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CONTACT PROBLEMS WITH BOUNDED FRICTION COERCIVE CASE

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Contact problems with friction remained for very long time unsolved in spite of their importance for practical applications. The reason consisted in difficulty how to treat the Coulomb law of friction in harmony with the non-penetration condition of the Signorini type. E.g. in the book by G. Duvaut and J. L. Lions [3] it is formulated as an open problem. The first result in terms of sufficient conditions for the existence of a solution was given by J. Nečas, J. Jarušek and J. Haslinger [8], who solved the case of a strip in R^2. The goal of the present paper is to extend the results of [8].

In order to explain the whole reiteration and renormation technique and the use of Tichonov's fixed point theorem, we shall first discuss the simplest case — the strip in R^3 (Sec. 2). The solution of the Signorini problem with friction can be defined by means of a fixed point of a certain operator constructed using a certain "auxiliary problem". The operator, defined on all "non-positive" elements of the dual space to the space of traces on the contact surface and acting into the same space, is continuous in the norm sense and gives "non-positive" results. For the fixed point method, however, the continuity with respect to the weak topology is required. At present, the weak continuity can be proved only for the spaces of more regular distributions by applying regularity results, the strong continuity on the original space and the compact imbedding theorem. The shift and renormation technique is essential for the regularity results as well as for the existence of a bounded set mapped into itself. Extensive calculations are performed to obtain the best possible conditions (using the given method) in the existence theorem.

In Sec. 3 we generalize the method for a general body in R^3 with a sufficiently smooth contact boundary. Here, moreover, some technique of "straightening of the boundary" and local coordinates are required. In Sec. 4, further generalization for a common contact of two elastic bodies with a $C^{2,1}$ smooth contact surface is made. The coercivity (i.e. a positive measure of the part of the boundary with a prescribed displacement) is supposed throughout the paper.

The results, existence theorems at the ends of sections, are formulated for homogeneous as well as non-homogeneous bodies. The estimations for admissible coefficients of friction are for the majority of materials in harmony with the technical praxis requirements.

1. CLASSICAL AND VARIATIONAL FORMULATION OF THE SIGNORINI PROBLEM WITH FRICTION, AUXILIARY PROBLEMS

The problem is formulated in [2] and [8]. Let Ω be an open domain in R^3 with a Lipschitz continuous boundary. Let us look for the displacement vector $u = [u^1, u^2, u^3]$. The strain tensor is given by the formula

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3, \quad x \in \Omega. \tag{1.1}$$

We consider the Hook law fulfilled on Ω in the form

$$\tau_{ij} = a_{ijkl} e_{kl}, \quad i, j = 1, 2, 3 \tag{1.2}$$

(in the sense of the obvious summation convention). The obvious symmetry $a_{ijkl}(x) = a_{jikl}(x) = a_{ijlk}(x) = a_{klij}(x)$ is supposed for every $x \in \Omega$ and $i, j, k, l = 1, 2, 3$. For the stress tensor the equilibrium conditions are considered,

$$-\frac{\partial \tau_{ij}}{\partial x_j} = f_i, \quad i = 1, 2, 3 \tag{1.3}$$

on Ω.

The boundary Γ of Ω consists of three parts. On Γ_u the displacement

$$u = u^0 \tag{1.4}$$

is given. On Γ_T the stress

$$T(u) = T^0 \tag{1.5}$$

is prescribed. On the contact part Γ_c we consider the Signorini conditions

$$T_n(u) \leqq 0, \quad u_n \leqq 0, \quad T_n(u)\, u_n = 0, \tag{1.6}$$

where u_n is the normal displacement ($u_n = u \,.\, n$), $T_n(u)$ is the normal stress ($T_n(u) = \tau_{ij}(u)\, n_i n_j$) and $n = (n_i)_{i=1}^3$ is the unit outer normal vector on Γ. Moreover, we require the conditions of the Coulomb law of friction on Γ_c to be fulfilled:

$$|T_t(u)| \leqq \mathscr{F}|T_n(u)|, \quad |u_t| \left(|T_t(u)| - \mathscr{F}|T_n(u)|\right) = 0, \tag{1.7}$$
$$(\mathscr{F} T_n(u))(x) < 0 \Rightarrow u_t(x) = (\lambda T_t)(x),$$

where $u_t = u - u_n n$ is the tangential displacement, $T_t(u) = T(u) - T_n(u)\, n$ is the tangential stress, $\mathscr{F}$ is the coefficient of friction and λ is a non-positive function on Γ_c. We look for such u that (1.1)–(1.7) are fulfilled.

Denote the Sobolev space $H^\alpha(\Omega) = W^{\alpha,2}(\Omega)$, $\alpha \in R^1$. Let $\mathscr{H} = (H^1(\Omega))^3$, $\mathscr{K} = \{u \in \mathscr{H};\ u = u^0$ on Γ_u, $u_n \leqq 0$ on Γ_c, both in the sense of traces$\}$, where $u^0 \in \mathscr{H}$ is a given function such that $u^0|_{\Gamma \setminus \Gamma_u} = 0$ in the sense of traces. Suppose $\Gamma_u, \Gamma_c, \Gamma_T$

are measurable subsets of Γ such that $\overline{\operatorname{Int}\Gamma_u} = \Gamma_u$, $\overline{\operatorname{Int}\Gamma_c} = \Gamma_c$, $\overline{\operatorname{Int}\Gamma_T} = \Gamma_T$, $(\operatorname{Int}\Gamma_u \cap \operatorname{Int}\Gamma_c) \cup (\operatorname{Int}\Gamma_c \cap \operatorname{Int}\Gamma_T) \cup (\operatorname{Int}\Gamma_T \cap \operatorname{Int}\Gamma_u) = \emptyset$ and $\partial\Gamma_c$, $\partial\Gamma_T$ and $\partial\Gamma_u$ are Lipschitz continuous (all in the sense of the relative topology on Γ). Let $\operatorname{mes}\Gamma_u > 0$. For $u, v \in \mathscr{H}$ put

$$a(u, v) = \int_\Omega a_{ijkl}\, e_{ij}(u)\, e_{kl}(v)\, dx\,. \tag{1.8}$$

We suppose $a_{ijkl}(x)$ to be Lipschitz continuous on Ω for every $i, j, k, l = 1, 2, 3$ and to satisfy

$$0 < a_0 \leqq (a_{ijkl}(x)\, \xi_{ij}\xi_{kl})\, |\xi|^{-2} \leqq A_0 < +\infty \tag{1.9}$$

for every $x \in \Omega$ and $\xi \equiv (\xi_{ij})_{ij=1}^3 \in R^9$. A_0, a_0 are constants independent of $x \in \Omega$ and $\xi \in R^9$. Let ${}^\circ\boldsymbol{H}^{1/2}(\Gamma_c)$ be the space of all functions w of $\boldsymbol{H}^{1/2}(\Gamma)$ such that $w|_{\overline{\Gamma \setminus \Gamma_c}} = 0$, provided with the norm of $\boldsymbol{H}^{1/2}(\Gamma)$. Let $\boldsymbol{H}^{-1/2}(\Gamma_c)$ be its dual space, denote $\boldsymbol{C}^{*-} = \{\varphi \in \boldsymbol{H}^{-1/2}(\Gamma_c);\ \varphi$ is non-positive in the dual sense to the ordering on ${}^\circ\boldsymbol{H}^{1/2}(\Gamma_c)$ given by the restriction of the canonical ordering on $\boldsymbol{L}_2(\Gamma_c)\}$. Denote by $(.,.)$ the duality pairing in $(\boldsymbol{L}_2(\Omega))^3$, $[.,.]$ the duality pairing in $(\boldsymbol{L}_2(\Gamma))^3$, $\langle .,. \rangle$ the duality pairing in $\boldsymbol{L}_2(\Gamma_c)$. Due to the bipolar theorem (which is very well known) it is possible to extend those pairings to the case of $v \in ({}^\circ\boldsymbol{H}^\alpha(\Omega))^3$ or $(\boldsymbol{H}^\alpha(\Gamma))^3$ or ${}^\circ\boldsymbol{H}^\alpha(\Gamma_c)$ and to φ belonging to the corresponding dual space; $\alpha > 0$. Let $\mathscr{F} \in \boldsymbol{C}^1(\Gamma_c)$ have a compact support, let $\operatorname{dist}(\operatorname{supp}\mathscr{F}, \Gamma \setminus \Gamma_c) > 0$. Let $f \in (\boldsymbol{L}_2(\Omega))^3$, $T^0 \in (\boldsymbol{H}^{-1/2}(\Gamma))^3$ be such that $[T^0, w] = 0$ for every $w \in (\boldsymbol{H}^{1/2}(\Gamma))^3$ with $w|_{\Gamma_T} = 0$. For an arbitrary $g_n \in \boldsymbol{C}^{*-}$ we introduce the problem

$(\mathrm{g_n})$ Let us look for $u \in \mathscr{K}$ such that for every $v \in \mathscr{K}$ the following inequality holds:

$$a(u, v - u) + \langle \mathscr{F}|g_n|, |v_t| - |u_t| \rangle \geqq (f, v - u) + [T^0, v - u]\,, \tag{1.10}$$

where $|g_n|$ means $-g_n$ for $g_n \in \boldsymbol{C}^{*-}$.

It is easy to show that the problem (g_n) is a weak formulation of the classical problem, where $T_n(u)$ is replaced by g_n in (1.7). For $u \in \mathscr{H}$ we define $T_n(u)$ by the formula

$$\langle T_n(u), w_n \rangle = a(u, w) - (f, w) \quad \text{for every} \quad w \in \mathscr{H} \quad \text{such that} \quad w|_{\overline{\Gamma \setminus \Gamma_c}} = 0\,, \quad w_t|_{\Gamma_c} = 0\,. \tag{1.11}$$

Definition 1.1. Let u be a solution of $(\mathrm{g_n})$ for some $g_n \in \boldsymbol{C}^{*-}$. Let $\mathscr{F} g_n = \mathscr{F} T_n(u)$. Then the solution u is called *a solution of the Signorini problem with friction.*

Put $\mathscr{H}_0 = \{v \in \mathscr{H};\ v = 0 \text{ on } \Gamma_u\}$, $\mathscr{K}_0 = \{v \in \mathscr{H}_0;\ v_n \leqq 0 \text{ on } \Gamma_c\}$. Clearly, $\mathscr{K} = u^0 + \mathscr{K}_0$. For $v \in \mathscr{K}_0$ let us put

$$\begin{aligned} J_{g_n}(v) &= J_0(v) + I_{g_n}(v)\,, \\ J_0(v) &= \tfrac{1}{2}a(v, v) + a(u^0, v) - (f, v) - [T^0, v]\,, \\ I_{g_n}(v) &= \langle \mathscr{F}|g_n|, |v_t| \rangle\,. \end{aligned} \tag{1.12}$$

Due to the Korn inequality $a(v, v) \geqq c\|v\|^2_{\mathscr{H}}$ satisfied on $\mathscr{H}_0$, we have $J_{g_n}(v)$ strictly convex and coercive for an arbitrary $g_n \in \mathbf{C}^{*-}$. Hence there exists a unique $u' \in \mathscr{K}_0$ such that

$$(1.13) \qquad J_{g_n}(u') = \min \{J_{g_n}(v), v \in \mathscr{K}_0\} .$$

Evidently J_0 is differentiable and I_{g_n} is non-differentiable. By the obvious condition for the occurrence of the minimum of J_{g_n} on the cone $\mathscr{K}_0$ at the point u' which is necessary and sufficient (see e.g. [5]), $D_1 J_0(u')(v - u') + I_{g_n}(v) - I_{g_n}(u') \geqq 0$ for every $v \in \mathscr{K}_0$. So we can easily see that $u = u' + u^0$ is the unique solution of the problem (g_n). We have

Lemma 1.1. *For every $g_n \in \mathbf{C}^{*-}$ there exists a unique solution of the problem (g_n). There exists a constant c independent of g_n such that, denoting $\|\cdot\|_{-1/2,\Gamma}$ the norm in $(H^{-1/2}(\Gamma))^3$, $\|\cdot\|_0$ the norm in $(L_2(\Omega))^3$, we have*

$$(1.14) \qquad \|u\|_{\mathscr{H}} \leqq c(\|u^0\|_{\mathscr{H}} + \|T^0\|_{-1/2,\Gamma} + \|f\|_0) .$$

The operator $\Phi_0 : \mathscr{F}g_n \mapsto T_n(u)$ is continuous on $H^{-1/2}(\Gamma_c)$ and $T_n(u) \in \mathbf{C}^{*-}$ for every $g_n \in \mathbf{C}^{*-}$.

Proof. Putting $v = u^0$ in (1.10), we obtain $a(u, u) \leqq a(u, u^0) + (f, u - u^0) + [T, u]$. The Korn inequality and the obvious considerations imply (1.14). Let $g_n^i \in \mathbf{C}^{*-}$, $i = 1, 2$, let u^i be the corresponding solutions of (g_n^i), $i = 1, 2$. Putting $v = u^2$ in (1.10) with g_n^1 and $v = u^1$ in (1.10) with g_n^2, we obtain

$$(1.15) \qquad \|u^2 - u^1\|^2_{\mathscr{H}} \leqq \langle \mathscr{F}g_n^2 - \mathscr{F}g_n^1, |u_t^2| - |u_t^1|\rangle \leqq$$
$$\leqq c_1 \|\mathscr{F}g_n^2 - \mathscr{F}g_n^1\|_{-1/2,\Gamma_c} [\|u^1\|_{\mathscr{H}} + \|u^2\|_{\mathscr{H}}] .$$

The continuity of Φ_0 is a consequence of (1.15), (1.14), (1.11) and of the usual traces theorem for the space $H^1(\Omega)$. Putting $v = u + w$, $w \in \mathscr{H}$ such that $w|_{\Gamma \setminus \Gamma_c} = 0$, $w_n|_{\Gamma_c} \leqq 0$ a.e. in Γ_c, $w_t|_{\Gamma_c} = 0$ in (1.10), we have $\langle T_n(u), w_n\rangle = a(u, w) - (f, w) \geqq 0$. Thanks to the arbitrariness of $w_n|_{\Gamma_c}$, $w_n \leqq 0$, $T_n(u) \in \mathbf{C}^{*-}$.

