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DOMATICALLY CRITICAL GRAPHS

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In this paper we shall study domatically critical graphs. This study was proposed by E. J. COCKAYNE [1]. We consider finite undirected graphs without loops and multiple edges.

A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that each vertex of $V(G) - D$ is adjacent to at least one vertex of D. A domatic partition is a partition of $V(G)$ into pairwise disjoint dominating sets. The domatic number $d(G)$ of G is the maximal cardinality of a domatic partition of G. (The domatic number is to be distinguished from the domination number.)

A graph G is called domatically critical, if after deleting an arbitrary edge from G a graph with a smaller domatic number than that of G is obtained.

We shall prove a theorem.

Theorem 1. *Let G be a domatically critical graph with the domatic number $d(G) = d$. Then the vertex set $V(G)$ of G is the union of d pairwise disjoint sets $V_1, V_2, \ldots, V_d$ with the property that for any two distinct numbers i, j from the numbers $1, 2, \ldots, d$ the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ is a bipartite graph on the sets V_i, V_j, all of whose connected components are stars.*

Remark. A graph consisting of one edge with its end vertices is considered a star; a graph consisting of one isolated vertex is not.

Proof. As G has the domatic number d, there exists a domatic partition $\{V_1, V_2, \ldots, V_d\}$. Suppose that there exist two vertices of the same class of this partition which are joined by an edge e and let G' be the graph obtained from G by deleting e. Then each V_i, being a dominating set in G, is a dominating set also in G', because all edges joining vertices of V_i with vertices of $V(G) - V_i$ in G exist in G' as well. Therefore $\{V_1, V_2, \ldots, V_d\}$ is a domatic partition of G' and $d(G') \geqq d$; the graph G is not critical, which is a contradiction. We have proved that all sets $V_1, V_2, \ldots, V_d$ are independent in G; therefore each G_{ij} is a bipartite graph on the sets V_i, V_j. Now let i, j be two distinct numbers from the numbers $1, 2, \ldots, d$. As V_i

is a dominating set in G and $V_i \cap V_j = \emptyset$, each vertex of V_j must be adjacent to at least one vertex of V_i. Analogously each vertex of V_i must be adjacent to at least one vertex of V_j. Therefore in G_{ij} each vertex has the degree at least 1. If two vertices of a degree greater than 1 are adjacent in G_{ij}, then after deleting the edge joining them again each vertex of G_{ij} has a degree at least 1 in the resulting graph and in the graph obtained in this way from G the sets V_i, V_j (and obviously also all V_k for $i \neq k \neq j$) are domating sets, this graph has the domatic number d and G is not critical. Therefore each edge of G_{ij} is incident with a vertex of degree 1 in G_{ij} and each connected component of G_{ij} is a star.

A graph G is called indominable, if its vertex set can be partitioned into independent dominating sets. We have a corollary.

Corollary. *Every domatically critical graph is indominable.*

We express a conjecture.

Conjecture. *Every graph having the structure described in Theorem 1 is domatically critical.*

E. J. Cockayne and S. T. Hedetniemi [2] have proved that $d(G) \leqq \varrho(G) + 1$, where $\varrho(G)$ is the minimal degree of a vertex of G. If the equality $d(G) = \varrho(G) + 1$ holds, the graph G is called domatically full. We shall prove a theorem concerning regular domatically full graphs.

Theorem 2. *A regular domatically full graph G with n vertices and with a domatic number d exists if and only if d divides n; such a graph is also domatically critical. Its structure is the following: The vertex set $V(G) = \bigcup_{i=1}^{d} V_i$, $V_i \cap V_j = \emptyset$, $|V_i| = n/d$ and the subgraph G_{ij} of G induced by $V_i \cup V_j$ is regular of degree 1 (for $i = 1, \ldots, d$; $j = 1, \ldots, d$; $i \neq j$).*

