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ON DETERMINATION OF A CYCLIC ORDER

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In [5] it is shown that there exists a close relation between cyclic orders and orders on a set G. The aim of this paper is the study of cyclic orders from this point of view. We show that any cyclic order is in a certain sense generated by a system of orders. Further, the so called cocyclic order is introduced and properties of this relation are studied.

1. ORDERS AND CYCLIC ORDERS

1.1. Remark. By an *ordered set* we mean a pair $(G, <)$ where G is a set and $<$ is an order on G, i.e. an irreflexive and transitive binary relation on G. If $(G, <)$ is an ordered set, then there exists the least (with respect to the set inclusion) subset H of G such that $< \subseteq H^2$. If $<$ is a linear order on this set H, then we shall call the order $<$ a *linear order in G*.

1.2. Definition. Let G be a set, C a ternary relation on G. C is called a *cyclic order* on G, iff it is:

(i) asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \overline{\in} C$,
(ii) transitive, i.e. $(x, y, z) \in C$, $(x, z, u) \in C \Rightarrow (x, y, u) \in C$,
(iii) cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$.

If G is a set and C a cyclic order on G, then the pair (G, C) is called a *cyclically ordered set*. If, moreover, card $G \geqq 3$ and C is

(iv) complete, i.e. $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$,

then C is called a *linear cyclic order* on G and (G, C) is called a *linearly cyclically ordered set* or a *cycle*. If $C = \emptyset$, then (G, C) is called a *discrete cyclically ordered set*.

1.3. Lemma. *Let $(G, <)$ be an ordered set. For any $x, y, z \in G$ put $(x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on G.*

Proof. Trivial; see also [5], 3.5.

1.4. Lemma. *Let (G, C) be a cyclically ordered set, $x \in G$. For any $y, z \in G$ put $y <_{C,x} z$ iff either $(x, y, z) \in C$ or $x = y \neq z$. Then $<_{C,x}$ is an order on G and x is the least element of $(G, <_{C,x})$.*

Proof. Trivial; see also [5], 3.1.

1.5. Lemma. *Let $(G, <)$ be an ordered set with the least element x. Then there exists a cyclic order C on G such that $< \; = \; <_{C,x}$.*

Proof. Put $C = C_<$. By 1.3, C is a cyclic order on G; it is not difficult to prove $< \; = \; <_{C,x}$ (see also [5], 3.8).

Now we can put an analogous question: Let (G, C) be a cyclically ordered set. Does there exist an order $<$ on G such that $C = C_<$? The following lemma shows that the answer is negative in general.

1.6. Lemma. *Let (G, C) be a cyclically ordered set. If there exists an order $<$ on G such that $C = C_<$, then there exists a linear extension of C on G, i.e. such a linear cyclic order D on G that $C \subseteq D$.*

Proof. According to Szpilrajn's theorem ([7]) there exists a linear extension $\prec$ of the order $<$ on G. Thus $< \; \subseteq \; \prec$ and hence $C_< \subseteq C_\prec$, i.e. $C \subseteq C_\prec$. But $C_\prec$ is evidently a linear cyclic order on G.

As there exist cyclic orders that have no linear extension ([3]), 1.6 generally implies the negative answer to the above question. Nevertheless, we shall show that any cyclic order is a union of cyclic orders, each of which is generated by an order according to 1.3.

1.7. Definition. Let G be a set, $(<_i)_{i \in I}$ an indexed family of orders on G. We call this family *harmonized* iff the following conditions hold:

(1) If $i \in I$ and $x, y, z \in G$ are such elements that $x <_i y <_i z$, then either $z \nless_j y$, $y \nless_j x$ or $y \nless_j x$, $x \nless_j z$ or $x \nless_j z$, $z \nless_j y$ for any $j \in I$.

(2) If $i, j \in I$ and $x, y, z, u \in G$ are such elements that $(x, y, z) \in C_{<_i}$, $(x, z, u) \in C_{<_j}$, then there exists $k \in I$ such that $(x, y, u) \in C_{<_k}$.

1.8. Theorem. *Let G be a set, $(<_i)_{i \in I}$ a family of orders on G. Then the following statements are equivalent:*

(A) The family $(<_i)_{i \in I}$ is harmonized.

