{R}_\sigma(G)$ is finite.

The subgroup generated by two σ-subgroups need not be a σ-subgroup, even for l-retractions. In Theorem 4.7, which was first proven by J. Jakubík for lattice-ordered groups, we give a sufficient condition that the subgroup generated by a collection of σ-subgroups be a σ-subgroup. In fact, the remaining propositions in this section represent generalizations of corresponding theorems for lattice-ordered groups (see [4, Chapter 1]).

Corollary 4.6. *If $\sigma \in \operatorname{Ret} G$, $H, J, K, L \in \mathcal{N}_\sigma(G)$ such that $G = H \otimes J = K \otimes L$, and $H \subseteq K$, then $J \supseteq L$.*

Proof. $L = L \cap G = L \cap (H \vee J) = (L \cap H) \vee (L \cap J) \subseteq (L \cap K) \vee (L \cap J) = \{i\} \vee (L \cap J) \subseteq J$.

Thus, a σ-complement of a σ-factor is unique.

Theorem 4.7. *Let $\sigma \in \operatorname{Ret} G$, H_λ be a solid σ-subgroup of G for each $\lambda \in \Lambda$, T_λ be a σ-subgroup of G contained in H_λ, and suppose that if $\lambda, \gamma \in \Lambda$ with $\lambda \neq \gamma$, then $H_\lambda \cap H_\gamma = \{i\}$. If T is the subgroup of G generated by $\bigcup T_\lambda$, then T is a σ-subgroup of G, $T = \sum \otimes T_\lambda$, the restricted σ_T-product of the T_λ's, and T_γ is a normal solid σ_T-subgroup of T for each $\gamma \in \Lambda$. If $T_\gamma = H_\gamma$ for each γ in Λ, then T is a solid σ-subgroup of G.*

Proof. By Theorem 4.2, $xy = yx$ for each $x \in H_\lambda$, $y \in H_\gamma$, with $\gamma \neq \lambda$. Therefore, for each $\gamma \in \Lambda$, H_γ and T_γ are normal in $[\bigcup H_\lambda]$ and $[\bigcup T_\lambda]$, respectively. By Theorem 4.4, $H_\gamma \cap [\bigcup_{\gamma \neq \lambda} H_\lambda] = [\bigcup_{\lambda \neq \gamma} (H_\gamma \cap H_\lambda)] = \{i\}$. Thus $T_\gamma \cap [\bigcup_{\lambda \neq \gamma} T_\lambda] = \{i\}$ and so T is the restricted direct product of the T_λ's.

If $\{t_1, \ldots, t_m\} \in F(T)$, then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $t_{ij} \in T_{\lambda_j}$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$ such that $t_i = t_{i1} \ldots t_{in}$. Since $T_{\lambda_j} \subseteq H_{\lambda_j}$, we have by Theorem 4.1 that $\{t_1, \ldots, t_m\}\, \sigma = (\{t_{11}, \ldots, t_{m1}\}\, \sigma)(\{t_{12}, \ldots, t_{1n}, \ldots, t_{m2}, \ldots, t_{mn}\}\, \sigma)$, and, by induction, $\{t_{12}, \ldots, t_{1n}, \ldots, t_{m2}, \ldots, t_{mn}\}\, \sigma = (\{t_{12}, \ldots, t_{m2}\}\, \sigma) \ldots (\{t_{1n}, \ldots, t_{mn}\}\, \sigma)$. Therefore, $\{t_1, \ldots, t_m\}\, \sigma \in T$, T is a σ-subgroup of G, and T is the σ_T-product of the T_λ's.

Since $T_\lambda = H_\lambda \cap T$, T_λ is a normal solid σ_T-subgroup of T [2, Theorem 4.9 (i)]. The last assertion of the theorem follows from the fact that the collection of solid σ-subgroups of G is a complete sublattice of the lattice of subgroups of G.

The following corollaries are immediate from Theorems 4.4 and 4.7.

