Czechoslovak Mathematical Journal

D. Hardy; Francis J. Pastijn

The maximal regular ideal of the semigroup of binary relations

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 2, 194–198

Persistent URL: http://dml.cz/dmlcz/101736

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

THE MAXIMAL REGULAR IDEAL OF THE SEMIGROUP OF BINARY RELATIONS

D. HARDY[⊥], Ft. Collins, and F. PASTIJN*, Gent

(Received March 19, 1979)

If a semigroup contains a right [left, two-sided] ideal which is also a regular subsemigroup, then there is a maximal right [left, two-sided] such ideal which we shall call the maximal regular right [left, two-sided] ideal of the semigroup. This is the case for example when the semigroup under consideration contains a kernel, or more in particular, a zero. This leads to the question of characterizing the elements of the maximal regular right [left, two-sided] ideal of the semigroup of all binary relations B_X on the set X.

For any $x \in X$ and any $\varrho \in B_X$, let

$$x\varrho = \{y \in X \mid x \varrho y\}, \quad \varrho x = \{y \in X \mid y \varrho x\}.$$

For any $A \subseteq X$ and any $\varrho \in B_X$, let

$$A\varrho = \bigcup_{x \in A} x\varrho, \quad \varrho A = \bigcup_{x \in A} \varrho x;$$

let $V(\varrho) = \{A\varrho \mid A \subseteq X\}$, $V(\varrho)' = \{\varrho A \mid A \subseteq X\}$. Clearly $V(\varrho)$ and $V(\varrho)'$ form complete lattices under the usual set-inclusion. It is well-known that $V(\varrho)$ and $V(\varrho)'$ are anti-isomorphic [10], and that the binary relation ϱ is a regular element of B_X if and only if $V(\varrho)$ (or $V(\varrho)'$) is a completely distributive lattice [9]. Another characterization of the regular elements of the semigroup B_X may be found in [7].

Let $R_X[L_X, M_X]$ denote the maximal regular right [left, two-sided] ideal of B_X. Clearly $M_X \subseteq L_X \cap R_X$. We shall show that $L_X = R_X = M_X$, and we shall characterize the elements of B_X which belong to M_X.

It can be readily verified that M_X is non-trivial. An easy computation shows that M_X contains all elements ϱ for which $V(\varrho)$ is a complete chain. In particular, M_X contains

---

[⊥]) The first author's research was done while he was a visiting professor at the University of Nebraska.

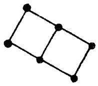
*) The second author's research was done while he was a visiting professor at the University of Nebraska, supported by a Fulbright-Hays Award.

the elements ϱ for which $V(\varrho)$ is a two-element chain; such elements ϱ are called the *rectangular binary relations*, and it can be shown that they form the least non-trivial ideal of B_X ([6], [8]).

Theorem 1. *The following statements are equivalent.*

(i) $\alpha \in R_X$,

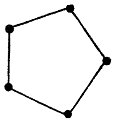
(ii) $V(\alpha)$ *is a completely distributive lattice which does not contain a sublattice*

of the form  , (1)

(iii) $V(\alpha)$ *is isomorphic to a lattice L which is a subdirect product of a complete chain C with itself such that*

(a) $(x, x) \in L$ *for all* $x \in C$,

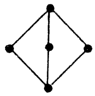
(b) *if* $(x, y) \in L$ *and* $x \neq y$, *then either x covers y or y covers x in C and* $(y, x) \in L$.

Proof. (i) $\Rightarrow$ (ii). Let $\alpha \in R_X$, and let us suppose that $V(\alpha)$ contains a sublattice of the form (1). Then there exist subsets A_1, A_2, A_3 of X, and elements 1, 2, 3 of X such that $1 \in A_1\alpha \setminus A_2\alpha$, $2 \in A_2\alpha \setminus A_1\alpha$, $A_2\alpha \subset A_3\alpha$ and $3 \in A_3\alpha \setminus (A_1 \cup A_2)\alpha$. Let $\gamma = \{(1, 1), (1, 3), (2, 2), (3, 2), (3, 3\}$. Then $A_1\alpha\gamma = \{1, 3\}$, $A_2\alpha\gamma = \{2\}$ and $A_3\alpha\gamma = \{2, 3\}$, and so $V(\alpha\gamma)$ is of the form  . Thus $\alpha\gamma$ is a non-regular element of B_X which contradicts $\alpha \in R_X$. Thus $V(\alpha)$ cannot contain a sublattice of the form (1).