2. SIGNORINI PROBLEM WITH BOUNDED FRICTION FOR A STRIP IN R^3

In this section $\Omega = R^2 \times (0, r)$, $r \in (0, +\infty)$, $\Gamma_u = R^2 \times \{0\}$, $\Gamma_c = R^2 \times \{r\}$, $\Gamma_T = \emptyset$. Our method of proof of the existence theorem for the problem is the following. By means of the shift technique in arguments using the Korn inequality and some traces lemmas, we show that $T_n(u) \in H^{-1/2+\alpha}(\Gamma_c)$ for $g_n \in H^{-1/2+\alpha}(\Gamma_c) \cap \mathbf{C}^{*-}$ and $\alpha \in (0, \frac{1}{2})$, because of the validity of the inequality

$$\|T_n(u)\|_{-1/2+\alpha,\Gamma_c} \leqq c(\mathscr{F}, a_0, A_0) \|g_n\|_{-1/2+\alpha,\Gamma_c} + \text{const.}$$

for a certain $c(\mathscr{F}, a_0, A_0)$ depending only on $\mathscr{F}$, a_0, A_0. In the case $c(\mathscr{F}, a_0, A_0) < 1$ we use the following fixed point theorem:

Theorem 2.1. (Tichonov, see [1].) *Let $\boldsymbol{X}$ be a locally convex space. Let $\boldsymbol{C} \subset \boldsymbol{X}$ be a convex compact set. Let $\boldsymbol{F}$ be a continuous mapping which maps $\boldsymbol{C}$ into $\boldsymbol{C}$. Then there exists a fixed point of $\boldsymbol{F}$ on $\boldsymbol{C}$.*

The method described will be used simultaneously with exact estimations that are important for the calculation of the maximal admissible magnitude of the coefficient of friction $\mathscr{F}$ (which corresponds to the minimal $c(\mathscr{F}, a_0, A_0)$). Without involving technicalities but in more detail than in the above introduction, the method is described in § 4 of [8].

Proposition 2.1 (reiteration). $\int_{R^2} |h|^{-2-2\alpha} |\exp(ih\xi) - 1|^2 \, dh = |\xi|^{2\alpha} c(\alpha)$ *for every* $\xi \in R^2$ *and* $\alpha \in (0, 1)$, *where*

$$c(\alpha) = 2^{2-2\alpha} \int_{-\infty}^{+\infty} \frac{\sin^2 t}{|t|^{1+2\alpha}} \, dt \int_{-\infty}^{+\infty} \frac{dv}{(1 + v^2)^{1+\alpha}} . \tag{2.1}$$

Proof. Denote $I(\alpha, \xi) = \int_{R^2} |h|^{-2-2\alpha} |\exp(ih\xi) - 1|^2 \, dh$. By the substitution

$$\frac{h_1\xi_1 + h_2\xi_2}{2} = t_1 , \quad \frac{h_1\xi_1 - h_2\xi_2}{2} = t_2$$

we can calculate for $\xi_1 \neq 0$, $\xi_2 \neq 0$ ($I(\alpha_j)$ is continuous)

$$I(\alpha, \xi) = |\xi_1\xi_2|^{1+2\alpha} \int_{R^2} \frac{8 \sin^2 t_1 \, dt_1 \, dt_2}{[(t_1^2 + t_2^2)(\xi_1^2 + \xi_2^2) + 2t_1t_2(\xi_2^2 - \xi_1^2)]^{1+\alpha}} .$$

Transforming the denominator in the form $[4\xi_1^2\xi_2^2t_1^2(\xi_1^2 + \xi_2^2)^{-1} [1 + (t_2(\xi_1^2 + \xi_2^2) \cdot (2|\xi_1\xi_2| \, t_1)^{-1} + (\xi_2^2 - \xi_1^2)(2|\xi_1\xi_2|)^{-1})^2]]^{1+\alpha}$ and substituting $v = t_2(\xi_1^2 + \xi_2^2) \cdot (2|\xi_1\xi_2| \, t_1)^{-1} + (\xi_2^2 - \xi_1^2)(2|\xi_1\xi_2|)^{-1}$, $t_1 = t$, we arrive at the desired result.

Using Proposition 2.1 we obtain the following expressions for the norms in $\boldsymbol{H}^\alpha(R^2)$, $0 < |\alpha| < 1$:

$$\|w\|^2_{\alpha,R^2} \equiv \int_{R^2} |w(x)|^2 \, dx + \int_{R^2}\int_{R^2} \frac{(w(x + h) - w(x))^2}{|h|^{2+2\alpha}} \, dx \, dh = \tag{2.2}$$

$$= (2\pi)^{-2} \int_{R^2} |\hat{w}(\xi)|^2 (1 + c(\alpha) |\xi|^{2\alpha}) \, d\xi \quad \text{for} \quad \alpha > 0 ,$$

$$\|w\|^2_{\alpha,R^2} = (2\pi)^{-2} \int_{R^2} |\hat{w}(\xi)|^2 (1 + c(-\alpha) |\xi|^{-2\alpha})^{-1} \, d\xi \quad \text{for} \quad \alpha < 0 , \tag{2.3}$$

where $\hat{w}$ denotes the Fourier transform of w. As in [8] we introduce the anisotropic Sobolev spaces $\boldsymbol{H}^{1,\alpha}(\Omega)$, $\alpha > 0$: $\boldsymbol{H}^{1,\alpha}(\Omega) \equiv \{w \in \boldsymbol{H}^1(\Omega); \|w\|_{1,\alpha,\Omega} < +\infty\}$ provided with the norm $\|\cdot\|_{1,\alpha,\Omega}$, where

$$\|w\|^2_{1,\alpha,\Omega} = \|w\|^2_{1,\Omega} + \int_0^r \int_{R^2} \int_{R^2} \left[\sum_{i=1}^3 \left[\frac{\partial w}{\partial x_i} (x_1 + h_1, x_2 + \right.\right. \tag{2.4}$$

$$+ h_2, x_3) - \frac{\partial w}{\partial x_i}(x_1, x_2, x_3)\Big]^2\Big] |h|^{-2-2\alpha} \, dh \, dx =$$

$$= (2\pi)^{-2} \int_0^r \int_{R^2} |\tilde{w}|^2 + \left(|\xi|^2 |\tilde{w}|^2 + \left|\frac{\partial \tilde{w}}{\partial x_3}\right|^2\right) (1 + c(\alpha) |\xi|^{2\alpha}) \, d\xi_1 \, d\xi_2 \, dx_3 ,$$

where $\tilde{w}$ is the Fourier transform of w involving the transformation of the first two variables.

Let us prove the following Korn inequality:

Proposition 2.2. $\int_\Omega e_{kl}(v) \, e_{kl}(v) \, dx \geqq \frac{1}{2} \|v\|'^2_{\mathscr{H}}$ *for every* $v \in \mathscr{H}$, $v|_{\Gamma_u} = 0$ *in the sense of traces, where*

$$\|v\|'^2_{\mathscr{H}} = \int_\Omega \sum_{l=1}^3 \sum_{k=1}^3 \left(\frac{\partial v_k}{\partial x_l}\right)^2 dx .$$

Proof. For the sake of simplicity we suppose $r = \pi$. Let us put $v_i(x_1, x_2, x_3) = -v_i(x_1, x_2, -x_3)$ for $i = 1, 2$, $v_3(x_1, x_2, x_3) = v_3(x_1, x_2, -x_3)$ for $x_3 \in (-\pi, 0)$. Let us put

$$(2.5) \qquad a_{k,m} = \pi^{-1/2} \int_{-\pi}^{+\pi} \tilde{v}_k(\xi, x_3) \sin m x_3 \, dx_3 , \quad k = 1, 2, ; \quad m = 1, 2, \ldots ,$$

$$b_m = \pi^{-1/2} \int_{-\pi}^{\pi} \tilde{v}_3(\xi, x_3) \cos m x_3 \, dx_3 , \quad m = 1, 2, \ldots ,$$

$$b_0 = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \tilde{v}_3(\xi, x_3) \, dx_3 , \quad a_{k,0} = 0 , \quad k = 1, 2 .$$

We have

$$\int_\Omega e_{kl}(v) \, e_{kl}(v) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \left[\int_{R^2} e_{kl}(v) \, e_{kl}(v) \, dx_1 \, dx_2\right] dx_3 =$$

$$= \frac{1}{2(2\pi)^2} \int_{R^2} \sum_{m=0}^{+\infty} [m^2 |b_m|^2 + \sum_{k=1}^{2} (\xi_k^2 |a_{k,m}|^2 + \tfrac{1}{2}(\xi_k^2 |b_m|^2 + m^2 |a_{k,m}|^2 +$$

$$+ 2 \operatorname{Re} i\xi_k a_{k,m} m \overline{b_m})) + \tfrac{1}{2}(\xi_1^2 |a_{2,m}|^2 + \xi_2^2 |a_{1,m}|^2 + 2 \operatorname{Re} \xi_1 \xi_2 \, i a_{1,m} \overline{i a_{2,m}})] \, d\xi .$$

Using the inequality $\frac{1}{2} \sum_{j=1}^3 |z_j|^2 + \sum_{1 \leqq j < k \leqq 3} \operatorname{Re} z_j \overline{z_k} = \frac{1}{2} \left|\sum_{j=1}^3 z_j\right|^2 \geqq 0$ for $z_j = i\xi_j a_{j,m}$, $j = 1, 2$, $z_3 = m b_m$, we obtain the result.

Lemma 2.1. *Let the coefficients of the bilinear form a fulfil* (1.9). *Then* $a(u, u) \geqq \frac{1}{2} a_0 \|u\|'^2_{\mathscr{H}}$ *for every* $u \in \mathscr{H}_0$ ($\mathscr{H}_0$ *has been defined in* § 1).

Let us return to the problem (g_n). The unique solution u of the problem fulfils (1.10). Put in (1.10) $v = u_{-h} - u^0_{-h} + u^0 \in \mathscr{K}$. (For an arbitrary function F on Ω and $h \in R^2 \times \{0\} \equiv R^2$ we denote $F_{-h}(x) = F(x + h)$.) Let us make in (1.10) the shift in arguments of all functions and distributions in the direction h. In the shifted inequality (1.10) we put $v_{-h} = u + u^0_{-h} - u^0 \in \mathscr{K}_{-h} = \{v_{-h}; v \in \mathscr{K}\}$. We finally obtain

$$(2.6) \qquad a(u_{-h} - u, u_{-h} - u) \leqq a(u_{-h} - u, u^0_{-h} - u^0) + \\ + (a_{-h} - a)(u_{-h}, u^0_{-h} - u^0 + u - u_{-h}) + (f_{-h} - f, u_{-h} - u) - \\ - \langle (\mathscr{F}|g_n|)_{-h} - \mathscr{F}|g_n|, |u_t|_{-h} - |u_t| \rangle .$$

Let us assume $u^0 \in (H^{1,\alpha}(\Omega))^3$ for a fixed $\alpha \in (0, \frac{1}{2})$. Denote by $\|\cdot\|_{1,\alpha}$ the norm in $(H^{1,\alpha}(\Omega))^3$. Because of the Lipschitz continuity of the coefficients a_{ijkl} and the fact that $\int_{|h|<\delta} |h|^{-2+\varepsilon} \mathrm{d}h < +\infty$ for every $\varepsilon \in (0, +\infty)$, $\delta \in (0, +\infty)$, the multiplication of (2.5) by $|h|^{-2-2\alpha}$ and its integration over R^2 yields the following inequality:

$$\int_{R^2} |h|^{-2-2\alpha} a(u_{-h} - u, u_{-h} - u) \, \mathrm{d}h \leqq \int_{R^2} |h|^{-2-2\alpha} a(u^0_{-h} - u^0, u_{-h} - u) \, \mathrm{d}h + \\ + c_1 \|u\|_{\mathscr{H}} (\|u^0\|_{\mathscr{H}} + \|f\|_0) + \int_{R^2} |h|^{-2-2\alpha} \langle (\mathscr{F}g_n)_{-h} - \mathscr{F}g_n, |u_t|_{-h} - |u_t| \rangle \, \mathrm{d}h .$$

Using Lemma 1.1, Cauchy and Hölder inequalities, we get for an arbitrary $\varepsilon > 0$

$$(2.7) \qquad \int_{R^2} |h|^{-2-2\alpha} a(u_{-h} - u, u_{-h} - u) \, \mathrm{d}h \leqq (1 + \varepsilon) \int_{R^2} |h|^{-2-2\alpha} \, . \\ . \langle (\mathscr{F}g_n)_{-h} - \mathscr{F}g_n, |u_t|_{-h} - |u_t| \rangle \, \mathrm{d}h + k(\varepsilon) [\|u^0\|^2_{1,\alpha} + \|f\|^2_0] .$$

To estimate the first term on the right hand side of (2.7), we denote $G = \mathscr{F}g_n$, $U = |u_t|$. Using Proposition 2.1, we obtain

$$(2.8) \qquad \int_{R^2} |h|^{-2-2\alpha} \langle G_{-h} - G, U_{-h} - U \rangle \, \mathrm{d}h = (2\pi)^{-2} \int_{R^2} \hat{G}(\xi) \overline{\hat{U}(\xi)} \, . \\ . \int_{R^2} |h|^{-2-2\alpha} |\exp(ih\xi) - 1)^2 \, \mathrm{d}h \, \mathrm{d}\xi = c(\alpha) (2\pi)^{-2} \int_{R^2} \hat{G}(\xi) \overline{\hat{U}(\xi)} \, . \\ . |\xi|^{2\alpha} \, \mathrm{d}\xi \leqq c(\alpha) (2\pi)^{-1} \|G\|_{-1/2+\alpha, \Gamma_c} \, . \\ . \left(\int_{R^2} |\hat{U}(\xi)|^2 \left[1 + c(\tfrac{1}{2} + \alpha) |\xi|^{1+2\alpha} + \frac{c(\frac{1}{2} + \alpha)}{c(\frac{1}{2} - \alpha)} |\xi|^{4\alpha} - 1 \right] \mathrm{d}\xi \right)^{1/2} \left(\frac{c(\frac{1}{2} - \alpha)}{c(\frac{1}{2} + \alpha)} \right)^{1/2} \leqq \\ \leqq (1 + \varepsilon) \, c(\alpha) \left(\frac{c(\frac{1}{2} - \alpha)}{c(\frac{1}{2} + \alpha)} \right)^{1/2} \|G\|_{-1/2+\alpha, \Gamma_c} \|U\|_{1/2+\alpha, \Gamma_c} + k(\varepsilon, \alpha) \|U\|^2_{1/2, \Gamma_c} ,$$

where for $\alpha \in \langle 0, \frac{1}{4} \rangle$ we estimate $|\xi|^{4\alpha} \leqq |\xi| + 1$ and put $\varepsilon = 0$, $k(0, \alpha) = k(\alpha)$. For $\alpha \in (\frac{1}{4}, \frac{1}{2})$ we use the estimate

$$yz \leqq \frac{y^p}{p} + \frac{z^q}{q} \quad \text{for} \quad p = \frac{1 + 2\alpha}{4\alpha}, \quad q = \frac{1 + 2\alpha}{1 - 2\alpha}, \quad y = (p\varepsilon)^{1/p} |\xi|^{4\alpha}, \quad z = (p\varepsilon)^{-1/p},$$

where $\varepsilon > 0$ is an arbitrary constant.