(B) The ternary relation $C = \bigcup_{i \in I} C_{<_i}$ is a cyclic order on G.

Proof. 1. Let (A) hold. If $(x, y, z) \in C$, then there exists $i \in I$ such that $(x, y, z) \in C_{<_i}$, i.e. either $x <_i y <_i z$ or $y <_i z <_i x$ or $z <_i x <_i y$. Suppose $(z, y, x) \in C$; then there exists $j \in I$ such that $(z, y, x) \in C_{<_j}$, i.e. either $z <_j y <_j x$ or $y <_j x <_j z$ or $x <_j z <_j y$. By a simple calculation we find that this contradicts (1) of 1.7 in all cases. The relation C is thus asymmetric. Trivially, (2) of 1.7 implies that the relation

C is transitive. Finally, as all relations $C_{<_i}$ $(i \in I)$ are cyclic, the union $C = \bigcup_{i \in I} C_{<_i}$ is cyclic as well. Hence C is a cyclic order on G and (B) holds.

2. Let (B) hold. If $i \in I$, $x, y, z \in G$ are such elements that $x <_i y <_i z$, then $(x, y, z) \in C_{<_i} \subseteq C$ so that $(z, y, x) \bar{\in} C$. This means $(z, y, x) \bar{\in} C_{<_j}$ for any $j \in I$, i.e. neither $z <_j y <_j x$ nor $y <_j x <_j z$ nor $x <_j z <_j y$ holds and this implies (1) of 1.7. Further, the transitivity of C implies (2) of 1.7. Thus, the family $(<_i)_{i \in I}$ is harmonized and (A) holds.

1.9. Theorem. *Let (G, C) be a cyclically ordered set. Then there exists a family $(<_i)_{i \in I}$ of orders on G such that $C = \bigcup_{i \in I} C_{<_i}$.*

Proof. Let $\mathcal{T}$ be the set of all ordered triples $(x, y, z) \in G^3$ such that $(x, y, z) \in C$. For any $\tau = (x, y, z) \in \mathcal{T}$ define an order $<_\tau$ on $\{x, y, z\}$ by $x <_\tau y <_\tau z$. Then $(<_\tau)_{\tau \in \mathcal{T}}$ is a family of orders on G and clearly $C = \bigcup_{\tau \in \mathcal{T}} C_{<_\tau}$ holds.

Let us note that any order $<_\tau$ in the proof of 1.9 is a linear order in G. Thus, a stronger result holds:

1.10. Corollary. *Let (G, C) be a cyclically ordered set. Then there exists a family $(<_i)_{i \in I}$ of linear orders in G such that $C = \bigcup_{i \in I} C_{<_i}$.*

From 1.8 it follows that the family $(<_\tau)_{\tau \in \mathcal{T}}$ in the proof of 1.9 is harmonized; naturally it is simple to prove it directly. But we prove also

1.11. Theorem. *Let (G, C) be a cyclically ordered set. Then $C = \bigcup_{x \in G} C_{<_{C,x}}$.*

Proof. It is not difficult to prove $C_{<_{C,x}} \subseteq C$ for any $x \in G$ (see also [5], 3.9). Thus we have $\bigcup_{x \in G} C_{<_{C,x}} \subseteq C$. On the other hand, if $(x, y, z) \in C$, then $x <_{C,x} y <_{C,x} z$, which implies $(x, y, z) \in C_{<_{C,x}}$. This yields $C \subseteq \bigcup_{x \in G} C_{<_{C,x}}$ and hence $C = \bigcup_{x \in G} C_{<_{C,x}}$.

1.12. Corollary. *Let (G, C) be a cyclically ordered set. Then the family $(<_{C,x})_{x \in G}$ of orders on G is harmonized.*

2. WIDTH OF A CYCLICALLY ORDERED SET

2.1. Definition. Let (G, C) be a cyclically ordered set. We put $w(G, C) =$ $= \min \{\operatorname{card} I;$ there exists a harmonized family $(<_i)_{i \in I}$ of orders on G such that $C = \bigcup_{i \in I} C_{<_i}\}$, $W(G, C) = \min \{\operatorname{card} I;$ there exists a harmonized family $(<_i)_{i \in I}$ of linear orders in G such that $C = \bigcup_{i \in I} C_{<_i}\}$. The number $w(G, C)$ will be called the *width*, the number $W(G, C)$ the *strong width* of (G, C).