(ii) $\Rightarrow$ (iii). Let us suppose that $V(\alpha)$ is a completely distributive lattice which does not contain a sublattice of the form (1). Let C be the set which consists of all elements of $V(\alpha)$ which are comparable to every other element of $V(\alpha)$. Let T be a maximal chain in $V(\alpha)$. Clearly C is a subchain of T.

Let A and B be any pair of incomparable elements of $V(\alpha)$ and suppose that $D < A$. $V(\alpha)$ contains a sublattice which consists of the elements $A \vee B$, $A \wedge B$, A, B, $B \vee D$ and $D \vee (A \wedge B) = A \wedge (B \vee D)$. We know that $A \vee B$, $A \wedge B$, A, B are four distinct elements of $V(\alpha)$. Since $V(\alpha)$ cannot contain a sublattice of the form (1), we have either $B \vee D = A \vee B$ or $B \vee D = B$. If $B \vee D = A \vee B$, then $A = D \vee (A \wedge B) = A \wedge (B \vee D)$ and in this case $V(\alpha)$ would contain a sublattice of the form (1) consisting of the six distinct elements A, B, $A \vee B$, $A \wedge B$, D, $B \wedge D$; this is impossible, and thus $B \vee D = B$; in other words $D \leqq A \wedge B$. In a dual way we can show that if A and B are incomparable in $V(\alpha)$ and $D > A$, then $D \geqq A \vee B$.

Let A and B be any pair of incomparable elements of $V(\alpha)$, and let D be any element of $V(\alpha)$. If D were not comparable to A nor B, then the foregoing reasoning shows that $A \vee B = A \vee D = B \vee D$ and $A \wedge B = A \wedge D = B \wedge D$: this is obviously impossible since the distributive lattice $V(\alpha)$ cannot contain a sublattice

of the form  . Thus D is comparable to A or B. From the above reasoning

it now follows that D is comparable to $A, B, A \wedge B$ and $A \vee B$. We conclude that $A \wedge B, A \vee B \in C$, where $[A \wedge B, A \vee B]$ consists of the four elements $A, B, A \wedge B, A \vee B$, and that $A \vee B$ covers $A \wedge B$ in C. Furthermore, either A or B belongs to T.

It is easy to see that C is a closed sublattice of $V(\alpha)$. Hence C is a complete chain. For any $A \in T \setminus C$ let A' be the unique element of $V(\alpha)$ which is not comparable to A. Let L be the subdirect product of C with itself which consists of the elements

$$\begin{array}{ll} (D, D), & D \in C\,, \\ (A \vee A', A \wedge A'), & A \in T \setminus C\,, \\ (A \wedge A', A \vee A'), & A \in T \setminus C\,. \end{array}$$

Obviously the mapping

$$\begin{array}{llll} V(\alpha) \to L, & D \to (D, D)\,, & & D \in C\,, \\ & A \to (A \vee A', A \wedge A')\,, & & A \in T \setminus C\,, \\ & A' \to (A \wedge A', A \vee A')\,, & & A' \in V(\alpha) \setminus T \end{array}$$

is an isomorphism. Thus (iii) is satisfied.

(iii) $\Rightarrow$ (i). Let R denote the set of the elements $\alpha \in B_X$ which satisfy condition (iii). From (i) $\Rightarrow$ (iii) it follows that $R_X \subseteq R$. Let α be any element of R. Then $V(\alpha)$ is isomorphic to a lattice L which is a subdirect product of a complete chain C with itself where the conditions (a) and (b) are satisfied. Since L is a closed sublattice of the direct product of C with itself it follows that $V(\alpha)$ and L are completely distributive ([1], V. 5 and [5]). It follows from Zaretskiĭ's characterization of the regular elements that R consists of elements which are regular in B_X.

Let α be any element of R and let β be any element of B_X. From the fact that $V(\alpha\beta)$ is a complete lattice and the fact that $V(\alpha) \to V(\alpha\beta)$, $Y\alpha \to Y\alpha\beta$ is an order-preserving mapping it easily follows that $V(\alpha\beta)$ can be constructed in the way described by (iii). Thus $\alpha\beta \in R$, and so R is a right ideal of B_X.