It is easy to see that $\|G\|_{\beta,\Gamma_c} = \sup\{\langle\mathscr{F}g_n, v\rangle;\ v \in \boldsymbol{H}^{-\beta}(\Gamma_c),\ \|v\|_{-\beta,\Gamma_c} \leqq 1\} \leqq \leqq \|\mathscr{F}\|_\infty \|g_n\|_{\beta,\Gamma_c}$ for every $\beta \in (-1, 0\rangle$, where $\|\cdot\|_\infty$ denotes the norm in $\boldsymbol{L}_\infty(\Gamma_c)$. Evidently, $\| |u_t| \|_{1/2+\alpha,\Gamma_c} \leqq \|u_t\|_{1/2+\alpha,\Gamma_c}$ for every $\alpha \in \langle -\frac{1}{2}, \frac{1}{2})$, where the left hand term is the norm in $\boldsymbol{H}^{1/2+\alpha}(\Gamma_c)$, the right hand term in $(\boldsymbol{H}^{1/2+\alpha}(\Gamma_c))^2$. So we have for an arbitrary $\varepsilon > 0$

$$(2.9)\quad \int_{R^2} |h|^{-2-2\alpha}\, a(u_{-h} - u,\ u_{-h} - u)\,\mathrm{d}h \leqq (1+\varepsilon)\, c(\alpha) \left(\frac{c(\frac{1}{2}-\alpha)}{c(\frac{1}{2}+\alpha)}\right)^{1/2} \|\mathscr{F}\|_\infty \,.$$

$$\cdot \|g_n\|_{-1/2+\alpha,\Gamma_c} \|u_t\|_{1/2+\alpha,\Gamma_c} + k(\varepsilon,\alpha)\,[\|u^0\|^2_{1,\alpha} + \|f\|^2_0]\,.$$

Proposition 2.3. *For every $y \in \mathscr{H}_0 \cap (\boldsymbol{H}^{1,\alpha}(\Omega))^3$ the following estimate holds (a fulfils (1.9)):*

$$(2.10)\quad \|y|_{\Gamma_c}\|_{-1/2+\alpha,\Gamma_c} \leqq$$

$$\leqq \left[2c(\tfrac{1}{2}+\alpha)\,(a_0(c(\alpha)))^{-1} \int_{R^2} |h|^{-2-2\alpha}\, a(y_{-h} - y,\ y_{-h} - y)\,\mathrm{d}h\right]^{1/2} + k(\alpha)\|y\|_{\mathscr{H}}\,.$$

Proof. By Lemma 2.1,

$$\int_{R^2} |h|^{-2-2\alpha}\, a(y_{-h} - y,\ y_{-h} - y)\,\mathrm{d}h \geqq$$

$$\geqq \tfrac{1}{2}a_0 \int_{R^2} |h|^{-2-2\alpha} \int_\Omega \sum_{j=1}^3 \sum_{k=1}^3 \left(\frac{\partial y_j}{\partial x_k}(x+h) - \frac{\partial y_j}{\partial x_k}(x)\right)^2 \mathrm{d}x\,\mathrm{d}h \geqq$$

$$\geqq \tfrac{1}{2}a_0 \int_{R^2} |h|^{-2-2\alpha} \int_\Omega \sum_{j=1}^3 \sum_{k=1}^3 \left(\frac{\partial Z_j}{\partial x_k}(x+h) - \frac{\partial Z_j}{\partial x_k}(x)\right)^2 \mathrm{d}x\,\mathrm{d}h\,,$$

where Z_j are the corresponding solutions of the problems

$$(2.11)\quad \Delta w_j = 0 \text{ on } \Omega\,,\quad j = 1, 2, 3\,,$$

$$w_j(x_1, x_2, 0) = 0\,,\quad w_j(x_1, x_2, r) = y_j(x_1, x_2, r) \quad \text{a.e. in } R^2\,,\quad j = 1, 2, 3\,.$$

Using the Fourier transformation in the first two variables ($\tilde{Z}_j(\xi, x_3) = \operatorname{sh}(x_3|\xi|)\,.\,\operatorname{sh}^{-1}(r|\xi|)\,\tilde{y}_j(\xi, r)$), we have

$$(2.12)\quad \int_{R^2} |h|^{-2-2\alpha}\, a(y_{-h} - y,\ y_{-h} - y)\,\mathrm{d}h \geqq (2\pi)^{-2}\,\tfrac{1}{2}a_0\, c(\alpha)\,.$$

$$\cdot \sum_{j=1}^3 \int_{R^2} \int_0^r \left(\left|\frac{\partial \tilde{Z}_j}{\partial x_3}\right|^2 + |\xi|^2 |\tilde{Z}_j|^2\right) |\xi|^{2\alpha}\,\mathrm{d}x_3\,\mathrm{d}\xi =$$

$$= (2\pi)^{-2}\,\tfrac{1}{2}a_0\, c(\alpha) \sum_{j=1}^3 \int_{R^2} |\xi|^{1+2\alpha} |\tilde{y}_j(\xi, r)|^2 (1 + \exp(-2r|\xi|))\,.$$

$$.\,(1 - \exp(-2r|\xi|))^{-1}\,\mathrm{d}\xi \geqq (2c(\tfrac{1}{2}+\alpha))^{-1}\, a_0 c(\alpha)\, \|y|_{\Gamma_c}\|^2_{1/2+\alpha} - k(\alpha)\|y|_{\Gamma_c}\|^2_0\,.$$

Remark. If Ω is a homogeneous isotropic body, we have the Hook law in the form

$$\tau_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}\,, \tag{2.13}$$

where $\mu > 0$, $\lambda \geqq 0$ are Lamé constants. Hence (2.10) is in the form

$$\|y|_{\Gamma_c}\|_{+1/2+\alpha,\Gamma_c} \leqq \tag{2.14}$$
$$\leqq \left[c(\tfrac{1}{2} + \alpha)\,(\mu\, c(\alpha))^{-1} \int_{R^2} |h|^{-2-2\alpha}\, a(y_{-h} - y,\, y_{-h} - y)\,\mathrm{d}h\right]^{1/2} + k(\alpha)\|y\|_{\mathscr{H}}\,.$$

Denote $u' = u - u^0$. Using (2.9) and (2.10) we obtain

$$\left(\int_{R^2} |h|^{-2-2\alpha}\, a(u'_{-h} - u',\, u'_{-h} - u')\,\mathrm{d}h\right)^{1/2} \leqq a_0^{-1/2}(1 + \varepsilon)\,\|\mathscr{F}\|_\infty\,. \tag{2.15}$$
$$\cdot\, \|g_n\|_{-1/2+\alpha,\Gamma_c}\,(2c(\alpha)\, c(\tfrac{1}{2} - \alpha))^{1/2} + k(\varepsilon, \alpha)\,[\|u^0\|_{1,\alpha} + \|f\|_0]\,,$$

where $\varepsilon > 0$ is an arbitrary given constant. Naturally, $k(\varepsilon, \alpha)$ in (2.15) is different from $k(\varepsilon, \alpha)$ of (2.9). In particular, we have

Lemma 2.2. *Let $g_n \in \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c) \cap \boldsymbol{C}^{*-}$, $u^0 \in (\boldsymbol{H}^{1,\alpha}(\Omega))^3$, $f \in (\boldsymbol{L}_2(\Omega))^3$, Ω being the given strip. Then u solving (g_n) belongs also to $(\boldsymbol{H}^{1,\alpha}(\Omega))^3$, if all $a_{ijkl} \in \boldsymbol{C}^{0,1}(\Omega)$ and (1.9) holds.*

Proof. From the bipolar theorem $\boldsymbol{C}^{*-} \cap \boldsymbol{C}^1(\Gamma_c)$ is dense in $\boldsymbol{H}^{-1/2+\alpha}(\Gamma_c) \cap \boldsymbol{C}^{*-}$. Estimation like (1.15) yields the statement for $g_n \in \boldsymbol{C}^1(\Gamma_c)$. The use of that, Prop. 2.3, the reflexivity of $(\boldsymbol{H}^{1,\alpha}(\Omega))^3$, (2.15) and Lemma 2.1 completes the proof.

Proposition 2.4. *Under the suppositions of Lemma 2.2 let us define $T_n(u)$ as in (1.11). Then the inequality*

$$\|T_n(u)\|_{-1/2+\alpha,\Gamma_c} \leqq (1 + \varepsilon)\, A_0^{1/2}(c(\alpha)\, c(\tfrac{1}{2} - \alpha))^{-1/2}\,. \tag{2.16}$$
$$\cdot \left(\int_{R^2} |h|^{-2-2\alpha}\, a(u'_h - u',\, u'_{-h} - u')\,\mathrm{d}h\right)^{1/2} + k(\varepsilon, \alpha)\,[\|u^0\|_{1,\alpha} + \|f\|_0]$$

is valid, where $\varepsilon > 0$ is an arbitrarily given constant.

Proof. By means of shifts in arguments in (1.11) we obtain the following inequality for an arbitrarily given $w \in \mathscr{H}_0$, $w_t|_{\Gamma_c} = 0$:

$$\langle (T_n(u))_{-h} - T_n(u), w_n \rangle = a(u'_{-h} - u', w) + a(u^0_{-h} - u^0, w) + \tag{2.17}$$
$$+ (a_{-h} - a)(u_{-h}, w) - (f, w_h - w) \leqq [(a(u'_{-h} - u', u'_{-h} - u'))^{1/2} +$$
$$+ (a(u^0_{-h} - u^0, u^0_{-h} - u^0))^{1/2}]\,(a(w, w))^{1/2} + \chi(|h|)\, c_1[\|f\|_0 + \|u^0\|_{\mathscr{H}}]\,\|w\|_{\mathscr{H}}\,,$$

where $\chi(t) = t$ for $t \in \langle 0, 1 \rangle$, $\chi(t) = 1$ for $t > 1$, c_1 is a suitable constant generally different from c_1 of the preceding inequalities. Let for an arbitrary $w_n \in \boldsymbol{H}^{1/2}(\Gamma_c)$, v_3 denote the extension of w_n by means of (2.11). Let $\Xi : \boldsymbol{H}^{1/2}(\Gamma_c) \to \mathscr{H}$ be such that $w_n \mapsto [0, 0, v_3]$. So we have

$$(2.18) \qquad \|\Xi(w_n)\|'^2_{\mathscr{H}} = (2\pi)^{-2} \int_{\Omega} |\xi|^2 |\tilde{v}_3|^2 + \left|\frac{\partial \tilde{v}_3}{\partial x_3}\right|^2 d\xi\, dx_3 =$$

$$= (2\pi)^{-2} \int_{R^2} |\xi|\, |\tilde{w}_n(\xi, r)|^2 (1 + \exp(-2r|\xi|)) (1 - \exp(-2r\xi))^{-1} \,.$$

$$. \, d\xi \leqq (c(\tfrac{1}{2}))^{-1} \|w_n\|^2_{1/2,\Gamma_c} + k\|w_n\|^2_{-1/2,\Gamma_c} \,.$$

Let us put $\hat{w}_n(\xi) = \widehat{((T_n(u))_{-h} - T_n(u))}(\xi)\, (1 + c(\tfrac{1}{2})\, |\xi|)^{-1}\, \|\widehat{((T_n(u))_{-h} - T_n(u))}(\xi)\, . \, (1 + c(\tfrac{1}{2})\, |\xi|)^{-1}\|^{-1}_{1/2,R^2}$. We obtain for an arbitrary constant $\varepsilon > 0$ and suitable $c_1(\varepsilon)$:

$$(2.19) \qquad \|(T_n(u))_{-h} - T_n(u)\|^2_{-1/2,\Gamma_c} \leqq (1 + \varepsilon)\, (c(\tfrac{1}{2}))^{-1} \,.$$

$$. \, [(A_0\, a(u'_{-h} - u', u'_{-h} - u'))^{1/2} + A_0^{1/2} \|u^0_{-h} - u^0\|_{\mathscr{H}}]^2 +$$

$$+ \chi(|h|)\, k_1(\varepsilon)\, [\|u^0\|^2_{\mathscr{H}} + \|f\|^2_0] \,.$$

Multiplying (2.19) by $|h|^{-2-2\alpha}$ and integrating in h over R^2 we obtain

$$(2.20) \qquad \int_{R^2} |h|^{-2-2\alpha} \|(T_n(u))_{-h} - T_n(u)\|^2_{-1/2,\Gamma_c} dh \leqq (1 + \varepsilon)\, (c(\tfrac{1}{2}))^{-1} \,.$$

$$. \, A_0 \int_{R^2} a(u'_{-h} - u', u'_{-h} - u')\, |h|^{-2-2\alpha} dh + k_2(\varepsilon)\, [\|u^0\|^2_{1,\alpha} + \|f\|^2_0] \,.$$

Denoting $\widehat{(T_n(u))} = \mathscr{T}$, we convert the left hand part of (2.20) by means of the Fourier transformation into the form

$$(2.21) \quad (2\pi)^{-2} \int_{R^2} |\mathscr{T}(\xi)|^2 (1 + |\xi|\, c(\tfrac{1}{2}))^{-1}\, d\xi \int_{R^2} |h|^{-2-2\alpha} |\exp(ih\xi) - 1|^2\, dh =$$

$$= (2\pi)^{-2} c(\alpha) \int_{R^2} |\mathscr{T}(\xi)|^2 (1 + c(\tfrac{1}{2})\, |\xi|)^{-1} |\xi|^{2\alpha} d\xi \geqq \frac{c(\alpha)\, c(\frac{1}{2} - \alpha)}{c(\frac{1}{2})} \|T_n(u)\|^2_{-1/2+\alpha,\Gamma_c} -$$

$$- k\|T_n(u)\|^2_{-1/2,\Gamma_c} \,.$$

Summing up (2.20) and (2.21), we obtain (2.16).