If T is a ternary relation on a set G, then we denote by T^c the cyclic hull of T, i.e. $T^c = \{(x, y, z) \in G^3$; there exists an even permutation (u, v, w) of the sequence (x, y, z) such that $(u, v, w) \in T\}$.

2.2 Example. Let $G = \{x, y, z, u, v\}$, $T = \{(x, y, z), (x, y, u), (x, y, v), (z, u, v)\}$, $C = T^c$ (Fig. 1). It is easy to see that C is a cyclic order on G; we shall show $w(G, C) = 2$, $W(G, C) = 4$.

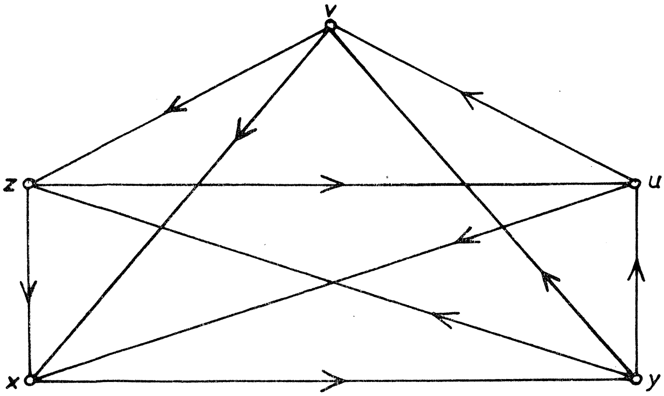

Fig. 1

First, we show that $w(G, C) > 1$. Suppose $w(G, C) = 1$, i.e. there exists an order $<$ on G such that $C = C_<$. Then $(x, y, z) \in C_<$, thus either $x < y < z$ or $y < z < x$ or $z < x < y$, and simultaneously $(x, y, u) \in C_<$, $(x, y, v) \in C_<$, $(z, u, v) \in C_<$.

Case 1. Let $x < y < z$. Then $y < u < x$ is impossible. If $u < x < y$, then $u < x < z$, thus $(u, x, z) \in C_< = C$, a contradiction. Hence we have $x < y < u$. For the same reason $x < y < v$ holds. If $z < u < v$, then $y < z < u$ and $(y, z, u) \in C_< = C$; if $u < v < z$, then $y < u < v$ and $(y, u, v) \in C$; if $v < z < u$, then $y < v < z$ and $(y, v, z) \in C$. Thus the case $x < y < z$ is impossible.

Case 2. Let $y < z < x$. Then $x < y < u$, $u < x < y$ are impossible, hence $y < u < x$. Analogously $y < v < x$ holds. If $z < u < v$, then $y < z < u$ and $(y, z, u) \in C$; if $u < v < z$, then $y < u < z$ and $(y, u, z) \in C$; if $v < z < u$, then $y < z < u$ and $(y, z, u) \in C$. Thus also the case $y < z < x$ is impossible.

Case 3. Let $z < x < y$. Analogously as in Case 1, we find that $u < x < y$, $v < x < y$ hold and any of the possibilities $z < u < v$, $u < v < z$, $v < z < u$ leads to a contradiction. Thus we have shown $w(G, C) > 1$. Now put $<_1 = \{(x, y), (x, z), (y, z), (x, u), (y, u), (x, v), (y, v)\}$, $<_2 = \{(z, u), (z, v), (u, v)\}$ (Fig. 2). We easily see that $C_{<_1} \cup C_{<_2} = C$. Thus $w(G, C) = 2$.

Further, put $G_1 = \{x, y, z\}$, $G_2 = \{x, y, u\}$, $G_3 = \{x, y, v\}$, $G_4 = \{z, u, v\}$ and let us define a linear order $<_i$ on G_i $(i = 1, 2, 3, 4)$ as follows: $x <_1 y <_1 z$, $x <_2$

