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QUOTIENT SPACES DEFINED BY LINEAR RELATIONS

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INTRODUCTION

In the present paper, quotient spaces of vector spaces defined by linear relations are introduced and investigated. It is shown that they can be reduced to those defined by linear equivalence relations, i.e., to the usual ones. The main tools are the linear selections and the inductive vector topologies. Moreover, using quotient spaces, a few selection theorems for linear relations are proved. Special vector topological properties of quotient spaces are to be treated in a subsequent paper.

Finally, to keept our paper self-contained as possible, we briefly sketch the main definitions from [8]. S is a linear relation from X into Y if X and Y are vector spaces over a field K and S is a linear subspace of $X \times Y$ such that $S(x) = \{y \in Y : (x, y) \in S\}$ is not empty for all $x \in X$. A function f from X into Y is a selection for S if $f(x) \in S(x)$ for all $x \in X$. Topological vector spaces are always supposed to be defined over $K = \mathbb{R}$ or $\mathbb{C}$.

1. QUOTIENT SPACES

Definition 1.1. If S is a linear relation from X into Y, then let

$$Y \mid S = \{S(x) : x \in X\},$$

and define addition and scalar multiplication in $Y \mid S$ by

$$S(x) + S(y) = S(x + y) \quad \text{and} \quad \lambda \cdot S(x) = S(\lambda x).$$

Moreover, let φ_S be the mapping defined on X by

$$\varphi_S(x) = S(x).$$

Theorem 1.2. *Let S be a linear relation from X into Y. Then $Y \mid S$ is a vector space, and φ_S is a linear mapping of X onto $Y \mid S$.*

Proof. From the linearity of S, it follows directly that S is compatible with the linear operations in the sense that $S(x) = S(y)$ and $S(z) = S(w)$ imply $S(y + z) =$

$= S(y + w)$, and $S(x) = S(y)$ implies $S(\lambda x) = S(\lambda y)$. Hence, it is clear that the definition of the algebraic operations in $Y \mid S$ is correct, and also that all the axioms of a vector space are satisfied. The linearity of φ_S follows at once from the corresponding definitions.

Remark 1.3. The vector space $Y \mid S$ and the linear map φ_S will be called the *quotient space* of Y defined by S and the projection of X onto $Y \mid S$, respectively.

If M is a subspace of a vector space X and R denotes the unique linear equivalence relation on X such that $R(0) = M$, then the vector space

$$X \mid M = X \mid R$$

is called the *quotient space of X modulo M.*

Theorem 1.4. *Let S be a linear relation from X into Y, f a selection for S, and φ the projection of Y onto $Y \mid S(0)$. Then*

$$\varphi_S = \varphi \circ f \quad \textit{and} \quad S = \varphi^{-1} \circ \varphi_S .$$

Proof. By Theorem 3.3 in [8], we have

$$\varphi_S(x) = S(x) = f(x) + S(0) = \varphi(f(x)) = \varphi^{-1}(\varphi(f(x)))$$

for all $x \in X$.

Theorem 1.5. *Let S be a linear relation from X onto Y. Then the vector spaces $Y \mid S$ and $Y \mid S(0)$ are identical.*

Proof. This follows immediately from the first assertion in Theorem 1.4 and from the fact that the linear operations in $Y \mid S$ coincide with the usual linear operations for sets with the only exception that $0 \cdot S(x) \neq \{0\}$ if S is not a function.

Theorem 1.6. *Let S be a linear relation from X into Y. Then the vector spaces $Y \mid S$ and $X \mid S^{-1}$ are isomorphic.*

Proof. By Corollary 3.10 in [8], it is clear that the mapping $S(x) \to S^{-1}(f(x))$, where f is a selection for S, is independent of f, and is an isomorphism of $Y \mid S$ onto $X \mid S^{-1}$.

Example 1.7. If f is a linear function from X onto Y then the vector spaces $X \mid f^{-1}$ and $X \mid f^{-1}(0)$ are identical, and the vector spaces $X \mid f^{-1}$ and Y are isomorphic.

2. INDUCTIVE VECTOR TOPOLOGIES

Theorem 2.1. *Let f be a linear mapping from a topological vector space X onto Y, and consider Y to be equipped with the finest topology for which f is continuous. Then Y is a topological vector space and f is an open mapping.*

Proof. A subset V of Y is open if and only if $f^{-1}(V)$ is open in X. Therefore, if U is an open subset of X, then $f(U)$ is open in Y since $f^{-1}(f(U)) = U + f^{-1}(0)$ is open in X. This shows that f is an open mapping. Now, it is easy to see that the vector space operations in Y are continuous. (Observe that this requires that f be onto Y.)