Lemma 2.3. *Let* $g_n \in \mathbf{C}^{*-} \cap \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$, $u^0 \in (\boldsymbol{H}^{1,\alpha}(\Omega))^3$, $f \in (\boldsymbol{L}_2(\Omega))^3$. *Then for an arbitrary* $\varepsilon > 0$ *there exists a constant* $k(\varepsilon, \alpha)$ *such that*

$$(2.22) \qquad \|T_n(u)\|_{-1/2+\alpha,\Gamma_c} \leqq (1 + \varepsilon)\, \|\mathscr{F}\|_\infty\, (a_0^{-1}\, 2A_0)^{1/2}\, \|g_n\|_{-1/2+\alpha,\Gamma_c} +$$

$$+ k(\varepsilon, \alpha)\, [\|u^0\|_{1,\alpha} + \|f\|_0] \,,$$

where u *is the corresponding solution of* (g_n).

The lemma is a direct consequence of (2.15) and (2.16).

Theorem 2.2. *Let* $\Omega = R^2 \times (0, r)$, $\Gamma_u = R^2 \times \{0\}$, $\Gamma_c = R^2 \times \{r\}$. *Let* $a_{ijkl}(x)$ *be Lipschitz continuous on* Ω, $i, j, k, l = 1, 2, 3$, *let the form a satisfy* (1.9). *Let* $u^0 \in (\boldsymbol{H}^{1,\alpha}(\Omega))^3$ *for some* $\alpha \in (0, \frac{1}{2})$, *let* $f \in (\boldsymbol{L}_2(\Omega))^3$. *Let us assume that* $\mathscr{F} \in \boldsymbol{C}^1(\Gamma_c)$ *with a compact support in* Int Γ_c *and* $\|\mathscr{F}\|_\infty < \sqrt{(a_0/2A_0)}$. *Then there exists a solution of the Signorini problem with friction.*

Remark 1. The respective position of Ω in R^3 is unimportant. Also the supposition $u^0 \in (\boldsymbol{H}^{1,\alpha}(\Omega))^3$ can be replaced by $u^0 \in \mathscr{H}$, as we can see in § 3 of this paper.

Proof of the theorem: Let $\varepsilon > 0$ be such a constant that $(1 + \varepsilon) \|\mathscr{F}\|_\infty \cdot (a_0^{-1} 2A_0)^{1/2} < 1$. Then there exists r_0 such that the operator $\boldsymbol{\Phi} : g_n \mapsto T_n(u)$ maps $M_{r_0} = \boldsymbol{C}^{*-} \cap \bar{B}_{r_0}(0)$ into itself, where $\bar{B}_{r_0}(0) = \{g \in \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c);\ \|g\|_{-1/2+\alpha,\Gamma_c} \leqq \leqq r_0\}$. By Theorem 2.1 it suffices to prove the weak continuity of $\boldsymbol{\Phi}$ on $\boldsymbol{C}^{*-} \cap \cap \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$ because of the reflexivity of $\boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$. Let $\{g_n^m\}_{m=1}^{+\infty}$ be an arbitrary sequence in $\boldsymbol{C}^{*-} \cap \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$ such that there exists $g_n \in \boldsymbol{C}^{*-} \cap \boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$, $g_n^m \rightharpoonup g_n$ in $\boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$. Denote $\mathfrak{F} = \{T_n(u^m)\}_{m=1}^{+\infty}$, where u^m are the corresponding solutions of (g_n^m), $m = 1, 2, \ldots$, denote by u the solution of (g_n). By (2.22), $\mathfrak{F}$ is weakly relatively compact in $\boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$. Let Q_0 be a weak limit of a subsequence $\mathfrak{F}' \subset \mathfrak{F}$. Because of the weak convergence $\mathscr{F} g_n^m \rightharpoonup \mathscr{F} g_n$ in $\boldsymbol{H}^{-1/2+\alpha}(\Gamma_c)$, the compact imbedding theorem (which is valid due to the compactness of supp $\mathscr{F}$) and Lemma 1.1 yield the strong convergence of $\mathfrak{F}'$ to $T_n(u)$ in $\boldsymbol{H}^{-1/2}(\Gamma_c)$. So $Q_0 = T_n(u)$ and evidently, the whole $\mathfrak{F}$ converges to $T_n(u)$.

Remark 2. Let Ω be homogeneous and isotropic (the Hook law is in the form (2.13)). Then a certain improvement of (2.16) is possible by means of the following procedure: Let us have (2.17), we need some exact estimate for $(a(w, w))^{1/2}$. Let us extend $w_n \in \boldsymbol{H}^{-1/2}(\Gamma_c)$ by means of the problem

$$\mu\left(\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2}\right) + (\lambda + 2\mu)\frac{\partial^2 v_3}{\partial x_3^2} = 0 \quad \text{on} \quad \Omega\,, \tag{2.23}$$

$$v_3(x_1, x_2, 0) = 0\,, \quad v_3(x_1, x_2, r) = w_n(x_1, x_2, r) \quad \text{a.e. in } R^2\,.$$

Put $\Xi_1(w_n) \equiv (0, 0, v_3) \equiv w$. Using the Fourier transformation in the first two variables $\tilde{v}_3(\xi, x_3) = \hat{w}_n(\xi, r)\,\mathrm{sh}\,(|\xi|\, x_3 \mu^{1/2}(\lambda + 2\mu)^{-1/2})\,\mathrm{sh}^{-1}\,(|\xi|\, r\mu^{1/2}(\lambda + 2\mu)^{-1/2})$ we have

$$a(w, w) = (2\pi)^{-2} \int_\Omega \mu|\tilde{v}_3|^2\,|\xi|^2 + (\lambda + 2\mu)\left|\frac{\partial \tilde{v}_3}{\partial x_3}\right|^2 \mathrm{d}\xi\,\mathrm{d}x_3 =$$

$$= \mu(2\pi)^{-2} \int_{R^2} |\xi|^2\,|\hat{w}_n(\xi, r)|^2\,\mathrm{sh}^{-2}\,(|\xi|\, r\mu^{1/2}(\lambda + 2\mu)^{-1/2})\,.$$

$$\cdot \int_0^r \mathrm{sh}^2\,(|\xi|\, x_3\mu^{1/2}(\lambda + 2\mu)^{-1/2}) + \mathrm{ch}^2\,(|\xi|\, x_3\mu^{1/2}(\lambda + 2\mu)^{-1/2})\,\mathrm{d}\xi\,\mathrm{d}x_3 =$$

$$= (\mu(\lambda + 2\mu))^{1/2}\,(2\pi)^{-2} \int_{R^2} |\xi|\,|\hat{w}_n(\xi, r)|^2\,[1 + \exp\,(-2r|\xi|\,\mu^{1/2}(\lambda + 2\mu)^{-1/2})]\,.$$

$$\cdot\,[1 - \exp(-2r|\xi|\,\mu^{1/2}(\lambda + 2\mu)^{-1/2})]^{-1}\,\mathrm{d}\xi \leqq (c(\tfrac{1}{2}))^{-1}\,(\mu(\lambda + 2\mu))^{1/2}\,.$$
$$\cdot\,\|w_n\|^2_{1/2,\Gamma_c} + k\|w_n\|^2_{-1/2,\Gamma_c}\,,$$

because the function

$$\psi(z) = \frac{2z\exp(-2r\mu^{1/2}(\lambda + 2\mu)^{-1/2}\,z)\,(1 + c(\frac{1}{2})\,z)}{1 - \exp(-2r\mu^{1/2}(\lambda + 2\mu)^{-1/2}\,z)}$$

is bounded on $\langle 0, +\infty)$. Using the same considerations as in the corresponding part of the proof of Proposition 2.4, we obtain instead of (2.16)

$$(2.24)\qquad \|T_n(u)\|_{-1/2+\alpha,\Gamma_c} \leqq (1 + \varepsilon)\,(\mu(\lambda + 2\mu))^{1/4}\,(c(\alpha)\,c(\tfrac{1}{2} - \alpha))^{-1/2}\,.$$
$$\cdot\left(\int_{R^2} |h|^{-2-2\alpha}\,a(u'_{-h} - u',\, u'_{-h} - u')\,\mathrm{d}h\right)^{1/2} + k(\varepsilon, \alpha)\,[\|u^0\|_{1,\alpha} + \|f\|_0]\,.$$

Using (2.24) and (2.15) modified by means of (2.14), we obtain

$$(2.25)\qquad \|T_n(u)\|_{-1/2+\alpha,\Gamma_c} \leqq (1 + \varepsilon)\,\|\mathscr{F}\|_\infty\,\mu^{-1/4}(\lambda + 2\mu)^{1/4}\,\|g_n\|_{-1/2+\alpha,\Gamma_c} +$$
$$+\,k(\varepsilon, \alpha)\,[\|u^0\|_{1,\alpha} + \|f\|_0]\,.$$

Thus we obtain

Corollary of Theorem 2.2. *Under the suppositions of the theorem with the exception of the estimate for* $\|\mathscr{F}\|_\infty$, *let* Ω *be homogeneous and isotropic and let* $\|\mathscr{F}\|_\infty < \sqrt[4]{(\mu/(\lambda + 2\mu))}$, *where* $\mu > 0$, $\lambda \geqq 0$ *are Lamé constants. Then there exists a solution of the Signorini problem with friction.*

Remark. Let

$$\mu = \frac{E}{2(1 + \sigma)}\,, \quad \lambda = \frac{E\sigma}{(1 + \sigma)(1 - 2\sigma)}\,.$$

Then

$$\sqrt[4]{\frac{\mu}{\lambda + 2\mu}} = \sqrt[4]{\frac{1 - 2\sigma}{2 - 2\sigma}}\,.$$

3. THE SIGNORINI PROBLEM WITH FRICTION FOR A GENERAL DOMAIN IN R^3 WITH SUFFICIENTLY SMOOTH CONTACT PART OF ITS BOUNDARY

The purpose of this part is the proof of the existence theorem for the Signorini problem with friction for a general domain in R^3 with a sufficiently smooth contact part of the boundary. The estimation of the admissible coefficient of friction will be the same as in Theorem 2.2, the method of the proof will be a generalization of the methods of Sec. 2 with the help of same "local straightening" of Γ_c. We require that Γ_c fulfil the following special conditions:

(3.1) There exists $\Delta_0 > 0$ such that for every $\delta \in (0, \Delta_0\rangle$ there exists a finite covering $\mathfrak{U}_\delta$ of the set $\Gamma_c \cap B_{\delta_0/2}(\operatorname{supp} \mathscr{F})$, where $B_{\delta_0/2}(\operatorname{supp} \mathscr{F}) = \{x \in R^3;\ \exists y \in \operatorname{supp} \mathscr{F},\ |x - y| < \delta_0/2\}$, $0 < \delta_0 = \operatorname{dist}(\operatorname{supp} \mathscr{F}, \Gamma \setminus \Gamma_c)$, satisfying the following conditions:

(3.1a) There exist constants $r, K_0 \in (0, +\infty)$ such that that for every $V \in \mathfrak{U}_\delta$ there exists $x_0 \in \operatorname{Int} V \cap \Gamma_c$ in such a way that we have the local description of the boundary by means of a function $\varphi_V : R^2 \to R^1$ such that (after a certain suitable rotation and shift of the coordinate system in R^3) $x_0 = [0, \varphi_V(0)]$, $\varphi_V(0) = 0$, $\varphi_V'(0) = 0$, $\|\varphi_V\|_{C^{2,1}(R^2)} < K_0$. Denoting $B_k(0) \equiv \{x \in R^2; |x| < k\}$, we suppose that the coordinate transformation $\Psi_V : [x, y] \mapsto [x, y - \varphi_V(x)]$ maps V onto $B_{\delta/2}(0) \times (-r, r)$ and simultaneously $\{[x, y] \in R^3;\ x \in B_\delta(0),\ y \in \langle \varphi_V(x) - r, \varphi_V(x))\} \subset R^3 \setminus \Omega$, $\{[x, y] \in R^3; x \in B_\delta(0), y \in (\varphi_V(x), \varphi_V(x) + r\rangle\} \subset \Omega$. Moreover, we suppose for every $\delta \in (0, \Delta_0\rangle$ and for each $V \in \mathfrak{U}_\delta$, $\operatorname{dist}(V, \Gamma \setminus \Gamma_c) > \delta_0/4$ and $\Psi_V^{-1}(B_\delta(0) \times (-r, r)) \cap \Gamma \setminus \Gamma_c = \emptyset$.

(3.1b) For every $\delta \in (0, \Delta_0\rangle$ and for each $V \in \mathfrak{U}_\delta$ there exists $\varrho_V \in C^{2,1}(R^3)$ such that $\varrho_V(x) \in \langle 0, 1\rangle$ everywhere on R^3, $\operatorname{dist}(\operatorname{supp} \varrho_V, R^3 \setminus V) > 0$ and $\sum_{V \in \mathfrak{U}_\delta} \varrho_V = 1$ holds on $\Gamma_c \cap B_{\delta_0/2}(\operatorname{supp} \mathscr{F})$.

(3.2) We suppose the existence of a function $\tilde{s} \in C^2(R^3)$ with a compact support contained in $B_{7\delta_0/8}(\operatorname{supp} \mathscr{F})$ such that $\tilde{s}(x) \in \langle 0, 1\rangle$ for every $x \in R^3$, $\tilde{s} \equiv 1$ on $\bigcup_{\delta \in (0, \Delta_0\rangle} \bigcup_{V \in \mathfrak{U}_\delta} \operatorname{supp} \varrho_V$. Let for every $\delta \in (0, \Delta_0)$ a finite extension $\mathfrak{V}_\delta$ of $\mathfrak{U}_\delta$ be possible with the following properties:

(3.2a) Every $\mathfrak{V}_\delta$, $\delta \in (0, \Delta_0\rangle$ is a covering of Γ_c. For every $\delta \in (0, \Delta_0\rangle$, each $V \in \mathfrak{V}_\delta$ is $C^{2,1}$ diffeomorphic to a set $M_V \times (-r, r)$ in the diffeomorphism Ψ_V given (after a suitable rotation and shift of the coordinate system in R^3) by a function φ_V by means of the same relations as in (3.1a). We suppose M_V in the form of an open bounded euclidean ball in R^2 having the center at 0 for every $V \in \mathfrak{V}_\delta$ such that $V \cap \operatorname{supp} \tilde{s} \neq \emptyset$. For the other $V \in \mathfrak{V}_\delta$ let M_V be an open euclidean ball in R^2 or an open (generally unbounded) polygon. For every $V \in \mathfrak{V}_\delta \setminus \mathfrak{U}_\delta$, $V \cap B_{\delta_0/4}(\operatorname{supp} \mathscr{F}) = \emptyset$ and, moreover, at least one of the sets $\bar{V} \cap \operatorname{supp} \tilde{s}$, $\bar{V} \cap (\overline{\Gamma \setminus \Gamma_c})$ is empty, $\delta \in (0, \Delta_0\rangle$.