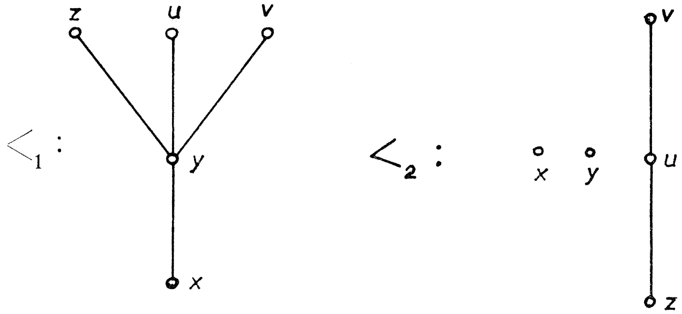

Fig. 2

$<_2 y <_2 u$, $x <_3 y <_3 v$, $z <_4 u <_4 v$. Each $<_i$ is a linear order in G and clearly $C = \bigcup_{i=1}^{4} C_{<_i}$. This implies $W(G, C) \leqq 4$. On the other hand, let $(<_i)_{i \in I}$ be a family of linear orders in G such that $C = \bigcup_{i \in I} C_{<_i}$ and let $i \in I$ be such an element that $(x, y, z) \in C_{<_i}$. Then $<_i$ is a linear order on $H \supseteq G_1$; if $H \neq G_1$, then either $u \in H$ or $v \in H$. In the first case we have either $(y, z, u) \in C_{<_i}$ or $(u, z, y) \in C_{<_i}$, which is a contradiction, for $(y, z, u) \notin C$, $(u, z, y) \notin C$; in the second, either $(y, z, v) \in C_{<_i}$ or $(v, z, y) \in C_{<_i}$, a contradiction. Thus $H = G_1$. For the same reason there exist $j \in I, j \neq i$, $k \in I$, $i \neq k \neq j$, $l \in I$, $l \notin \{i, j, k\}$ such that $<_j$ is a linear order on G_2, $<_k$ a linear order on G_3, $<_l$ a linear order on G_4. Thus card $I \geqq 4$, $W(G, C) \geqq 4$ and we have $W(G, C) = 4$.

The definition directly yields

2.3. Lemma. *Let (G, C) be a cyclically ordered set. Then*
(1) $w(G, C) \leqq W(G, C)$,
(2) $w(G, C) = 1$ *iff there exists an order $<$ on G such that $C = C_<$.*

In [5], the following notion was introduced (3.12): A cyclically ordered set (G, C) is called *x-stable* (where $x \in G$) iff the following condition is satisfied: $y, z \in G - \{x\}$, $(u, y, z) \in C$ for some $u \in G \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$. Further, it is proved that (3.15) if (G, C) is a cyclically ordered set which is x-stable for some $x \in G$, then $C = C_{<_{C,x}}$. As a consequence, we obtain

2.4. Corollary. *Let (G, C) be a cyclically ordered set. If there exists $x \in G$ such that (G, C) is x-stable, then $w(G, C) = 1$.*

Let us recall the definition of the direct sum of cyclically ordered sets ([5], 2.7): Let $(G_i, C_i)_{i\in I}$ be a family of cyclically ordered sets and let the sets G_i be pairwise disjoint. The *direct sum* of sets (G_i, C_i) $(i \in I)$ is the cyclically ordered set (G, C) where $G = \bigcup_{i\in I} G_i$, $C = \bigcup_{i\in I} C_i$; we write $(G, C) = \sum_{i\in I} (G_i, C_i)$. If $I = \{1, \ldots, n\}$, we write $\sum_{i\in I} (G_i, C_i) = (G_1, C_1) + \ldots + (G_n, C_n)$. Now, let (G, C) be a cyclically ordered set with $W(G, C) = 1$. Then there exists a linear order $<$ on a subset $G_1 \subseteq G$ such that $C = C_<$. If card $G_1 \leqq 2$, then $C_< = \emptyset$ so that (G, C) is discrete. If $G_1 = G$ and card $G \geqq 3$, then $C_<$ is linear, so that (G, C) is a cycle. In the other cases, put $G_2 = G - G_1$, $C_1 = C = C_<$, $C_2 = \emptyset$; then clearly $(G, C) = (G_1, C_1) + (G_2, C_2)$. On the other hand, if $(G, C) = (G_1, C_1) + (G_2, C_2)$ where (G_1, C_1) is a cycle and (G_2, C_2) is discrete, then $C = C_{<_{C,x}}$ for any $x \in G_1$ and $<_{C,x}$ is a linear order in G. Thus, we have

2.5. Lemma. *Let (G, C) be a cyclically ordered set. Then $W(G, C) = 1$ iff (G, C) is either a cycle or a discrete cyclically ordered set or $(G, C) = (G_1, C_1) + (G_2, C_2)$ where (G_1, C_1) is a cycle and (G_2, C_2) is discrete.*

2.6. Theorem. *Let (G, C) be a cyclically ordered set. Then $w(G, C) \leqq$ card G.*

Proof follows from 1.11.