Theorem 2.2. *For each $\alpha \in \Gamma$, let f_α be a linear mapping of a topological vector space X_α into Y. Then there exists a finest vector topology on Y for which f_α is continuous for all $\alpha \in \Gamma$.*

Proof. Let $\mathscr{P}$ be the family of all pre-semi-norms p on Y for which $p \circ f_\alpha$ is continuous for all $\alpha \in \Gamma$. (We use the term 'pre-semi-norm' instead of 'J-semi-norm' [9]. A real functional p on Y is a pre-semi-norm on Y if (1) $p(x + y) \leqq p(x) + p(y)$ for all $x, y \in Y$; (2) $p(\lambda x) \leqq p(x)$ for all $|\lambda| \leqq 1$ and $x \in Y$; and (3) $\lim_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in Y$.) Denote $\mathscr{T}$ the vector topology on Y induced by $\mathscr{P}$. (This is the unique vector topology on Y for which the family of all sets $U_p(\varepsilon) = \{x \in Y : p(x) \leqq \varepsilon\}$, where $p \in \mathscr{P}$ and $\varepsilon > 0$, is a subbase of the neighbourhood system of 0.) Then, by the definition of $\mathscr{P}$, it is clear that each f_α is continuous for $\mathscr{T}$.

On the other hand, if $\mathscr{T}'$ is another such vector topology on Y, and $\mathscr{P}'$ denotes the family of all pre-seminorms on Y which are continuous for $\mathscr{T}'$, then $p' \circ f_\alpha$ is continuous for all $p' \in \mathscr{P}'$ and $\alpha \in \Gamma$. Thus $\mathscr{P}' \subset \mathscr{P}$, and hence $\mathscr{T}' \subset \mathscr{T}$ since $\mathscr{P}'$ induces $\mathscr{T}'$.

Theorem 2.3. *For each $\alpha \in \Gamma$, let f_α be a linear mapping of a topological vector space X_α into Y, and consider Y to be equipped with the finest vector topology for which f_α is continuous for all $\alpha \in \Gamma$. Moreover, let φ be a linear mapping from Y into a topological vector space Z. Then φ is continuous if and only if $\varphi \circ f_\alpha$ is continuous for all $\alpha \in \Gamma$.*

Proof. If φ is continuous, then it is clear that each $\varphi \circ f_\alpha$ is continuous. Suppose now that each $\varphi \circ f_\alpha$ is continuous. Let q be a continuous pre-semi-norm on Z. Then $(q \circ \varphi) \circ f_\alpha = q \circ (\varphi \circ f_\alpha)$ is continuous for all $\alpha \in \Gamma$. Thus, by the definition of the topology of Y, the pre-semi-norm $p = q \circ \varphi$ is continuous. This implies that φ is continuous.

Theorem 2.4. *For each $\alpha \in \Gamma$ and $\beta \in \Gamma_\alpha$, let $f_{\alpha\beta}$ be a linear mapping from a topological vector space $X_{\alpha\beta}$ into Y_α, and φ_α be a linear mapping of Y_α into Z. Moreover, for each $\alpha \in \Gamma$, let Y_α be equipped with the finest vector topology for which $f_{\alpha\beta}$ is continuous for all $\beta \in \Gamma_\alpha$. Then the finest vector topology on Z for which φ_α is continuous for all $\alpha \in \Gamma$ coincides with the finest vector topology on Z for which $\varphi_\alpha \circ f_{\alpha\beta}$ is continuous for all $\alpha \in \Gamma$ and $\beta \in \Gamma_\alpha$.*

Proof. Let p be a pre-semi-norm on Z. Then, by Theorem 2.3, $p \circ \varphi_\alpha$ is continuous for all $\alpha \in \Gamma$ if and only if $p \circ (\varphi_\alpha \circ f_{\alpha\beta})$ is continuous for all $\alpha \in \Gamma$ and $\beta \in \Gamma_\alpha$. Hence, the theorem is quite obvious.

Notes 2.5. Theorem 2.1 seems to have been overlooked in the standard references on topological vector spaces, since quotient spaces are directly treated there.

Theorems 2.2, 2.3 and 2.4 for locally convex topologies can be found in [4]. Theorems 2.2 and 2.3 for Hausdroff vector topologies can be found in [1]. Our proofs seem to be more simple than those given in [4] and [1].

3. QUOTIENT TOPOLOGIES

Definition 3.1. If S is a linear relation from a topological vector space X into Y, then consider $Y \mid S$ to be equipped with the finest topology for which φ_S is continuous.