(3.2b) For every $\delta \in (0, \Delta_0\rangle$ the partition of unity on $\Gamma_c \cap B_{\delta_0/2}(\operatorname{supp} \mathscr{F})$ has a corresponding extension to the partition of unity $\{\varrho_V, V \in \mathfrak{V}_\delta\}$ on Γ_c of the class $C^{2,1}$ such that for every $V \in \mathfrak{V}_\delta$, $\operatorname{dist}(\operatorname{supp} \varrho_V, R^3 \setminus V) > 0$.

To prove the existence theorem we need the following lemma:

Lemma 3.1. *Let Q be a domain in R^N, $N \geqq 1$. Let $\zeta \in C^1(R^N)$ be a non-negative function such that* $\operatorname{dist}(\operatorname{supp} \zeta, R^N \setminus Q) > 0$. *Let $g \in H^\alpha(Q)$ be arbitrary, $|\alpha| < 1$. Denote for $\alpha \in \langle 0, 1\rangle$ $g_\zeta = g\zeta$ on Q, $g_\zeta \equiv 0$ on $R^N \setminus Q$; for $\alpha \in (-1, 0)$ let $\langle g_\zeta, v\rangle_{L_2(R^N)} = \langle g, v\zeta\rangle_{L_2(Q)}$ for every $v \in H^{-\alpha}(R^N)$. Then there exist constants*

$c^{\Omega}(\alpha, \zeta)$, $d_i^{\Omega}(\alpha, \zeta)$, $i = 1, 2$, *independent of* g *and such that*

$$\|g_\zeta\|_{\alpha,\Omega} \leqq c^{\Omega}(\alpha, \zeta) \|g\|_{\alpha,\Omega}, \tag{3.3}$$

$$d_1^{\Omega}(\alpha, \zeta) \|g_\zeta\|_{\alpha,R^N} \leqq \|g_\zeta\|_{\alpha,\Omega} \leqq d_2^{\Omega}(\alpha, \zeta) \|g_\zeta\|_{\alpha,R^N}, \tag{3.4}$$

where for a domain $M \subset R^N$, $\|\cdot\|_{\alpha,M}$ *denotes the norm in* $\boldsymbol{H}^\alpha(M)$.

The proof is obvious.

Let us suppose that $\Omega \in \boldsymbol{C}^{0,1}$ satisfies (3.1) and (3.2). For a fixed $\delta \in (0, \Delta_0\rangle$ and $V \in \mathfrak{U}_\delta$ we can write φ, ϱ, Ψ instead of φ_V, ϱ_V, Ψ_V. The matrix of the coordinate transformation corresponding to Ψ has the form

$$\mathsf{E} - \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ \dfrac{\partial\varphi}{\partial x_1}, & \dfrac{\partial\varphi}{\partial x_2}, & 0 \end{pmatrix},$$

where E is the unit matrix. Let us write the coefficients of the form a after the rotation and the shift of the coordinate system according to (3.1) also as a_{ijkl}. Denote $B^+(\delta, r) = B_\delta(0) \times (0, r)$, $B(\delta, r) = B_\delta(0) \times (-r, r)$. For every $w \in \mathscr{H} \equiv \mathscr{H}(\Omega)$ such that supp $w \subset \Psi^{-1}(B(\delta, r))$ we have (denoting for an arbitrary function F on V, $\bar{F} = F \circ (\Psi^{-1})$):

$$\begin{aligned} a(u, w) = & \int_{B^+(\delta,r)} \tfrac{1}{4} \bar{a}_{ijkl} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\left(\frac{\partial \bar{w}_k}{\partial x_l} + \frac{\partial \bar{w}_l}{\partial x_k}\right) \mathrm{d}x + \\ & + \int_{B^+(\delta,r)} \tfrac{1}{4} \left[(-\bar{a}_{ijkl}) \left(\frac{\partial \bar{u}_i}{\partial x_3}\frac{\partial\varphi}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_3}\frac{\partial\varphi}{\partial x_i}\right)\left(\frac{\partial \bar{w}_k}{\partial x_l} + \frac{\partial \bar{w}_l}{\partial x_k}\right) - \right. \\ & - \bar{a}_{ijkl}\left(\frac{\partial \bar{u}_l}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\left(\frac{\partial \bar{w}_k}{\partial x_3}\frac{\partial\varphi}{\partial x_l} + \frac{\partial \bar{w}_l}{\partial x_3}\frac{\partial\varphi}{\partial x_k}\right) + \\ & \left. + \bar{a}_{ijkl}\left(\frac{\partial \bar{u}_i}{\partial x_3}\frac{\partial\varphi}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_3}\frac{\partial\varphi}{\partial x_i}\right)\left(\frac{\partial \bar{w}_k}{\partial x_3}\frac{\partial\varphi}{\partial x_l} + \frac{\partial \bar{w}_l}{\partial x_3}\frac{\partial\varphi}{\partial x_k}\right)\right] \mathrm{d}x = \\ & = \bar{a}(\bar{u}, \bar{w}) + b(\bar{u}, \bar{w}), \end{aligned} \tag{3.5}$$

where $\bar{a}(\bar{u}, \bar{w})$ is the first and $b(\bar{u}, \bar{w})$ the second integral in (3.5). We have

$$\int_{B^+(\delta,r)} b_{ijkl} \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{w}_k}{\partial x_l} \mathrm{d}x \equiv b(\bar{u}, \bar{w}) \leqq \text{const. } \delta \|u\|_{\mathscr{H}} \|v\|_{\mathscr{H}} \text{ for every } u, v \in \mathscr{H} \equiv \mathscr{H}(\Omega).$$

The following inequality is a consequence of (1.10) for an arbitrary $w \in \mathscr{K} \subset \mathscr{H}(\Omega)$ with the support in $\Psi^{-1}(B(\delta, r))$:

$$\bar{a}(\bar{u}, \bar{w}) + b(\bar{u}, \bar{w}) + \int_{B_\delta(0)} \mathscr{F}|\bar{g}_n| \left[|\bar{u}_t + \bar{w}_t| - |\bar{u}_t|\right] J(x) \, \mathrm{d}x \geqq \int_{B^+(\delta,r)} \bar{f}\bar{w} \, \mathrm{d}x, \tag{3.6}$$

where $J(x) = \sqrt{(1 + (\partial\varphi/\partial x_1)^2 + (\partial\varphi/\partial x_2)^2)}$. Of course, the "tangentiality" of u_t, w_t is considered with respect to the surface $\Gamma \cap V$. Let us write $a, \varrho, u, w, f, \mathscr{F}, g_n$ instead of $\bar{a}, \bar{\varrho}, \bar{u}, \bar{w}, \bar{f}, \bar{\mathscr{F}}, \bar{g}_n$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L_2(R^2)$, by $(\cdot, \cdot)$ the scalar product in $(L_2(\Theta))^3$, $\Theta = R^2 \times (0, r)$. Then

$$(3.7) \qquad a(u, w) + b(u, w) + \langle \mathscr{F} |g_n| \, J\,, \quad |u_t + w_t| - |u_t| \rangle \geqq (f, w)$$

for every $w \in \mathscr{H}(\Theta) = (H^1(\Theta))^3$ with the support in $B(\delta, r)$ such that $w_n \leqq 0$ on $R^2 \times \{0\}$, $w_t = 0$ on $R^2 \times \{0\}$, where naturally $w_n = w \,.\, n$ and $n(x)$ is the unit outer normal vector of Γ_c at the point $[x, \varphi(x)]$, $x \in B_\delta(0) \equiv B_\delta(0) \times \{0\}$. Define $n_{-h}(x) = n(x + h)$ for $h \in R^2$ such that $(x + h) \in B_\delta(0)$. For $x \in B(\delta, r)$, $x = [x', x'']$ define $L_{-h}(x)$, the rotation of the coordinate system in R^3 with the axis perpendicular to $n(x')$ and $n_{-h}(x')$ transforming $n_{-h}(x')$ into $n(x')$.

Lemma 3.2. *For every $w \in \mathscr{H}(B^+(\delta, r)) \equiv (H^1(B^+(\delta, r)))^3$ such that* supp $w \subset B(\delta, r)$, *we have $\|L_{-h}w - w\|_{\mathscr{H}(B^+(\delta,r))} = O(|h|)$ for $|h| \to 0$.*

Proof. The coefficients of the matrix (d_{ij}) describing the rotation of a unit vector $z = [z_1, z_2, z_3]$ into a unit vector $y = [y_1, y_2, y_3]$ with the axis perpendicular to y and to z have the form

$$(3.8) \qquad d_{ij} = \delta_{ij} - 2(4 - \mathfrak{G}^2)^{-1} \left(-2\mathfrak{g}_i z_j + 2\mathfrak{g}_j z_i + \mathfrak{g}_i\mathfrak{g}_j + \mathfrak{G}^2 z_j(z_i + \mathfrak{g}_i)\right), \quad i, j = 1, 2, 3\,,$$

where $\mathfrak{g} \equiv (\mathfrak{g}_i)_{i=1}^3 = y - z$, $\mathfrak{G} = |\mathfrak{g}|$ and δ_{ij} is the Kronecker delta. For $\mathfrak{g}(x, h) = -n_{-h}(x') + n(x')$ we have $\mathfrak{g}_i(x, h) = O(|h|)$, $|h| \to 0$, $\partial\mathfrak{g}_i/\partial x_k = O(|h|)$, $|h| \to 0$, $i, k = 1, 2, 3$, for every $x = [x', x''] \in B_\delta(0) \times R^1$. So $d_{ij} = \delta_{ij} + O(|h|)$, $|h| \to 0$, $\partial d_{ij}/\partial x_k = O(|h|)$, $|h| \to 0$, for $i, j, k = 1, 2, 3$.

Let us take an arbitrary but fixed $g_n \in C^{*-} \cap H^{-1/2+\alpha}(\Gamma_c)$, where $\alpha > 0$ is a given constant, let u be the solution of $(\mathfrak{g}_n)$. Put $w(x) = \varrho^2(x)\left(L_{-h}(x)\, u_{-h}(x) - u(x)\right)$ into (3.7) for $h \in B_{\delta/4}(0) \subset R^2$. Into the shifted inequality (3.7) (in the direction h, $h \in B_{\delta/4}(0) \subset R^2$) we put $w_{-h}(x) = \varrho^2_{-h}(x)\left((L_h)_{-h}(x)\, u(x) - u_{-h}(x)\right)$, where $(L_h)_{-h}(x) = L_h(x + h)$. Using also the equality

$$a(\psi p, q) = a(p, \psi q) + \frac{1}{2}\int_{B^+(\delta,r)} a_{ijkl}\left(\frac{\partial\psi}{\partial x_j}\, p_i + \frac{\partial\psi}{\partial x_i}\, p_j\right) e_{kl}(q) - a_{ijkl} e_{ij}(p)\left(\frac{\partial\psi}{\partial x_l}\, q_k + \frac{\partial\psi}{\partial x_k}\, q_l\right) \mathrm{d}x\,,$$

which is valid for arbitrary $p, q \in \mathscr{H}(B^+(\delta, r))$ and $\psi \in C^1(B(\delta, r))$ with the support in $B(\delta, r)$, and using the corresponding equality for the form b, we obtain

$$(3.9) \qquad a(\varrho_{-h}u_{-h} - \varrho u,\, \varrho_{-h}u_{-h} - \varrho u) + b(\varrho_{-h}u_{-h} - \varrho u,\, \varrho_{-h}u_{-h} - \varrho u) \leqq$$
$$\leqq \int_{B^+(\delta,r)} a_{ijkl}\frac{1}{2}\left[\left(\left(\frac{\partial\varrho_{-h}}{\partial x_j} - \frac{\partial\varrho}{\partial x_j}\right) u_i + \left(\frac{\partial\varrho_{-h}}{\partial x_i} - \frac{\partial\varrho}{\partial x_i}\right) u_j\right) + \right.$$

$$+\left(\frac{\partial \varrho_{-h}}{\partial x_j}((u_i)_{-h}-u_i)+\frac{\partial \varrho_{-h}}{\partial x_l}((u_j)_{-h}-u_j)\right)\Bigg]\, e_{kl}(\varrho_{-h}u_{-h}-\varrho u)\,\mathrm{d}x\,+$$

$$+\int_{B^+(\delta,r)} a_{ijkl}\,\tfrac{1}{2}e_{ij}(u_{-h})\left[\frac{\partial \varrho_{-h}}{\partial x_k}(\varrho u_l-(\varrho u_l)_{-h})\,+\right.$$

$$\left.+\frac{\partial \varrho_{-h}}{\partial x_l}(\varrho u_k-(\varrho u_k)_{-h})\right]\mathrm{d}x+\int_{B^+(\delta,r)} a_{ijkl}\,\tfrac{1}{2}e_{ij}(u)\,.$$

$$\cdot\left[\frac{\partial \varrho}{\partial x_k}((\varrho u_l)_{-h}-\varrho u_l)+\frac{\partial \varrho}{\partial x_l}((\varrho u_k)_{-h}-\varrho u_k)\right]\mathrm{d}x\,+$$

$$+\int_{B^+(\delta,r)} b_{ijkl}\left[\frac{\partial \varrho_{-h}}{\partial x_j}((u_i)_{-h}-u_i)+u_i\left(\frac{\partial \varrho_{-h}}{\partial x_j}-\frac{\partial \varrho}{\partial x_j}\right)\right]\cdot$$

$$\cdot\frac{\partial}{\partial x_l}((\varrho u_k)_{-h}-\varrho u_k)\,\mathrm{d}x+\int_{B^+(\delta,r)} b_{ijkl}\left[\frac{\partial u_i}{\partial x_j}\frac{\partial \varrho}{\partial x_l}-\left(\frac{\partial u_i}{\partial x_j}\frac{\partial \varrho}{\partial x_l}\right)_{-h}\right]\cdot$$

$$\cdot(\varrho_{-h}(u_k)_{-h}-\varrho u_k)\,\mathrm{d}x+a(u,\varrho(\varrho-\varrho_{-h})\,u_{-h})+b(u,\varrho(\varrho-\varrho_{-h})\,u_{-h})\,+$$

$$+\,a(u_{-h},\varrho_{-h}(\varrho_{-h}-\varrho)\,u)+b(u_{-h},\varrho_{-h}(\varrho_{-h}-\varrho)\,u)\,+$$

$$+\,a(u,\varrho^2(L_{-h}u_{-h}-u_{-h}))+b(u,\varrho^2(L_{-h}u_{-h}-u_{-h}))\,+$$

$$+\,(a_{-h}-a)\,(u_{-h},\varrho^2_{-h}(u-u_{-h}))+(b_{-h}-b)\,(u_{-h},\varrho^2_{-h}(u-u_{-h}))\,+$$

$$+\,a_{-h}(u_{-h},\varrho^2_{-h}((L_h)_{-h}\,u-u))+b_{-h}(u_{-h},\varrho^2_{-h}((L_h)_{-h}\,u-u))\,+$$

$$+\,(f,\varrho^2(u_{-h}-L_{-h}u_{-h}))+(f_{-h}(\varrho^2_{-h}-\varrho^2),\,u_{-h}-u)\,+$$

$$+\,(f_{-h},\varrho^2_{-h}(u-(L_h)_{-h}\,u))+(f_{-h}-f,\varrho^2(u_{-h}-u))\,+$$

$$+\,\langle\mathscr{F}|g_n|\,J,\varrho^2|(L_{-h}u_{-h})_t-(u_t)_{-h}|\rangle\,+$$

$$+\,\langle\mathscr{F}_{-h}|g_n|_{-h}\,J_{-h},\varrho^2_{-h}(|((L_h)_{-h}\,u)_{(t)-h}-u_t|)\rangle\,+$$

$$+\,\langle\mathscr{F}_{-h}|g_n|_{-h}\,J_{-h},|\varrho^2_{-h}u_t+(1-\varrho^2_{-h})\,(u_t)_{-h}|-|u_t|_{-h}\rangle\,+$$

$$+\,\langle\mathscr{F}|g_n|\,J,|\varrho^2(u_t)_{-h}+(1-\varrho^2)\,u_t|-|u_t|\rangle\,.$$

