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THE MAXIMAL REGULAR IDEAL OF THE SEMIGROUP
OF BINARY RELATIONS

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If a semigroup contains a right [left, two-sided] ideal which is also a regular sub-semigroup, then there is a maximal right [left, two-sided] such ideal which we shall call the maximal regular right [left, two-sided] ideal of the semigroup. This is the case for example when the semigroup under consideration contains a kernel, or more in particular, a zero. This leads to the question of characterizing the elements of the maximal regular right [left, two-sided] ideal of the semigroup of all binary relations B_X on the set X .

For any $x \in X$ and any $\varrho \in B_X$, let

$$x\varrho = \{y \in X \mid x \varrho y\}, \quad \varrho x = \{y \in X \mid y \varrho x\}.$$

For any $A \subseteq X$ and any $\varrho \in B_X$, let

$$A\varrho = \bigcup_{x \in A} x\varrho, \quad \varrho A = \bigcup_{x \in A} \varrho x;$$

let $V(\varrho) = \{A\varrho \mid A \subseteq X\}$, $V(\varrho)' = \{\varrho A \mid A \subseteq X\}$. Clearly $V(\varrho)$ and $V(\varrho)'$ form complete lattices under the usual set-inclusion. It is well-known that $V(\varrho)$ and $V(\varrho)'$ are anti-isomorphic [10], and that the binary relation ϱ is a regular element of B_X if and only if $V(\varrho)$ (or $V(\varrho)'$) is a completely distributive lattice [9]. Another characterization of the regular elements of the semigroup B_X may be found in [7].

Let $R_X[L_X, M_X]$ denote the maximal regular right [left, two-sided] ideal of B_X . Clearly $M_X \subseteq L_X \cap R_X$. We shall show that $L_X = R_X = M_X$, and we shall characterize the elements of B_X which belong to M_X .

It can be readily verified that M_X is non-trivial. An easy computation shows that M_X contains all elements ϱ for which $V(\varrho)$ is a complete chain. In particular, M_X contains

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the elements ϱ for which $V(\varrho)$ is a two-element chain; such elements ϱ are called the *rectangular binary relations*, and it can be shown that they form the least non-trivial ideal of B_X ([6], [8]).

Theorem 1. *The following statements are equivalent.*

- (i) $\alpha \in R_X$,
- (ii) $V(\alpha)$ is a completely distributive lattice which does not contain a sublattice



- (iii) $V(\alpha)$ is isomorphic to a lattice L which is a subdirect product of a complete chain C with itself such that

- (a) $(x, x) \in L$ for all $x \in C$,
- (b) if $(x, y) \in L$ and $x \neq y$, then either x covers y or y covers x in C and $(y, x) \in L$.

Proof. (i) \Rightarrow (ii). Let $\alpha \in R_X$, and let us suppose that $V(\alpha)$ contains a sublattice of the form (1). Then there exist subsets A_1, A_2, A_3 of X , and elements 1, 2, 3 of X such that $1 \in A_1\alpha \setminus A_2\alpha$, $2 \in A_2\alpha \setminus A_1\alpha$, $A_2\alpha \subset A_3\alpha$ and $3 \in A_3\alpha \setminus (A_1 \cup A_2)\alpha$. Let $\gamma = \{(1, 1), (1, 3), (2, 2), (3, 2), (3, 3)\}$. Then $A_1\alpha\gamma = \{1, 3\}$, $A_2\alpha\gamma = \{2\}$ and $A_3\alpha\gamma =$



B_X which contradicts $\alpha \in R_X$. Thus $V(\alpha)$ cannot contain a sublattice of the form (1).

(ii) \Rightarrow (iii). Let us suppose that $V(\alpha)$ is a completely distributive lattice which does not contain a sublattice of the form (1). Let C be the set which consists of all elements of $V(\alpha)$ which are comparable to every other element of $V(\alpha)$. Let T be a maximal chain in $V(\alpha)$. Clearly C is a subchain of T .

Let A and B be any pair of incomparable elements of $V(\alpha)$ and suppose that $D < A$. $V(\alpha)$ contains a sublattice which consists of the elements $A \vee B, A \wedge B, A, B, B \vee D$ and $D \vee (A \wedge B) = A \wedge (B \vee D)$. We know that $A \vee B, A \wedge B, A, B$ are four distinct elements of $V(\alpha)$. Since $V(\alpha)$ cannot contain a sublattice of the form (1), we have either $B \vee D = A \vee B$ or $B \vee D = B$. If $B \vee D = A \vee B$, then $A = D \vee (A \wedge B) = A \wedge (B \vee D)$ and in this case $V(\alpha)$ would contain a sublattice of the form (1) consisting of the six distinct elements $A, B, A \vee B, A \wedge B, D, B \wedge D$; this is impossible, and thus $B \vee D = B$; in other words $D \leq A \wedge B$. In a dual way we can show that if A and B are incomparable in $V(\alpha)$ and $D > A$, then $D \geq A \vee B$.