If (G, C) is a cyclically ordered set and $H \subseteq G$ is a subset such that $D = C \cap H^3$ is a linear cyclic order on H, then (H, D) is called a *cycle in* (G, C).

2.7. Theorem. *Let (G, C) be a cyclically ordered set which is not discrete. Then $W(G, C) = \min\{\text{card}\, I$; there exists a family $(G_i, C_i)_{i\in I}$ of cycles in (G, C) such that $C = \bigcup_{i\in I} C_i$.*

Proof. Put $\min\{\text{card}\, I$; there exists a family $(G_i, C_i)_{i\in I}$ of cycles in (G, C) such that $C = \bigcup_{i\in I} C_i\} = m$. Let $(<_j)_{j\in J}$ be a harmonized family of linear orders in G such that $C = \bigcup_{j\in J} C_{<_j}$ and card $J = W(G, C)$. Each $<_j$ is a linear order on a certain (maximal) subset $G_j \subseteq G$ and we may assume card $G_j \geqq 3$ (otherwise $C_{<_j} = \emptyset$ and $<_j$ can be omitted). Thus $(G_j, C_{<_j})$ is a cycle in (G, C) and we obtain $m \leqq W(G, C)$. On the other hand, let $(G_i, C_i)_{i\in I}$ be a family of cycles in (G, C) such that $C = \bigcup_{i\in I} C_i$ and card $I = m$. By 2.5, $W(G_i, C_i) = 1$ for each $i \in I$, i.e. there exists a linear order $<_i$ in G_i such that $C_i = C_{<_i}$. Then each $<_i$ is a linear order in G and $C = \bigcup_{i\in I} C_{<_i}$, which implies $W(G, C) \leqq m$.

2.8. Corollary. *Let $(G, <)$ be an ordered set and $(<_i)_{i\in I}$ a family of all maximal linear orders in G that are included in $<$. Then $W(G, C_<) \leqq$ card I.*

Proof. Clearly, $\bigcup_{i\in I} <_i = <$ which implies $\bigcup_{i\in I} C_{<_i} \subseteq C_<$. On the other hand, if $(x, y, z) \in C_<$, then either $x < y < z$ or $y < z < x$ or $z < x < y$. Then there exists a maximal chain $(G_i, <_i)$ in $(G, <)$ containing $\{x, y, z\}$ and hence $(x, y, z) \in C_{<_i}$. Thus $C_< = \bigcup_{i\in I} C_{<_i}$ and the assertion follows from 2.7.

2.9. Lemma. *Let (G, C) be a cyclically ordered set, let $H \subseteq G$ and $D = C \cap H^3$. Then $w(H, D) \leqq w(G, C)$, $W(H, D) \leqq W(G, C)$.*

Proof. Let $(<_i)_{i\in I}$ be a harmonized family of orders on G such that $C = \bigcup_{i\in I} C_{<_i}$ and $\operatorname{card} I = w(G, C)$. Put $\prec_i = <_i \cap H^2$; then $\prec_i$ is an order on H and it is easy to prove $D = \bigcup_{i\in I} C_{\prec_i}$. Thus $w(H, D) \leqq \operatorname{card} I = w(G, C)$. If $<_i$ is a liner order in G, then $\prec_i$ is a linear order in H so that also $W(H, D) \leqq W(G, C)$.