For the last two terms in (3.9) we have the following estimate:

$$(3.10)\quad \langle\mathscr{F}_{-h}|g_n|_{-h}\,J_{-h},\varrho^2_{-h}(|u_t|-|u_t|_{-h})\rangle+\langle\mathscr{F}|g_n|\,J,\varrho^2(|u_t|_{-h}-|u_t|)\rangle\leqq$$

$$\leqq\langle(\mathscr{F}g_nJ\varrho)_{-h}-\mathscr{F}g_nJ\varrho,|\varrho u_t|_{-h}-|\varrho u_t|\rangle\,+$$

$$+\,\langle(\mathscr{F}|g_n|\,J\varrho)_{-h},|\varrho_{-h}-\varrho|\,|u|\rangle+\langle\mathscr{F}|g_n|\,J\varrho,|\varrho_{-h}-\varrho|\,|u_{-h}|\rangle\,.$$

Let us sum up (3.9) and (3.10). We multiply the obtained inequality by $|h|^{-2-2\alpha}$ and integrate in h over $B_{\delta/4}(0)\subset R^2$. The most important term for the estimation is the following one (see the corresponding part of Sec. 2):

$$(3.11)\qquad \int_{B_{\delta/4}(0)} |h|^{-2-2\alpha} \langle (\mathscr{F}G)_{-h} - \mathscr{F}G,\ |\varrho u_t|_{-h} - |\varrho u_t| \rangle \, \mathrm{d}h\,,$$

where $G = \varrho g_n J$. The other terms will be estimated in the following way: the terms containing the expressions

$$a_{-h} - a,\ b_{-h} - b,\ \varrho_{-h} - \varrho \quad \text{or} \quad \frac{\partial \varrho_{-h}}{\partial x_i} - \frac{\partial \varrho}{\partial x_i}\,, \quad i = 1, 2, 3\,,$$

can be estimated by using (1.14), due to the Lipschitz continuity of the above mentioned expressions. Using Lemma 3.2, we estimate the terms in which $(L_h)_{-h}\, u - u$ or $L_{-h} u_{-h} - u_{-h}$ occur. The terms containing $(u_{-h} - u)\,|h|^{-2-2\alpha}$ in product with some X can be extimated by

$$k \left[\int_{\substack{(B^+(\delta,r))^2 \\ (x-y)\in R^2\times 0}} |x - y|^{\eta-4}\, |u(x) - u(y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right]^{1/2} \left[\int_{B_{\delta/4}(0)} |h|^{-4\alpha-\eta} \int_{B^+(\delta,r)} X^2 \, \mathrm{d}x \, \mathrm{d}h \right]^{1/2}$$

and we estimate the first factor by $\|u\|_{\mathscr{H}(\Omega)}$, the second one using the supposition $\eta \in (0, 2 - 4\alpha)$ and the respective form of X. The terms in which g_n or $(g_n)_{-h}$ appear can be estimated by applying the Cauchy inequality, Lemma 3.1 and the traces theorem for $\boldsymbol{H}^1(\Omega)$; some special terms of that type require also the use of the supposed continuous differentiability of the function n. Denoting $U = \varrho u$, we finally obtain for every $\varepsilon > 0$

$$(3.12)\quad \int_{B_{\delta/4}(0)} |h|^{-2-2\alpha}\, [a(U_{-h} - U,\ U_{-h} - U) + b(U_{-h} - U,\ U_{-h} - U)] \, \mathrm{d}h \leqq$$

$$\leqq (1 + \varepsilon) \int_{B_{\delta/4}(0)} |h|^{-2-2\alpha} \langle \mathscr{F}_{-h} G_{-h} - \mathscr{F}G,\ |U_t|_{-h} - |U_t| \rangle \, \mathrm{d}h +$$

$$+ k(\varepsilon, \alpha)\, K(\varrho, \delta, A_0, K_0, \mathscr{F}, u^0, f, T^0)\,,$$

where the constant K depends on $\delta, A_0, K_0, \|\varrho\|_{C^2(R^3)}, \|\mathscr{F}\|_{C^1(\Gamma_c)}, \|u^0\|_{\mathscr{H}}, \|f\|_0$, $\|T^0\|_{-1/2,\Gamma}$, the norm of $(\boldsymbol{H}^{-1/2}(\Gamma))^3$. According to Lemma 3.1 and (1.14) we have

$$(3.13)\quad \int_{R^2} |h|^{-2-2\alpha} \langle \mathscr{F}_{-h} G_{-h} - \mathscr{F}G,\ |U_t|_{-h} - |U_t| \rangle \, \mathrm{d}h \leqq (1 + \varepsilon)\, c(\alpha)\,.$$

$$.\left(\frac{c(\frac{1}{2} - \alpha)}{c(\frac{1}{2} + \alpha)} \right)^{1/2} \|\mathscr{F}\|_\infty \, \|G\|_{-1/2+\alpha,R^2} \, \|U_t\|_{1/2+\alpha,R^2} + k(\varepsilon, \alpha)\,.$$

$$.\, K(\varrho, \delta, A_0, K_0, \mathscr{F}, u^0, f, T^0)\,.$$

Let us extend the coefficients $a_{ijkl}(x)$ and $b_{ijkl}(x)$, $i, j, k, l = 1, 2, 3$ defined on $B^+(\delta, r)$ onto the strip Θ in the following way: for $x \in R^3$, $x = [\varrho_0 \cos v, \varrho_0 \sin v, z]$

define

$$(3.14)\qquad P_\delta(x) = \left\langle \begin{array}{l} x \quad \text{for} \quad 0 \leqq \varrho_0 \leqq \delta\,, \\ [\delta \cos v,\ \delta \sin v,\ z] \quad \text{for} \quad \varrho_0 \geqq \delta \end{array}\right.$$

and put $a_{ijkl}(x) \equiv a_{ijkl}(P_\delta(x))$, $b_{ijkl}(x) \equiv b_{ijkl}(P_\delta(x))$ on Θ, $i, j, k, l = 1, 2, 3$. Putting $M = \{|\xi|^{-2}\, a_{ijkl}(x)\, \xi_{ij}\xi_{kl};\ x \in \Theta,\ \xi \in R^9\}$, we have

$$(3.15)\qquad a_0 \leqq \inf M \leqq \sup M \leqq A_0$$

and for b we have $b_{ijkl}(x)\, \xi_{ij}\xi_{kl} \leqq \delta \operatorname{const.} |\xi|^2$ for every $x \in \Theta$, $\xi \in R^9$. The coefficients of the extended a, b are Lipschitz continuous with the same constant of continuity as the original ones. Also $\partial\varphi/\partial x_i$, $i = 1, 2$, defined on $B_\delta(0) \times \{0\}$ can be extended by means of $P_\delta(x)$ from (3.14) and in that way we can extend the unit outer normal vector n onto the strip Θ so that for an arbitrary $x = |y, z| \in \Theta$ we define $n([y, z]) \equiv n([y, 0]) \equiv n(P_\delta([y, 0]))$.

Putting $U_t = [U_1, U_2, 0] + \mathring{U}$, where

$$(3.16)\qquad \mathring{U} = \left[1 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2\right]^{-1} \left[\left(U_1 \frac{\partial\varphi}{\partial x_1} + U_2 \frac{\partial\varphi}{\partial x_2}\right) \cdot \right.$$
$$\left. \cdot \left[-\frac{\partial\varphi}{\partial x_1}, -\frac{\partial\varphi}{\partial x_2}, 1\right] + U_3 \left[\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2\right]\right],$$

we get

$$(3.17)\qquad \|U_t\|_{1/2+\alpha,R^2} \leqq \|U\|_{1/2+\alpha,R^2}(1 + \varepsilon(\delta)) + \|U\|_{0,R^2}\, k(\delta)\,,$$

where $\varepsilon(\delta) \to 0$ for $\delta \to 0$, $\|\cdot\|_{0,R^2}$ is the norm of $(L_2(R^2))^3$. (3.17) is a consequence of the estimate

$$\|\mathring{U}\|_{1/2+\alpha,R^2} \leqq k \sum_{i=1}^{2} \left[\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{\infty,R^2} \|U\|_{1/2+\alpha,R^2} + \left\|\frac{\partial\varphi}{\partial x_i}\right\|_{1,\infty,R^2} \|U\|_{0,R^2}\right],$$

where $\|\cdot\|_{1,\infty,R^2}$ is the norm of $W^{1,\infty}(R^2)$.

Using Proposition 2.3 (the modification for the case $\Gamma_c = R^2 \times \{0\}$, $\Gamma_u = R^2 \times \times \{r\}$ cannot change the constants in (2.10)), we convert the inequality (3.13) into the form

$$(3.18)\qquad \int_{R^2} |h|^{-2-2\alpha} \langle \mathscr{F}_{-h} G_{-h} - \mathscr{F} G,\ |U_t|_{-h} - |U_t| \rangle \,\mathrm{d}h \leqq (1 + \varepsilon(\delta) + \varepsilon)\,.$$
$$\cdot \left(\frac{2}{a_0}\, c(\alpha)\, c(\tfrac{1}{2} - \alpha)\right)^{1/2} \|\mathscr{F}\|_\infty \|G\|_{-1/2+\alpha,R^2} \left(\int_{R^2} |h|^{-2-2\alpha} a(U_{-h} - U, U_{-h} - U)\,\mathrm{d}h\right)^{1/2} +$$
$$+ k(\varepsilon, \alpha)\, K(\varrho, \delta, A_0, K_0, \mathscr{F}, u^0, f, T^0)\,.$$

Using (3.12) and a procedure analogous to the proof of Lemma 2.2 we obtain the inequality

$$(3.19)\qquad \left[\int_{R^2} |h|^{-2-2\alpha} a(U_{-h} - U,\ U_{-h} - U) + b(U_{-h} - U,\ U_{-h} - U)\,\mathrm{d}h\right]^{1/2} \leqq$$

$$\leqq (1 + \varepsilon + \varepsilon(\delta)) \left(\frac{2}{a_0} c(\alpha)\, c(\tfrac{1}{2} - \alpha) \right)^{1/2} \|\mathscr{F}\|_\infty \|G\|_{-1/2+\alpha, R^2} +$$

$$+ k(\varepsilon, \alpha) K(\varrho, \delta, A_0, K_0, u^0, \mathscr{F}, T^0),$$

corresponding to (2.15). On Ω we have

$$\text{(3.20)} \qquad - \frac{\partial \varrho\, \tau_{kl}(u)}{\partial x_l} = \varrho f_k - \tau_{kl}(u) \frac{\partial \varrho}{\partial x_l}, \quad k = 1, 2, 3.$$

So it is possible to define $\varrho T_n(u) \in \boldsymbol{H}^{-1/2}(\Gamma_c)$ by the formula

$$\text{(3.21)} \qquad \langle \varrho T_n(u), w_n \rangle_{\Gamma_c} = a(u\varrho, w) - (\varrho f, w) +$$

$$+ \int_{\Psi^{-1}(B^+(\delta, r))} a_{ijkl} \left[e_{ij}(u) \frac{1}{2} \left(w_l \frac{\partial \varrho}{\partial x_k} + w_k \frac{\partial \varrho}{\partial x_l} \right) - \frac{1}{2} \left(u_i \frac{\partial \varrho}{\partial x_j} + u_j \frac{\partial \varrho}{\partial x_i} \right) e_{kl}(w) \right] dx$$

for every $w \in \mathscr{H}(\Omega)$ such that $w_t = 0$ on Γ_c, $w = 0$ on $\overline{\Gamma \setminus \Gamma_c}$. Naturally, we can restrict ourselves to $w \in \mathscr{H}(\Omega)$ such that $\operatorname{supp} w \subset \Psi^{-1}(B^+(\delta, r))$ and the above mentioned boundary value conditions are fulfilled. Using the transformation Ψ we obtain

$$\text{(3.22)} \qquad \langle \varrho T_n(u)\, J, w_n \rangle_{B_\delta(0)} = a(\varrho u, w) + b(\varrho u, w) - (\varrho f, w) +$$

$$+ \int_{B^+(\delta, r)} \tfrac{1}{2} a_{ijkl} \left[e_{ij}(u) \left(\frac{\partial \varrho}{\partial x_k} w_l + \frac{\partial \varrho}{\partial x_l} w_k \right) - \left(\frac{\partial \varrho}{\partial x_i} u_j + \frac{\partial \varrho}{\partial x_j} u_i \right) e_{kl}(w) \right] +$$

$$+ b_{ijkl} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial \varrho}{\partial x_l} w_k - u_i \frac{\partial \varrho}{\partial x_j} \frac{\partial w_k}{\partial x_l} \right) dx$$

for an arbitrary $w \in \mathscr{H}(B^+(\delta, r))$ with $w_t = 0$ on $B_\delta(0) \times \{0\}$, $w \equiv 0$ on $\partial(B^+(\delta, r)) \setminus (B_\delta(0) \times \{0\})$. Extending all the appropriate terms in (3.22) like e.g.