If α and β are $\mathcal{D}$-related elements of B_X, then $V(\alpha) \cong V(\beta)$ ([4], [10]). Thus R is a union of $\mathcal{D}$-classes of B_X, and we can now conclude that R is also a regular subsemigroup of B_X. Consequently $R = R_X$.

Theorem 2. $R_X = L_X = M_X$.

Proof. Let α be any element of B_X. By the dual of Theorem 1 we have that $\alpha \in L_X$ if and only if $V(\alpha)'$ does not contain a sublattice of the form (1). Since $V(\alpha)$ and

$V(\alpha)'$ are isomorphic, we have by Theorem 1 that $\alpha \in L_X$ if and only if $\alpha \in R_X$. Thus $L_X = R_X$ is a two-sided ideal, and so $L_X = R_X \subseteq M_X$. Since obviously $M_X \subseteq L_X \cap \cap R_X$ the equality holds.

Theorem 3. *The automorphism group of M_X is isomorphic to the symmetric group* Sym X.

Proof. The semigroup M_X contains the relations of the form $\{(x, x)\}$, $x \in X$: M_X is a r-semigroup. Furthermore, for every $\mu \in \text{Sym } X$ and every $\alpha \in M_X$ we must have $\mu^{-1}\alpha\mu \in M_X$. It then follows from [2], Corollary 7, that the automorphism group of M_X is isomorphic to Sym X.

Theorem 4. *M_X is a subdirectly irreducible regular semigroup. The equality is the greatest idempotent-separating congruence on M_X.*

Proof. From [2], Proposition 2, it follows that a congruence π on M_X is trivial if and only if the π-class containing the empty relation is trivial. Therefore there exists a least non-trivial congruence on M_X if and only if there exists a least non-trivial ideal of M_X, and if this is the case, then the least non-trivial congruence on M_X is precisely the Rees congruence which is associated with this least non-trivial ideal. Since M_X is an ideal and a regular subsemigroup of B_X, every ideal of M_X must also be an ideal of B_X. The ideal of B_X which consists of the rectangular binary relations is contained in M_X, and we know that this ideal is the least non-trivial ideal of B_X. Thus the rectangular binary relations constitute the least non-trivial ideal of M_X. We conclude that M_X is subdirectly irreducible ([3], I. 3.7).

Remarks.

1. If $|X| = 2$, then the identity mapping Δ_X belongs to M_X since $V(\Delta_X)$ satisfies (ii) of Theorem 1. Thus we know without any computation that $M_X = B_X$ is regular in this case (B_X contains 16 elements, 11 of which are idempotents).
2. M_X is contained in the intersection of all maximal regular subsemigroups of B_X. If $|X| > 2$, then M_X is properly contained in this intersection since the identity mapping Δ_X belongs to every maximal regular subsemigroup of B_X.

References

[1] *Birkhoff, G.:* Lattice Theory, Providence (1967).

[2] *McAlister, D. B.:* Homomorphisms of semigroups of binary relations, Semigroup Forum *3* (1971), 185—188.

[3] *Petrich, M.:* Lectures in Semigroups, London (1977).

[4] *Plemmons, R. J.,* and *M. T. West:* On the semigroup of binary relations, Pac. J. Math. *35* (1970), 743—753.

[5] *Raney, G. N.:* A subdirect union representation for completely distributive complete lattices, Proc. Amer. Math. Soc. *4* (1953), 518—522.

[6] *Schein, B. M.:* Semigroups of binary relations, Mini-Conference on Semigroup Theory, Szeged (1972), 17—24.
[7] *Schein, B. M.:* Regular elements of the semigroup of all binary relations, Semigroup Forum *13* (1976), 95—102.
[8] *Zaretskiĭ, K. A.:* Abstract characterization of the semigroup of all binary relations, Proc. Leningrad Pedagogical Inst. *183* (1958), 251—263 (in Russian).
[9] *Zaretskiĭ, K. A.:* Regular elements of the semigroup of all binary relations, Usp. Mat. Nauk *17* (1962), 177—179 (in Russian).
[10] *Zaretskiĭ, K. A.:* The semigroup of binary relations, Mat. Sbornik *61* (1963), 291—305 (in Russian).

Authors' addresses: D. Hardy, Department of Mathematics, Colorado State University, Ft. Collins, CO. 80523; F. Pastijn, Dienst Hogere Meetkunde, Rijksuniversiteit Gent, Krijgslaan 271 B-9000 Gent, Belgium.