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EVERY GRAPH IS AN INDUCED ISOPART OF A CIRCULANT

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1. INTRODUCTION

We will say that a graph is nonempty if it contains at least one edge.

By a *decomposition* of a nonempty graph G is meant a family of subgraphs $G_1, G_2, \ldots, G_k$ of G such that their edge sets form a partition of the edge set of G. This is denoted by

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_k .$$

Each member of the family is called a *part* of the decomposition. A graph G is said to be *H-decomposable* (or has an *H-decomposition*) if G has a decomposition in which all of its parts are isomorphic to the graph H. If G is H-decomposable, then H is referred to as an *isopart* of G. We also say that G has an *isomorphic decomposition* into the graph H. An obvious necessary condition for a graph G to be H-decomposable is that the size of G is a multiple of the size of H.

Wilson [5], using algebraic techniques, proved that for every nonempty graph H there exists an integer λ (depending on H) such that if $n \geqq \lambda$ and n satisfies certain divisibility conditions, then the complete graph K_n is H-decomposable.

Fink [2] showed that every nonempty graph H is an induced isopart of a regular (not necessarily complete) graph G. In this note we prove a similar result with the added condition that the graph G is a circulant (its adjacency matrix is a circulant).

2. LABELINGS AND ISOMORPHIC DECOMPOSITIONS OF GRAPHS

Let $n \geqq 3$ be an integer and S a nonempty subset of $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. The *circulant* $G = G(n; S)$ has vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $v_i v_j \in E(G)$, the edge set of G, if and only if either $j - i$ or $i - j$ is congruent, modulo n, to an element of S. Then set S is called the *length set* of G and the *length of any pair* v_i, v_j of vertices is defined as $l(v_i, v_j) = \min \{|i - j|, |n - (i - j)|\}$. By the length $l(e)$ of an edge $e = v_i v_j$ we mean $l(e) = l(v_i, v_j)$.

Circulants can be drawn in the Euclidean plane with its n vertices $v_0, v_1, \ldots, v_{n-1}$ regularly distributed counterclockwise about a circle, where the edges are represented by chords joining the appropriate vertices.

Note that in $G = G(n; S)$, for any vertex $v_i \in V(G)$ and any $s \in S$, the vertex v_i is adjacent to both v_{i+s} and v_{i-s} (where the subscripts are expressed modulo n). Moreover, $v_{i+s} \neq v_{i-s}$ unless $s = n/2$. Therefore if $n/2 \in S$, then $G(n; S)$ is regular of degree $2|S| - 1$; otherwise, it is regular of degree $2|S|$. The cyclic permutation $\varphi = (v_0 v_1 \ldots v_{n-1})$ is an automorphism of G. Associated with φ we have an induced permutation φ_E defined on the edges of G as follows: The image of an edge xy of G under φ_E is the edge $\varphi(x)\,\varphi(y)$. Considering the action of the permutation group

$$\{\varphi_E, \varphi_E^2, \ldots, \varphi_E^n\}$$

on $E(G)$, we observe that this action partitions the edge set of G into $|S|$ orbits. Two edges belong to the same orbit E_s if and only if they have the same length s. If $l(e) = n/2$ then the orbit containing e has $n/2$ members; otherwise it has n members.

The following lemma will enable us to construct some isomorphic decompositions of a given circulant. Henceforth N_n will denote the set $\{0, 1, \ldots, n - 1\}$, for $n \geqq 1$.

Lemma 1. *Let $G = G(n; S)$ be a circulant such that $n/2 \notin S$. For each $s \in S$, let e_s be an edge in the orbit E_s. If H is the subgraph of G induced by the edges e_i, $i \in S$, then G is H-decomposable.*

Proof. For each $k \in N_n$, let $A_k = \{\varphi_E^k(e_s) \mid s \in S\}$ and let G_k be the subgraph of G induced by the set $\varphi_E^k(A_0)$ of edges. Thus, $G_0 = H$.

We now show that φ^k is an isomorphism between G_0 and G_k, for each $k \in N_n$. Let v_s and v_t be two adjacent vertices of G_0. Note that $\varphi^k(v_s)$ and $\varphi^k(v_t)$ are adjacent in G_k since $\varphi_E^k(v_s v_t) = \varphi^k(v_s)\,\varphi^k(v_t)$ and $\varphi_E^k(v_s v_t) \in A_k$; thus $G_0 \cong G_k$.

It remains only to prove that $\{A_k \mid k \in N_n\}$ forms a partition of $E(G)$. If $e \in E(G)$, then $l(e) \in S$, and by the way that G_0 was defined, there exists an edge e_0 in G_0 such that $l(e_0) = l(e)$. However, there exists $k \in N_n$ such that $\varphi_E^k(e_0) = e$; therefore $e \in A_k$ and consequently $E = \bigcup_{i=0}^{n-1} E_k$.