2.10. Theorem. *Let $(G_i, C_i)_{i\in I}$ be a family of cyclically ordered sets and let the sets G_ι be pairwise disjoint. Let $(G, C) = \sum_{i\in I} (G_i, C_i)$. Then $w(G, C) = \sup \{w(G_i, C_i);\ i \in I\}$, $W(G, C) \leqq \sum_{i\in I} W(G_i, C_i)$. If, moreover, no (G_i, C_i) is discrete, then $W(G, C) = \sum_{i\in I} W(G_i, C_\iota)$.*

Proof. As $G_i \subseteq G$, $C_i = C \cap G_i^3$ for any $i \in I$, 2.9. implies $w(G_i, C_i) \leqq w(G, C)$ for any $i \in I$ and thus $\sup \{w(G_i, C_i);\ i \in I\} \leqq w(G, C)$. Put $\sup \{w(G_i, C_i);\ i \in I\} = m$ and let J be a set with $\operatorname{card} J = m$. For any $i \in I$ find a family $(<_{i,j})_{j\in J}$ of orders on G_i such that $C_i = \bigcup_{j\in J} C_{<_{i,j}}$ and for a given $j \in J$ put $<_j = \bigcup_{i\in I} <_{i,j}$. Then $<_j$ is an order on G (in fact, $<_j$ is the cardinal sum of orders $<_{i,j}$; $i \in I$). We show that $C = \bigcup_{j\in J} C_{<_j}$. Let $(x, y, z) \in C$. Then there exists (just one) $i \in I$ such that $x, y, z \in G_i$ and $(x, y, z) \in C_i$. This implies the existence of $j \in J$ such that $(x, y, z) \in C_{<_{i,j}}$. As $<_{i,j} \subseteq <_j$, we have $(x, y, z) \in C_{<_j} \subseteq \bigcup_{j\in J} C_{<_j}$. We have proved $C \subseteq \bigcup_{j\in J} C_{<_j}$. Let $(x, y, z) \in \bigcup_{j\in J} C_{<_j}$. Then there exists $j \in J$ such that $(x, y, z) \in C_{<_j}$. By definition of the order $<_j$ there exists (just one) $i \in I$ such that $(x, y, z) \in C_{<_{i,j}}$. Thus $(x, y, z) \in C_i$ and $(x, y, z) \in C$. Hence $C = \bigcup_{j\in J} C_{<_j}$, which implies $w(G, C) \leqq \operatorname{card} J = m$ and we have $w(G, C) = m = \sup \{w(G_i, C_i);\ i \in I\}$. The assertion on $W(G, C)$ follows from 2.7.

3. COCYCLICALLY ORDERED SETS

3.1. Definition. Let G be a set, T a ternary relation on G. T is called a *cocyclic order* on G, iff it is

(v) reflexive, i.e. $x, y, z \in G$, $\operatorname{card} \{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \in T$,
cyclic, complete and satisfies the condition

(vi) $x, y, z, u \in G$, pairwise distinct, $(x, y, z) \in T \Rightarrow (x, y, u) \in T$ or $(x, u, z) \in T$.

If G is a set and T a cocyclic order on G, then the pair (G, T) is called a *cocyclically ordered set.*

If G is a set and T a ternary relation on G, then we denote by Co T the complement of T in G^3, i.e. Co $T = G^3 - T$.

3.2. Theorem. *Let G be a set, T a ternary relation on G. T is a cocyclic order on G iff* Co *T is a cyclic order on G.*

Proof. 1. Let T be a cocyclic order on G and denote Co $T = C$. Assume that there exist $x, y, z \in G$ with $(x, y, z) \in C$, $(z, y, x) \in C$. Then $(x, y, z) \,\bar{\in}\, T$, $(z, y, x) \,\bar{\in}\, T$ which implies $x \neq y \neq z \neq x$ and thus T is not complete. This is a contradiction and hence C is asymmetric. Let $(x, y, z) \in C$ and assume $(y, z, x) \,\bar{\in}\, C$. Then $(y, z, x) \in T$ and as T is cyclic, $(x, y, z) \in T$, a contradiction. Thus C is cyclic. Let $(x, y, z) \in C$, $(x, z, u) \in C$. Then $x \neq y \neq z \neq x$, $x \neq z \neq u \neq x$ and we shall show $y \neq u$. If $y = u$, then $(x, z, y) \in C$, thus $(z, y, x) \in C$ as C is cyclic and this contradicts the asymmetry of C. Thus the elements x, y, z, u are pairwise distinct. Assume $(x, y, u) \,\bar{\in}\, C$. Then $(x, y, u) \in T$ and, by (vi), either $(x, y, z) \in T$ or $(x, z, u) \in T$. But this contradicts the assumption $(x, y, z) \in C$, $(x, z, u) \in C$. We have shown that C is transitive and thus $C =$ Co T is a cyclic order on G.