Proof. Suppose that there exists a regular domatically full graph G with n vertices and with the domatic number d. As G is regular and domatically full, each vertex of G has degree $d - 1$. After deleting an arbitrary edge from G a graph G' is obtained in which two vertices have degree $d - 2$; hence $d(G') \leqq d - 1$ and G is domatically critical. This implies that G has the structure described in Theorem 1. Consider an integer i such that $1 \leqq i \leqq d$. Each vertex $x \in V_i$ must be adjacent to at least one vertex of V_j for each $j \in \{1, \ldots, d\} - \{i\}$. As these sets are pairwise disjoint, for each $j \neq i$ there exists exactly one edge joining x with a vertex of V_j. Therefore in each G_{ij} all vertices have degree 1. As G_{ij} is a bipartite graph on the sets V_i, V_j it is a complete matching of these sets and $|V_i| = |V_j|$. As i, j were chosen arbitrarily, all classes of the partition $\{V_1, V_2, \ldots, V_d\}$ have equal cardinalities and $|V_i| = n/d$ for each $i = 1, \ldots, d$. This is possible only if n/d is an integer, i.e. if d divides n. Therefore a regular domatically full graph with n vertices and with the domatic number d exists only if d divides n, and if it exists, it has the described structure. On the other hand, if d divides n, then obviously there exists a graph G with the described structure (e.g. the graph with n/d connected components which are all isomorphic to K_d). Thus let G be a graph with the described structure. Then it is evidently regular of degree $d - 1$. Its domatic number is at least d, because there exists a domatic partition $\{V_1, V_2, \ldots, V_d\}$. The inequality $d(G) \leqq \varrho(G) + 1$ implies that this domatic number cannot be greater than d, therefore it is equal to d and G is domatically full.

We shall now solve Problem 9 from [1]. An indivisible dominating set in a graph G is such a dominating set in G which is not a union of two distinct dominating sets. The least cardinality of a partition of the vertex set of G into indivisible dominating sets is called *the adomatic number of* G and denoted by $ad(G)$. (This is an analogue of the achromatic number of a graph.) Obviously $ad(G) \leqq d(G)$. Problem 9 in [1] is the following:

Do there exist vertex partitions into indivisible dominating sets of all orders between $ad(G)$ and $d(G)$?

The answer is negative.

Theorem 3. *To each positive integer n there exists a graph G for which*

$$d(G) - ad(G) = n$$

holds and which has the property that each partition of its vertex set into indivisible dominating sets has the cardinality either $d(G)$, or $ad(G)$.

Proof. Let G be the complete bipartite graph on sets A, B such that $|A| = |B| = n + 2$. The set A is evidently a dominating set in G. If A' is a proper subset of A, then no vertex of $A - A'$ is adjacent to a vertex of A', therefore A' is not a dominating set in G and A is an indivisible dominating set in G. Analogously B is an indivisible dominating set in G. Each two-element set $\{a, b\}$, where $a \in A$, $b \in B$, is a dominating set in G, because each vertex of $A - \{a\}$ is adjacent to b and each vertex of $B - \{b\}$ is adjacent to a. Evidently neither $\{a\}$ nor $\{b\}$ is a dominating set in G, therefore $\{a, b\}$ is a indivisible dominating set in G. Now let D be an indivisible dominating set in G. If $D \cap A = \emptyset$, then $D < B$. As shown above, D cannot be a proper subset of B, therefore $D = B$. Analogously $D \cap B = \emptyset$ implies $D = A$. If $D \cap A \neq \emptyset$, $D \cap B \neq \emptyset$, then D is the union of the sets $\{a, b\}$ for all $a \in D \cap A$ and all $b \in D \cap B$. As all these sets are indivisible dominating sets and D is also an indivisible dominating set, we must have $D = \{a, b\}$ for some $a \in A$ and $b \in B$. We have proved that each indivisible dominating set in G is equal either to A, or to B, or to some set $\{a, b\}$, where $a \in A$, $b \in B$. Each partition of the vertex set of G into indivisible dominating sets either is $\{A, B\}$, or consists of two-elements sets $\{a, b\}$ with the property that the edges joining these pairs $\{a, b\}$ form a complete matching of G. Therefore the cardinality of such a partition is either $ad(G) = 2$ or $d(G) = n + 2$ and the assertion is proved.

References

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