$$U = \varrho u, \varrho f, \tfrac{1}{2} a_{ijkl}\, e_{ij}(u) \frac{\partial \varrho}{\partial x_k}, \quad b_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial \varrho}{\partial x_k}, \quad l = 1, 2, 3,$$

by 0 onto the strip Θ, we obtain the natural extension of $\varrho\, T_n(u)$ to the form from $\boldsymbol{H}^{-1/2}(R^2)$ by means of (3.22), where, of course, a and b are defined by the corresponding integrals over the whole strip Θ and the equality must hold for every $w \in \mathscr{H}(\Theta)$ such that $w \equiv 0$ on $R^2 \times \{r\}$, $w_t \equiv 0$ on $R^2 \times \{0\}$. Putting $\mathscr{S} = \varrho\, T_n(u)\, J$ and using the procedure described in Proposition 2.4 (the only modification is that the prescribed w_n on $R^2 \times \{0\}$ are extended to $v \in \boldsymbol{H}^1(\Theta)$ harmonically so that $v|_{R^2 \times \{r\}} \equiv 0$, we put $w = v \cdot n$ and we have the estimate $(a(w, w))^{1/2} \leqq (1 + \varepsilon(\delta))\,.\, A_0^{1/2} \|w_n\|_{1/2, R^2} (c(\tfrac{1}{2}))^{-1/2} + k(\delta) \|w_n\|_{-1/2, R^2}$). Finally we obtain

$$\text{(3.23)} \qquad \|\mathscr{S}\|_{-1/2+\sigma, R^2} \leqq (1 + \varepsilon(\delta) + \varepsilon)\, A_0^{1/2} (c(\alpha)\, c(\tfrac{1}{2} - \alpha))^{-1/2} \,.$$

$$. \left(\int_{R^2} |h|^{-2-2\alpha} [a(U_{-h} - U, U_{-h} - U) + b(U_{-h} - U, U_{-h} - U)]\, dh \right)^{1/2} +$$

$$+ k(\varepsilon, \alpha) K(\varrho, \delta, A_0, K_0, \mathscr{F}, u^0, f, T^0).$$

Putting $K(\delta, A_0, K_0, \mathscr{F}, u^0, f, T^0, \alpha, \varepsilon) = \max_{V \in \mathfrak{U}_\delta} K(\varrho_V, \delta, A_0, K_0, \mathscr{F}, u^0, f, T^0)\ k(\varepsilon, \alpha)$ and considering (3.15) and (3.19), we have

$$(3.24) \qquad \|\mathscr{S}\|_{-1/2+\alpha,R^2} \leqq (1 + \varepsilon(\delta) + \varepsilon)\ \|\mathscr{F}\|_\infty \left(\frac{2A_0}{a_0}\right)^{1/2} \|G\|_{-1/2+\alpha,R^2} + \\ + K(\delta, A_0, K_0, \mathscr{F}, u^0, f, T^0, \alpha, \varepsilon)\,.$$

Let us define $S_n(u) = \tilde{s}\ T_n'(u)$, where $\tilde{s}$ was introduce in (3.2). Clearly for $V \in \mathfrak{B}_\delta \setminus \mathfrak{U}_\delta$ such that $V \cap \operatorname{supp} \tilde{s} \neq \emptyset$ and for $\mathscr{S} = ((\varrho_V S_n(u)) \circ \Psi_V^{-1})\ J_V$ we have

$$(3.25) \qquad \|\mathscr{S}\|_{-1/2+\alpha,R^2} \leqq K'(\varrho_V, u^0, f, T^0, \delta) \leqq \max_{V \in \mathfrak{B}_\delta \setminus \mathfrak{U}_\delta} K'(\varrho_V, u^0, f, T^0, \delta) \equiv \\ \equiv K'(\delta, u^0, f, T^0)\,,$$

because of the fact that $\operatorname{supp} \mathscr{F} \cap \operatorname{supp} \varrho_V = \emptyset$. Let us define on $H^{-1/2+\alpha}(\Gamma_c)$ the equivalent norms ${}_\delta|||\cdot|||_{-1/2+\alpha,\Gamma_c}$ according to Lemma 3.1 in the following way

$$(3.26) \qquad {}_\delta|||\tilde{s}\vartheta|||_{-1/2+\alpha,\Gamma_c} = \Big(\sum_{V \in \mathfrak{B}_\delta} \|((\tilde{s}\varrho_V\vartheta) \circ \Psi_V^{-1})\ J_V\|^2_{-1/2+\alpha,R^2}\Big)^{1/2}\,.$$

Due to (3.24) and (3.25) we have

$$(3.27) \quad {}_\delta|||S_n(u)|||_{-1/2+\alpha,\Gamma_c} \leqq (1 + \varepsilon(\delta) + \varepsilon)\ \|\mathscr{F}\|_\infty \left(\frac{2A_0}{a_0}\right)^{1/2} {}_\delta|||\tilde{s}g_n|||_{-1/2+\alpha,\Gamma_c} + \\ + K(\delta, A_0, K_0, \mathscr{F}, u^0, f, T^0, \varepsilon, \alpha)\,.$$

Using the considerations analogous to those in the proof of Theorem 2.2 and applying Theorem 2.1 we obtain (considering also $\tilde{s} \equiv 1$ on $\operatorname{supp} \mathscr{F}$) the following theorem.

Theorem 3.1. *Let the domain Ω have the boundary Γ of the class $C^{0,1}$ consisting of $\Gamma_u, \Gamma_c, \Gamma_T$, which satisfy the corresponding suppositions of Sec. 1, (3.1) and (3.2). Let for $i, j, k, l = 1, 2, 3$, $a_{ijkl}(x)$ be Lipschitz continuous and satisfy (1.9) on Ω. Let $u^0 \in \mathscr{H} \equiv \mathscr{H}(\Omega)$ such that $u^0|_{\overline{\Gamma \setminus \Gamma_u}} = 0$. Let $f \in (L_2(\Omega))^3$, let $T^0 \in (H^{-1/2}(\Gamma))^3$ such that $[T^0, w] = 0$ for every $w \in (H^{1/2}(\Gamma))^3$ such that $w|_{\overline{\Gamma \setminus \Gamma_T}} = 0$. Let $\mathscr{F} \in C^1(\Gamma_c)$ have a compact support in* Int Γ_c, *let $\|\mathscr{F}\|_\infty < \sqrt{(a_0/2A_0)}$. Then there exists a solution of the Signorini problem with friction.*

Remark. If Ω is a homogeneous isotropic body (see (2.13)), then it is possible to use the improvement of the estimate (3.27) analogously to Remark 2 to Theorem 2.2. The constants λ, μ of the form a have the same value after the rotation, shift and "straightening" of the coordinate system as at the beginning. Using the same procedure as in the corresponding part of this section we derive (3.22) and the estimate (3.19) with $\mu^{-1/2}$ instead of $\sqrt{(2/a_0)}$. The estimate of the term $(a(w, w))^{1/2}$ which is important for (3.23) will be established by mans of the procedure from the just mentioned remark to Theorem 2.2, hence

$$(a(w, w))^{1/2} \leqq (c(\tfrac{1}{2}))^{-1/2}\ (\mu(\lambda + 2\mu))^{1/4}\ \|w_n\|_{1/2,R^2}(1 + \varepsilon(\delta) + \varepsilon) + \\ + k(\varepsilon, \delta)\ \|w_n\|_{-1/2,R^2}\,.$$

So we have $(\mu(\lambda + 2\mu))^{1/4}$ instead of $A_0^{1/2}$ in (3.23).

Corollary of Theorem 3.1. *Let Ω be homogeneous isotropic. Let all the suppositions of Theorem* 3.1 *be fulfilled with the exception of the estimate for $\|\mathscr{F}\|_\infty$. Let*

$$\|\mathscr{F}\|_\infty < \sqrt[4]{\left(\frac{\mu}{\lambda + 2\mu}\right)} = \sqrt[4]{\left(\frac{1 - 2\sigma}{2 - 2\sigma}\right)},$$

where λ, μ are the Lamé constants, σ is the Poisson ratio. Then there exists at least one solution of the Signorini problem with friction.

Remark. For steel we have $\sigma = 0{\cdot}30$, i.e. the existence of the solution is ensured for $\|\mathscr{F}\|_\infty \leqq 0{\cdot}73$, for aluminium ($\sigma = 0{\cdot}34$) it suffices to take $\|\mathscr{F}\|_\infty \leqq 0{\cdot}70$. By the book by R. Stephenson [9] the majority of materials have $\sigma \sim 0{\cdot}3$. Greater σ occurs e.g. by plumbine ($\sigma = 0{\cdot}45$, $\|\mathscr{F}\|_\infty \leqq 0{\cdot}638$) and by india-rubber ($\sigma = 0{\cdot}49$, $\|\mathscr{F}\|_\infty \leqq \leqq 0{\cdot}37$). Clearly, the results for the majority of materials are fully in harmony with practical requests. Unsatisfactory results for the case of india-rubber are probably caused by big deformations of this material, which make it impossible to use the model with a small strain tensor (e_{ij}).

4. THE GENERAL CONTACT PROBLEM WITH FRICTION

In this part we shall study the contact problem in the following classical formulation. Let us have two bodies $\Omega_1 \subset R^3$, $\Omega_2 \subset R^3$. For the displacements $u^\iota : \Omega_\iota \to R^3$, $\iota = 1, 2$, let the strain tensors be given by (1.1), let on Ω_ι the Hook law

$$\tau^\iota_{ij} = a^\iota_{ijkl}\, e_{kl}(u^\iota)\,, \quad i, j = 1, 2, 3\,, \quad \iota = 1, 2 \tag{4.1}$$

be fulfilled, where a^ι_{ijkl} fulfil the obvious symmetric conditions on Ω_ι, $\iota = 1, 2$. Let for each $\iota \in \{1, 2\}$ the equilibrium conditions of the type (1.3) be satisfied on Ω_ι with the corresponding τ^ι_{ij}, f_ι. For $\iota = 1, 2$ we suppose $\Gamma^\iota \equiv \partial\Omega_\iota$ consisting of 3 parts like in Sec. 1 so that on Γ^ι_u the condition of the type (1.4) with u^0_ι is prescribed, on Γ^ι_T the condition of the type (1.5) with T^0_ι is given. On $\Gamma_c = \bar{\Omega}_1 \cap \bar{\Omega}_2$ (the common contact part of Γ^1 and Γ^2) we suppose that the following conditions are fulfilled:

$$u^1_n - u^2_n \leqq 0\,, \quad T_n = T^1_n = -T^2_n \leqq 0\,, \quad T_n(u^1_n - u^2_n) = 0 \tag{4.2}$$

(contact conditions of the Signorini type),

$$T^1_t = -T^2_t = T_t\,, \quad |T_t| \leqq \mathscr{F}|T_n|\,, \quad (|T_t| - \mathscr{F}|T_n|)\,|u^1_t - u^2_t| = 0\,, \tag{4.3}$$

$$u^1_t - u^2_t = \lambda T_t \text{ on } \{x \in \Gamma_c;\, \mathscr{F}T_n(x) < 0\} \text{ (The Coulomb law of friction)},$$

where n is the unit outer normal vector to Γ_c with respect to Ω_1, the normal components of u^ι, T^ι are considered with respect to this vector, λ is a suitable non-positive function on Γ_c. This formulation can be found in [4].

Let Ω_ι be disjoint domains in R^3 with the $C^{0,1}$-smooth boundary, $\iota = 1, 2$. The suppositions for Γ_c, Γ_T^ι, Γ_n^ι, $\iota = 1, 2$, are the same as in Sec. 1, particularly, mes $\Gamma_u^\iota > 0$, $\iota = 1, 2$. Moreover, let Γ_c be of the class $C^{2,1}$ fulfilling (3.1) and (3.2) with respect to Ω_1 such that the set $\{[x, y] \in R^3,\ y \in \langle \varphi_V(x) - r,\ \varphi_V(x)) \} \subset \Omega_2 \subset R^3 \setminus \Omega_1$ for every $V \in \mathfrak{U}_\delta$ and for every $V \in \mathfrak{V}_\delta$ such that $V \cap \operatorname{supp} \tilde{s} \neq \emptyset$, $\delta \in (0, \Delta_0\rangle$. Let $\Omega = \Omega_1 \cup \Omega_2$, $\mathscr{H}'(\Omega) = \mathscr{H}(\Omega_1) \times \mathscr{H}(\Omega_2)$, where $\mathscr{H}(\Omega_\iota) = (\boldsymbol{H}^1(\Omega_\iota))^3$, $\iota = 1, 2$. Let $\mathscr{K} = \{v = [v^1, v^2] \in \mathscr{H}'(\Omega);\ v_\iota^0 = u_\iota^0 \text{ on } \Gamma_u^\iota,\ \iota = 1, 2,\ v_n^1 - v_n^2 \leqq 0 \text{ on } \Gamma_c\}$, where $u_\iota^0 \in \mathscr{H}(\Omega_\iota)$ is such that $u_\iota^0 \equiv 0$ on $\overline{\Gamma^\iota \setminus \Gamma_u^\iota}$, $\iota = 1, 2$. Let $\mathscr{F} \in C^1(\Gamma_c)$ with a compact support fulfilling (3.1), (3.2), all a_{ijkl}^ι are supposed to be Lipschitz continuous on Ω_ι and satisfy (1.9) with the corresponding $a_{0,\iota} A_{0,\iota}$ on Ω_ι, $\iota = 1, 2$. Denote $a(u, v) = a^1(u^1, v^1) + a^2(u^2, v^2)$, where $a^\iota(u^\iota, v^\iota)$ is given as $a(u, v)$ in Sec. 1, $\iota = 1, 2$. Let $L(v) = \int_{\Omega_1} (f_1 v^1)\, dx + \int_{\Omega_2} f_2 v^2\, dx + \int_{\Gamma_T^1} T_1^0 v^1\, ds + \int_{\Gamma_T^2} T_2^0 v^2\, ds$, where $f_\iota \in (\boldsymbol{L}_2(\Omega_\iota))^3$, $\iota = 1, 2$, T_ι^0 satisfies the corresponding condition of Sec. 1 for $\iota = 1, 2$. For $g_n \in \boldsymbol{C}^{*-}$ defined in Sec. 1 we introduce the problem $\langle g_n \rangle$: Let us look for $u \in \mathscr{K}$ such that for every $v \in \mathscr{K}$,

$$(4.4)\qquad a(u, v - u) + \int_{\Gamma_c} \mathscr{F} |g_n| \left(|v_t^1 - v_t^2| - |u_t^1 - u_t^2| \right) ds \geqq L(v - u)\,.$$

In accordance with the preceding, a solution u of $\langle g_n \rangle$ for arbitrary $g_n \in \boldsymbol{C}^{*-}$ will be called the solution of the contact problem with friction iff $\mathscr{F} g_n = \mathscr{F} T_n(u)$, where $T_n(u)|_{\Gamma_c} = T_n^1(u^1)|_{\Gamma_c} = -T_n^2(u^2)|_{\Gamma_c}$ and $T_n^\iota(u)|_{\Gamma_c}$ is defined by the formula corresponding to (1.11), $\iota = 1, 2$.