2. Let $C =$ Co T be a cyclic order on G. From the asymmetry and cyclicity of C we easily derive $(x, y, z) \in C \Rightarrow x \neq y \neq z \neq x$. Thus $x, y, z \in G$, card $\{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \,\bar{\in}\, C$, hence $(x, y, z) \in T$ and the relation T is reflexive. Let $(x, y, z) \in T$ and assume $(y, z, x) \,\bar{\in}\, T$. Then $(y, z, x) \in C$ and by the cyclicity of C, $(x, y, z) \in C$ which is a contradiction. Thus T is cyclic. Let $x, y, z \in G$, $x \neq y \neq z \neq x$ and assume $(x, y, z) \,\bar{\in}\, T$, $(z, y, x) \,\bar{\in}\, T$. Then $(x, y, z) \in C$, $(z, y, x) \in C$, which contradicts the asymmetry of C. Hence T is complete. Let $x, y, z, u \in G$ be pairwise distinct elements such that $(x, y, z) \in T$ and assume $(x, y, u) \,\bar{\in}\, T$, $(x, u, z) \,\bar{\in}\, T$. Then $(x, y, u) \in C$, $(x, u, z) \in C$ and hence $(x, y, z) \in C$ by the transitivity of C, which is a contradiction. Thus $(x, y, u) \in T$ or $(x, u, z) \in T$, T satisfies (vi) and is, therefore, a cocyclic order on G.

3.3 Corollary. *Let G be a set, $<$ an order on G. Then* Co $C_<$ *is a cocyclic order on G.*

3.4. Theorem. *Let G be a set, $(<_i)_{i\in I}$ a family of orders on G. Then $\bigcap_{i\in I}$* Co $C_{<_i}$ *is a cocyclic order on G iff the family $(<_i)_{i\in I}$ is harmonized.*

Proof. Clearly $\bigcap_{i\in I} \text{Co}\, C_{<_i} = \text{Co}\left(\bigcup_{i\in I} C_{<_i}\right)$ so that – by 3.2 – $\bigcap_{i\in I} \text{Co}\, C_{<_i}$ is a cocyclic order on G iff $\bigcup_{i\in I} C_{<_i}$ is a cyclic order on G. But this holds by 1.8 iff the family $(<_i)_{i\in I}$ is harmonized.

3.5. Theorem. *Let (G, T) be a cocyclically ordered set. Then there exists a harmonized family $(<_i)_{i\in I}$ of orders on G such that* $T = \bigcap_{i\in I} \text{Co } C_{<_i}$.

Proof. As Co T is a cyclic order on G, by 1.9 there exists a harmonized family $(<_i)_{i\in I}$ of orders on G such that Co $T = \bigcup_{i\in I} C_{<_i}$. But then $T = \bigcap_{i\in I} \text{Co } C_{<_i}$.

Analogously, from 1.10 we obtain

3.6. Corollary. *Let (G, T) be a cocyclically ordered set. Then there exists a harmonized family $(<_i)_{i\in I}$ of linear orders in G such that* $T = \bigcap_{i\in I} \text{Co } C_{<_i}$.

3.7. Definition. Let (G, T) be a cocyclically ordered set. Put $d(G, T) = \min \{\text{card } I;$ there exists a harmonized family $(<_i)_{i\in I}$ of orders on G such that $T = \bigcap_{i\in I} \text{Co } C_{<_i}\}$, $D(G, T) = \min \{\text{card } I;$ there exists a harmonized family $(<_i)_{i\in I}$ of linear orders in G such that $T = \bigcap_{i\in I} \text{Co } C_{<_i}\}$.

3.8. Theorem. *Let (G, T) be a cocyclically ordered set. Then* $d(G, T) = w(G, \text{Co } T)$, $D(G, T) = W(G, \text{Co } T)$.

Proof. For any harmonized family $(<_i)_{i\in I}$ of orders on G the relation $T = \bigcap_{i\in I} \text{Co } C_{<_i}$ is equivalent to the relation Co $T = \bigcup_{i\in I} C_{<_i}$. This yields both the assertions.

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