Corollary 4.8. *If $\sigma \in \operatorname{Ret} G$, $G = \sum \otimes H_\lambda$, where $\{H_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{N}_\sigma(G)$ and $H \in \mathcal{N}_\sigma(G)$, then $H = \sum \otimes (H \cap H_\lambda)$.*

Corollary 4.9. *If $\sigma \in \operatorname{Ret} G$ and $G = \sum_{\lambda \in \Lambda} \otimes H_\lambda = \sum_{\gamma \in \Gamma} \otimes J_\gamma$, where $\{H_\lambda \mid \lambda \in \Lambda\} \cup \{J_\gamma \mid \gamma \in \Gamma\} \subseteq \mathcal{N}_\sigma(G)$, then $G = \sum_{(\lambda,\gamma) \in \Lambda \times \Gamma} \otimes (H_\lambda \cap J_\gamma)$.*

5. Example. If G is a lattice-ordered group and M is a convex l-subgroup of G that is maximal with respect to not containing some g in G, then M is called a *regular* subgroup. It is well known that the collection of convex l-subgroups that contain a regular subgroup is a chain. It is trivial that the property of being a convex l-subgroup is transitive. The following example shows that even though a solid σ-subgroup represents a generalization of a convex l-subgroup, neither of the above properties is true for retractable groups.

Example 5.1. Let K and σ be as given in Example 3.6, and ϕ be the endomorphism of K given by $(a, b, c)\, \phi = (b + c, 0, 0)$. If $H_1 = \{(a, 0, 0) \mid a \in Q\}$, $H_2 =$

$= \{(a, b, 0) | a, b \in Q\}$, and $H_3 = \{(a, 0, c) | a, c \in Q\}$, then H_1, H_2, and H_3 are convex σ-subgroups of K and are ϕ-invariant. Thus, by Theorem 3.5, H_1, H_2, and H_3 are solid $\sigma^\wedge$-subgroups of K. We assert that these are the only proper solid $\sigma^\wedge$-subgroups of K. Suppose (by way of contradiction) that H is a proper solid $\sigma^\wedge$-subgroup of G, where $H \notin \{H_1, H_2, H_3\}$. Since K is divisible, H must be a subspace of the rational vector space K. Note that if $A = \{(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)\} \in F(K)$, then from the definition of $\sigma^\wedge$, $A\sigma^\wedge = (\bigvee a_i + \bigvee b_i - \bigwedge b_i + \bigvee c_i - \bigwedge c_i, \bigvee b_i, \bigvee c_i)$.

Case 1. The dimension of H is 1. Then $H = \{r(a, b, c) | r \in Q\}$, for some $(0, 0, 0) \neq (a, b, c)$ in K. There are numerous subcases and we present only one of these to indicate how a proof could follow. If $a \neq 0$, then, since $H \neq H_1$, H has a basis of the form $(1, b, c)$, where $b \neq 0$ or $c \neq 0$. If $b > 0$ and $c > 0$, then $\{(0, 0, 0), (1, b, c)\}\, \sigma^\wedge = (1 + b + c, b, c) = r(1, b, c)$ for some $r \in Q$. But then, $b + c = 0$, a contradiction. The other subcases are done similarly.

Case 2. The dimension of H is 2. Then $H \cap H_2$ and $H \cap H_3$ are solid $\sigma^\wedge$-subgroups of K of dimension 1. Since H_1 is the only solid $\sigma^\wedge$-subgroup of dimension 1, $H \cap H_2 = H_1 = H \cap H_3$. But then, $H_1 = (H \cap H_2) \vee (H \cap H_3) = H \cap (H_2 \vee H_3) = H \cap K = H$, a contradiction.

Therefore, the lattice of solid $\sigma^\wedge$-subgroups of K is $\{\{(0, 0, 0)\}, H_1, H_2, H_3, K\}$. Now $\{(0, 0, 0)\}$ is a maximal solid $\sigma^\wedge$-subgroup with respect to not containing $(1, 0, 0)$ and the solid $\sigma^\wedge$-subgroups that contain $\{(0, 0, 0)\}$ do not form a chain. It is easily verified that H_1 is the smallest convex $\sigma^\wedge$-subgroup of K (or see [2, Corollary 4.6]). Also, $\{(0, 0, 0)\}$ is a convex $\sigma^\wedge_{H_1}$-subgroup of H_1, but not a convex $\sigma^\wedge$-subgroup of K. Therefore, the property of being a "convex σ-subgroup" is not transitive. Note that the restriction of $\sigma^\wedge$ to $F(H_1)$ is an l-retraction and the retraction of K/H_1 induced by $\sigma^\wedge$ is an l-retraction, but $\sigma^\wedge$ is not an l-retraction.

Finally, we note that the lattice of solid $\sigma^\wedge$-subgroups of K cannot be isomorphic to the lattice of all convex l-subgroups of a lattice-ordered group. Recalling Corollary 4.5, we ask the following question:

If L is a finite distributive lattice, is there a retractable group G and $\sigma \in \text{Ret } G$ such that L is isomorphic to the lattice of normal solid σ-subgroups of G?

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