The existence, uniqueness, a priori estimates and the other properties declared in Lemma 1.1 can be proved in this case in the same way (using the Korn inequality on each Ω_ι, $\iota = 1, 2$). Our purpose is to prove some estimates analogous to (3.27) applying the methods of § 3.

Let us take a fixed $\delta \in (0, \Delta_0\rangle$ and $V \in \mathfrak{U}_\delta$ of (3.1). Using the transformation $\Psi \equiv \Psi_V$, we convert (4.4) to

$$(4.5)\qquad a(u, w) + b(u, w) + \langle \mathscr{F} |g_n|\, J\,,\ |u_t^1 - u_t^2 + w_t^1 - w_t^2| - |u_t^1 - u_t^2| \rangle \geqq$$
$$\geqq (f_1, w^1) + (f_2, w^2)\,,$$

where

$$(4.6)\quad a(u, w) = \int_{B^+(\delta, r)} a_{ijkl}^1\, e_{ij}(u^1)\, e_{kl}(w^1)\, dx + \int_{B^-(\delta, r)} a_{ijkl}^2\, e_{ij}(u^2)\, e_{kl}(w^2)\, dx\,,$$

$$b(u, w) = \sum_{\iota=1}^{2} b^\iota(u^\iota, w^\iota)\,,$$

$b^\iota(u^\iota, w^\iota)$ are defined in accordance with (3.5), $(\cdot, \cdot)$ are the corresponding duality pairings in $(\boldsymbol{L}_2(R^2 \times (0, r)))^3$ or $(\boldsymbol{L}_2(R^2 \times (-r, 0)))^3$, $\langle \cdot, \cdot \rangle$ is the duality pairing in $\boldsymbol{L}_2(R^2)$. Naturally, $B^-(\delta, r) = B_\delta(0) \times (-r, 0)$.

The inequality (4.5) holds for every $w = [w^1, w^2] \in \mathscr{H}_{\delta,r} \equiv \mathscr{H}(B^+(\delta, r)) \times \mathscr{H}(B^-(\delta, r))$ such that $\operatorname{supp} w^1 \cup \operatorname{supp} w^2 \subset B(\delta, r)$, $(w^1 . n - w^2 . n) \leqq 0$ for n

being the unit outer normal vector on Γ_c with respect to Ω_1. The integration for such w can be extended to the integration over $R^2 \times (0, r)$ or over $R^2 \times (-r, 0)$.

Using the rotation L_{-h} defined in Sec. 3, we put $w^\iota = \varrho^2(L_{-h}\, u^\iota_{-h} - u^\iota)$ in (4.5). In the corresponding shifted inequality we put $w^\iota_{-h} = \varrho^2_{-h}((L_h)_{-h}\, u^\iota - u^\iota_{-h})$. The inequality obtained by the addition is analogous to (3.9), Using the corresponding estimates based on the Lipschitz continuity of ϱ, a^ι_{ijkl}, b^ι_{ijkl}, $\partial\varrho/\partial x_i$, $\partial^2\varphi/\partial x_i^2$, $i, j, k, l = 1, 2, 3$, $\iota = 1, 2$, and on Lemmas 1.1 and 3.2, we obtain for $G = \varrho g_n J$ and $U = \varrho u$

$$(4.7)\quad \int_{B_{\delta/4}(0)} |h|^{-2-2\alpha}[a(U_{-h} - U,\, U_{-h} - U) + b(U_{-h} - U,\, U_{-h} - U)]\, \mathrm{d}h \leqq$$
$$\leqq \int_{R^2} |h|^{-2-2\alpha}\langle \mathscr{F}_{-h}G_{-h} - \mathscr{F}G,\, |U^1_t - U^2_t|_{-h} - |U^1_t - U^2_t|\rangle\, \mathrm{d}h +$$
$$+ K(\varrho, \delta, A_{0,1}, A_{0,2}, K_0, \mathscr{F}, u^0_1, u^0_2, f_1, f_2, T^0_1, T^0_2)\,.$$

By the corresponding extension of n, a^ι_{ijkl}, b^ι_{ijkl}, $\iota = 1, 2$, $i, j, k, l = 1, 2, 3$, by means of P_δ defined in (3.14) and using the expression of $U^\iota_t = [U^\iota_1, U^\iota_2, 0] + U^\iota$ and Proposition 2.3, we finally obtain

$$(4.8)$$
$$\sum_{\iota=1,2} \int_{R^2} |h|^{-2-2\alpha}[a^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota) + b^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota)]\, \mathrm{d}h \leqq$$
$$\leqq (1 + \varepsilon(\delta))\,(c(\alpha)\, c(\tfrac{1}{2} - \alpha))^{1/2}\, \|\mathscr{F}\|_\infty\, \|G\|_{-1/2+\alpha, R^2} \sum_{\iota=1,2} \left(\frac{2}{a_{0,\iota}}\right)^{1/2} \cdot$$
$$\cdot \left(\int_{R^2} [a^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota) + b^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota)]\, |h|^{2-2\alpha}\, \mathrm{d}h\right)^{1/2} +$$
$$+ K(A_{0,1}, A_{0,2}, K_0, \mathscr{F}, u^0_1, u^0_2, f_1, f_2, T^0_1, T^0_2, \varepsilon, \delta, \alpha)\,.$$

Converting (1.11) ($\iota = 1, 2$) by means of Ψ and using the corresponding considerations and calculations from Sec. 3, we obtain for $\mathscr{S} = \varrho T_n(u)\, J$ the following inequality holding for $\iota = 1, 2$:

$$(4.9)\qquad \|\mathscr{S}\|_{-1/2+\alpha, R^2} \leqq (1 + \varepsilon(\delta) + \varepsilon)\,(A_{0,\iota})^{1/2}\,(c(\alpha)\, c(\tfrac{1}{2} - \alpha))^{-1/2}\,.$$
$$\cdot \left(\int_{R^2} |h|^{-2-2\alpha}[a^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota) + b^\iota(U^\iota_{-h} - U^\iota,\, U^\iota_{-h} - U^\iota)]\, \mathrm{d}h\right)^{1/2} +$$
$$+ K(\delta, A_{0,1}, A_{0,2}, K_0, \mathscr{F}, u^0_1, u^0_2, f_1, f_2, T^0_1, T^0_2, \varepsilon, \alpha)\,.$$

Using (4.8) and (4.9) (which are independent of $V \in \mathfrak{U}_\delta$ for each $\delta \in (0, \Delta_0\rangle$) we need to calculate an estimate for $\mathscr{S}$ in the form

$$(4.10)\qquad \|\mathscr{S}\|_{-1/2+\alpha, R^2} \leqq \tilde{c}(1 + \varepsilon + \varepsilon(\delta))\, \|\mathscr{F}\|_\infty\, \|G\|_{-1/2+\alpha, R^2} +$$
$$+ K(\delta, A_{0,1}, A_{0,2}, K_0, \mathscr{F}, u^0_1, u^0_2, f_1, f_2, T^0_1, T^0_2, \varepsilon, \alpha)\,,$$

which yields the estimate $\|\mathscr{F}\|_\infty < \tilde{c}^{-1}$ for the admissible coefficient of friction. It is possible to calculate the following estimates supposing $A_{0,1} \leqq A_{0,2}$ in (4.11) and (4.12) (the converse case is symmetric):

(4.11)
$$\|\mathscr{F}\|_\infty < \sqrt{\left(\frac{a_{0,1}a_{0,2}}{2A_{0,1}A_{0,2}}\right)}\frac{A_{0,1} + A_{0,2}}{\sqrt{(a_{0,1}A_{0,1})} + \sqrt{(a_{0,2}A_{0,2})}} \quad \text{if } \frac{a_{0,2}}{a_{0,1}} \geqq \frac{(A_{0,2} - A_{0,1})^2}{4A_{0,1}A_{0,2}},$$

$$(4.12)\quad \|\mathscr{F}\|_\infty < \sqrt{\left(\frac{2a_{0,1}a_{0,2}}{A_{0,1}}\right)}\frac{1}{\sqrt{a_{0,2}} + \sqrt{(a_{0,1} + a_{0,2})}} \quad \text{if } \frac{a_{0,2}}{a_{0,1}} \leqq \frac{(A_{0,2} - A_{0,1})^2}{4A_{0,1}A_{0,2}},$$

or

$$(4.13)\qquad \|\mathscr{F}\|_\infty < \sqrt{\left(\frac{a_{0,1}a_{0,2}}{2A_{0,1}A_{0,2}}\frac{A_{0,1} + A_{0,2}}{a_{0,1} + a_{0,2}}\right)} \quad \text{for arbitrary } a_{0,1} > 0\,,$$

$$a_{0,2} > 0\,, \quad A_{0,1} > 0\,, \quad A_{0,2} > 0\,.$$

Of course, the estimate (4.13) is a little worse than (4.11) or (4.12) if the corresponding inequalities for $a_{0,\iota}$, $A_{0,\iota}$, $\iota = 1, 2$ hold.

If Ω_1 and Ω_2 are homogeneous isotropic, then we can use the estimate

$$(4.14)\qquad \|\mathscr{F}\|_\infty < \mathfrak{a}$$

according to the corresponding remarks of Secs. 2 and 3. $\mathfrak{a}$ is stated in the following table, where

$$\mathfrak{v} = \left(\frac{1 - \sigma_1}{1 - \sigma_2}\frac{1 - 2\sigma_2}{1 - 2\sigma_1}\right)^{1/2}, \quad \tilde{T} = \frac{E_2}{E_1}\frac{1 + \sigma_1}{1 + \sigma_2},$$

(4.15)

$\mathfrak{v}$	$\tilde{T}$	$\mathfrak{a}$
$\langle 4, +\infty)$	$\langle \mathfrak{v} - 2\sqrt{\mathfrak{v}}, +\infty)$	$\sqrt[4]{\left(\frac{(1 - 2\sigma_1)(1 - 2\sigma_2)}{2(1 - \sigma_1(1 - \sigma_2)}\right)} \cdot \dfrac{\frac{E_1}{1 + \sigma_1}\sqrt{\left(\frac{1 - \sigma_1}{1 - 2\sigma_1}\right)} + \frac{E_2}{1 + \sigma_2}\sqrt{\left(\frac{1 - \sigma_2}{1 - 2\sigma_2}\right)}}{\frac{E_1}{1 + \sigma_1}\sqrt[4]{\left(\frac{1 - \sigma_1}{1 - 2\sigma_1}\right)} + \frac{E_2}{1 + \sigma_2}\sqrt[4]{\left(\frac{1 - \sigma_2}{1 - 2\sigma_2}\right)}}$
$\langle \frac{1}{4}, 4\rangle$	$\langle 0, +\infty)$	
$\langle 0, \frac{1}{4}\rangle$	$\left\langle 0, \frac{\mathfrak{v}}{1 - 2\sqrt{\mathfrak{v}}}\right\rangle$	
$\langle 4, +\infty)$	$\langle 0, \mathfrak{v} - 2\sqrt{\mathfrak{v}}\rangle$	$\sqrt[4]{\left(\frac{1 - 2\sigma_2}{2 - 2\sigma_2}\right)}\dfrac{2}{1 + \sqrt{\left(1 + \frac{E_2}{E_1}\frac{1 + \sigma_1}{1 + \sigma_2}\right)}}$
$\langle 0, \frac{1}{4}\rangle$	$\left\langle \frac{\mathfrak{v}}{1 - 2\sqrt{\mathfrak{v}}}, +\infty\right)$	$\sqrt[4]{\left(\frac{1 - 2\sigma_1}{2 - 2\sigma_1}\right)}\dfrac{2}{1 + \sqrt{\left(1 + \frac{E_1}{E_2}\frac{1 + \sigma_2}{1 + \sigma_1}\right)}}$

$\langle 0, +\infty) \langle 0, +\infty)$
$$\sqrt[4]{\left(\frac{(1-2\sigma_1)(1-2\sigma_2)}{2(1-\sigma_1)(1-\sigma_2)}\right)} \cdot$$
$$\cdot \sqrt{\left(\frac{\frac{E_1}{1+\sigma_1}\sqrt{\left(\frac{1-\sigma_1}{1-2\sigma_1}\right)} + \frac{E_2}{1+\sigma_2}\sqrt{\left(\frac{1-\sigma_2}{1-2\sigma_2}\right)}}{\frac{E_1}{1+\sigma_1} + \frac{E_2}{1+\sigma_2}}\right)} .$$

Of course, the last value of $\mathfrak{a}$ in (4.15), which corresponds to the case of (4.13), is smaller than the former values of $\mathfrak{a}$ for $\mathfrak{v}$, $\tilde{T}$ belonging to the corresponding intervals. The first value of $\mathfrak{a}$ is the best and can be used in all cases, in which $\max(\sigma_1, \sigma_2) \leqq \leqq 15/31$. We have

Theorem 4.1. *Under the suppositions mentioned before the introduction of the problem* $\langle g_n \rangle$ *there exists a solution of the contact problem with friction if one of the condition* (4.11)–(4.13) *holds. Particularly, if both* Ω_1 *and* Ω_2 *are homogeneous isotropic* (*with the Poisson ratio* σ_ι, *and the Young modulus of elasticity* E_ι, $\iota = 1, 2$) *then* (4.14), *with* $\mathfrak{a}$ *from any condition of* (4.15) *for which both* $\mathfrak{v}$ *and* $\tilde{T}$ *belong to the prescribed intervals, is sufficient for the existence of a solution.*

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