Theorem 3.2. *Let S be a linear relation from a topological vector space X into Y. Then $Y \mid S$ is a topological vector space and φ_S is an open mapping.*

Proof. This follows at once from Theorem 2.1.

Corollary 3.3. *Let S be a linear relation from a topological vector space X into another topological vector space Y, and suppose that there exists a continuous (resp. an open) selection f for S. Then S is a lower semi-continuous (resp. an open) relation.*

Proof. By Theorem 1.4, we have $S = \varphi^{-1} \circ \varphi \circ f$, where φ is the projection of Y onto $Y \mid S(0)$. Moreover, by Definition 3.1 and Theorem 3.2, φ is a continuous open mapping. Thus, $S^{-1}(V) = f^{-1}(\varphi^{-1}(\varphi(V)))$ is open in X if V is open in Y (resp. $S(U) = \varphi^{-1}(\varphi(f(U)))$ is open in Y if U is open in X), i.e., S is lower semi-continuous (resp. open).

Corollary 3.4. *Let S be a linear relation from a topological vector space X onto another topological vector space Y such that there exists a continuous selection f for S. Then the topology of $Y \mid S$ is finer than that of $Y \mid S(0)$.*

Proof. In this case, by Theorem 1.5, the vector spaces $Y \mid S$ and $Y \mid S(0)$ are identical, and moreover by the first assertion in Theorem 1.4, φ_S is a continuous mapping of X onto $Y \mid S(0)$.

Theorem 3.5. *Let S be a linear relation from a topological vector space X onto Y. Let $\mathscr{F}$ be a nonvoid family of linear selections for S, and consider Y to be equipped with the finest vector topology for which each $f \in \mathscr{F}$ is continuous. Then the topological vector spaces $Y \mid S$ and $Y \mid S(0)$ are identical.*

Proof. This follows directly from Theorems 1.4 and 2.4.

Corollary 3.6. *Let S be a linear relation from a topological vector space X onto Y, and consider Y to be equipped with the coarsest topology for which the projection φ of Y onto $Y \mid S$ is continuous. Then Y is a topological vector space and S is a lower semicontinuous open relation such that every linear selection for S is continuous.*

Proof. By the assertion proved at the beginning of § 11 in Chapter 2 of [4], it is clear that Y is a topological vector space.

Now if f is a linear selection for S, then Theorem 3.5 implies that the finest vector topology on Y for which f is continuous is finer than the original topology of Y, and thus f is continuous. Moreover, by Corollary 3.3, S is lower semi-continuous.

Finally, if U is an open subset of X, then by Theorem 1.4, $S(U) = \varphi^{-1}(\varphi_S(U))$, whence we can infer that $S(U)$ is open in Y, since φ_S is open and φ is continuous. This proves that S is open.

Corollary 3.7. *Let S be a linear relation from a topological vector space X onto Y. Let $\mathscr{F}$ be a nonvoid family of linear selections for S, and consider Y to be equipped with the finest vector topology for which each $f \in \mathscr{F}$ is continuous. Then S is a lower semi-continuous open relation.*

Proof. Theorem 3.5 implies that the original topology of Y is finer than the coarsest topology on Y for which the projection φ of Y onto $Y | S$ is continuous. Thus, by Corollary 3.6, S is open. Finally, again by Corollary 3.3, it is clear that S is lower semi-continuous.

Problems 3.8. Let S be a linear relation from a topological vector space X onto Y, and denote $\mathscr{F}$ be the family of all linear selections for S. (By Theorem 4.1 in [8], $\mathscr{F}$ is not empty.)

Let $\mathscr{T}_{\mathscr{F}}$ be the finest vector topology on Y for which each $f \in \mathscr{F}$ is continuous; and for $f \in \mathscr{F}$, let $\mathscr{T}_f$ be the finest vector topology on Y for which f is continuous. Moreover, let $\mathscr{T}_\varphi$ be the coarsest topology on Y for which the projection φ of Y onto $Y | S$ is continuous.

Then, it is clear that $\mathscr{T}_{\mathscr{F}} \subset \mathscr{T}_f$. Moreover, by Theorem 3.5, $\mathscr{T}_\varphi \subset \mathscr{T}_{\mathscr{F}}$. However, we do not know when the above inclusions are proper. (If S is a function, then the above inclusions are not proper.)

Notes 3.9. This paper was presented at the Conference on Functional Equations and Inequalities in Debrecen (Hungary), August 20–25, 1979.

Corollaries 3.3 and 3.6 give some partial answers to a problem on the existence of continuous linear selections for a linear relation from a topological vector space into another posed by the first author at the Symposium on Functional Equations in Retzhof (Austria), September 3–10, 1978.

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