Let A and B be any pair of incomparable elements of $V(\alpha)$, and let D be any element of $V(\alpha)$. If D were not comparable to A nor B , then the foregoing reasoning shows that $A \vee B = A \vee D = B \vee D$ and $A \wedge B = A \wedge D = B \wedge D$: this is

obviously impossible since the distributive lattice $V(\alpha)$ cannot contain a sublattice

of the form . Thus D is comparable to A or B . From the above reasoning

it now follows that D is comparable to $A, B, A \wedge B$ and $A \vee B$. We conclude that $A \wedge B, A \vee B \in C$, where $[A \wedge B, A \vee B]$ consists of the four elements $A, B, A \wedge B, A \vee B$, and that $A \vee B$ covers $A \wedge B$ in C . Furthermore, either A or B belongs to T .

It is easy to see that C is a closed sublattice of $V(\alpha)$. Hence C is a complete chain. For any $A \in T \setminus C$ let A' be the unique element of $V(\alpha)$ which is not comparable to A . Let L be the subdirect product of C with itself which consists of the elements

$$\begin{aligned} &(D, D), && D \in C, \\ &(A \vee A', A \wedge A'), && A \in T \setminus C, \\ &(A \wedge A', A \vee A'), && A \in T \setminus C. \end{aligned}$$

Obviously the mapping

$$\begin{aligned} V(\alpha) \rightarrow L, \quad D &\rightarrow (D, D), && D \in C, \\ A &\rightarrow (A \vee A', A \wedge A'), && A \in T \setminus C, \\ A' &\rightarrow (A \wedge A', A \vee A'), && A' \in V(\alpha) \setminus T \end{aligned}$$

is an isomorphism. Thus (iii) is satisfied.

(iii) \Rightarrow (i). Let R denote the set of the elements $\alpha \in B_X$ which satisfy condition (iii). From (i) \Rightarrow (iii) it follows that $R_X \subseteq R$. Let α be any element of R . Then $V(\alpha)$ is isomorphic to a lattice L which is a subdirect product of a complete chain C with itself where the conditions (a) and (b) are satisfied. Since L is a closed sublattice of the direct product of C with itself it follows that $V(\alpha)$ and L are completely distributive ([1], V. 5 and [5]). It follows from Zaretskii's characterization of the regular elements that R consists of elements which are regular in B_X .

Let α be any element of R and let β be any element of B_X . From the fact that $V(\alpha\beta)$ is a complete lattice and the fact that $V(\alpha) \rightarrow V(\alpha\beta), Y\alpha \rightarrow Y\alpha\beta$ is an order-preserving mapping it easily follows that $V(\alpha\beta)$ can be constructed in the way described by (iii). Thus $\alpha\beta \in R$, and so R is a right ideal of B_X .

If α and β are \mathcal{D} -related elements of B_X , then $V(\alpha) \cong V(\beta)$ ([4], [10]). Thus R is a union of \mathcal{D} -classes of B_X , and we can now conclude that R is also a regular subsemigroup of B_X . Consequently $R = R_X$.

Theorem 2. $R_X = L_X = M_X$.

Proof. Let α be any element of B_X . By the dual of Theorem 1 we have that $\alpha \in L_X$ if and only if $V(\alpha')$ does not contain a sublattice of the form (1). Since $V(\alpha)$ and

$V(\alpha')$ are isomorphic, we have by Theorem 1 that $\alpha \in L_X$ if and only if $\alpha \in R_X$. Thus $L_X = R_X$ is a two-sided ideal, and so $L_X = R_X \subseteq M_X$. Since obviously $M_X \subseteq L_X \cap R_X$ the equality holds.

Theorem 3. *The automorphism group of M_X is isomorphic to the symmetric group $\text{Sym } X$.*

Proof. The semigroup M_X contains the relations of the form $\{(x, x)\}$, $x \in X$: M_X is a r -semigroup. Furthermore, for every $\mu \in \text{Sym } X$ and every $\alpha \in M_X$ we must have $\mu^{-1}\alpha\mu \in M_X$. It then follows from [2], Corollary 7, that the automorphism group of M_X is isomorphic to $\text{Sym } X$.

Theorem 4. *M_X is a subdirectly irreducible regular semigroup. The equality is the greatest idempotent-separating congruence on M_X .*

Proof. From [2], Proposition 2, it follows that a congruence π on M_X is trivial if and only if the π -class containing the empty relation is trivial. Therefore there exists a least non-trivial congruence on M_X if and only if there exists a least non-trivial ideal of M_X , and if this is the case, then the least non-trivial congruence on M_X is precisely the Rees congruence which is associated with this least non-trivial ideal. Since M_X is an ideal and a regular subsemigroup of B_X , every ideal of M_X must also be an ideal of B_X . The ideal of B_X which consists of the rectangular binary relations is contained in M_X , and we know that this ideal is the least non-trivial ideal of B_X . Thus the rectangular binary relations constitute the least non-trivial ideal of M_X . We conclude that M_X is subdirectly irreducible ([3], I. 3.7).

Remarks.

1. If $|X| = 2$, then the identity mapping Δ_X belongs to M_X since $V(\Delta_X)$ satisfies (ii) of Theorem 1. Thus we know without any computation that $M_X = B_X$ is regular in this case (B_X contains 16 elements, 11 of which are idempotents).
2. M_X is contained in the intersection of all maximal regular subsemigroups of B_X . If $|X| > 2$, then M_X is properly contained in this intersection since the identity mapping Δ_X belongs to every maximal regular subsemigroup of B_X .

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