To show that the sets $A_0, A_1, \ldots, A_{n-1}$ are pairwise disjoint, we proceed by a counting argument. Since G is regular of degree $2|S|$, it has $n|S|$ edges. If $A_0, A_1, \ldots$ $\ldots, A_{n-1}$ were not pairwise disjoint, then

$$n|S| = |E(G)| < \sum_{i=0}^{n-1} |A_i| = n|S|,$$

which is a contradiction.

Therefore $G = G_0 \oplus \ldots \oplus G_{n-1}$ and G is H-decomposable.

We say that G is *rotationally H-decomposable* if G possesses the kind of decomposition described in the preceding lemma. Loosely speaking, each part G_{i+1} is obtained from G_i by rotating G_i counterclockwise through an angle of $2\pi/n$ radians about its center. Now, the basic idea leading to the main results is to imbed a preassigned graph H into a circulant such that H is a part of the H-decomposition. To achieve this we will extend a labeling technique introduced by Rosa [4] when he rotationally decomposed K_{2n+1} into a graph of size n admitting a so called ϱ-*valuation*.

Given a graph $G = G(V, E)$, with vertex set V and edge set E, a *labeling* f of G is a one-to-one map from V into the set $\mathbb{N}$ of nonnegative integers. The labeling f induces a map $\bar{f}$ from E into $\mathbb{N}$, called *induced edge labeling*, such that $\bar{f}(uv) = |f(u) - f(v)|$.

By using powers of 2 as labels, we can easily verify the following lemma.

Lemma 2. *Every nonempty graph has a labelling such that its induced edge labeling is one-to-one and there exist two adjacent vertices with labels* 1 *and* 2.

Theorem 1. *For every nonempty graph H, there exists a connected circulant G such that G is rotationally H-decomposable.*

Proof. We assume, without lost of generality, that $V(H) = \{v_0, v_1, \ldots, v_m\}$ and that $v_0v_1 \in E(H)$. By Lemma 2, there exists a labeling f of H for which $\bar{f}$ is one-to-one, $f(v_0) = 1$ and $f(v_1) = 2$.

Let $S = \{\bar{f}(e) \mid e \in E(H)\}$ and $r = \max S$. Then by Lemma 1, the circulant $G(n; S)$ is H-decomposable, where $n = \max\{2r + 1, |V(H)|\}$. Note that $1 \in S$ so that G contains a hamiltonian cycle and is connected. Moreover, from the proof of Lemma 1, it follows that G is rotationally H-decomposable.

With this result at hand we can prove an even stronger result.

Corollary 1. *For every nonempty graph H there exists a connected circulant G such that G is rotationally H-decomposable and every part of such a decomposition is an induced subgraph of G.*

Proof. Case 1. If H is a complete graph then a labeling f described in the proof of Theorem 1 will give the required decomposition.

Case 2. If H is not complete, we can add to H all the edges of its complement obtaining a complete graph H^*. Next, we do as in Case 1 to obtain a labeled graph H^* and a connected circulant G^* which is rotationally decomposed into (induced) copies of H^*. Deleting from G^* all edges having lengths not present in the set of lengths corresponding to the edges of H, it is obtained a circulant G with the properties stated in the Corollary.

3. APPLICATIONS TO DIGRAPHS

If F is a digraph and $F_1, F_2, \ldots, F_n (n \geqq 1)$ are nonempty arc-disjoint subdigraphs of F satisfying the property that

$$E(F) = \bigcup_{i=1}^{n} E(F_i),$$

then we say that F is the *arc sum* of the *parts* $F_1, F_2, \ldots, F_n$ and write $F = F_1 \oplus F_2 \oplus \ldots \oplus F_n$. If there is a digraph D that is isomorphic to each of the parts

$F_1, F_2, \ldots, F_n$ then we say that F has an *isomorphic decomposition* into the digraph D or that F is *D-decomposable.*

A digraph is r-regular (or simply regular) if each of its vertices has both indegree and outdegree equal to r. For example, a regular digraph is obtained from a complete graph K_p by replacing each edge uv by two arcs (u, v) and (v, u). This digraph is called the *complete symmetric digraph* K_p^*. Isomorphic decompositions of K_p^* have been considered, among others, by Bermond and Sotteau [1]. Harary, Robinson and Wormald [3] showed that if t divides $p(p - 1)$, then K_p^* has an isomorphic decomposition into t copies of some digraph. Using algebraic techniques, Wilson [5] proved that for every nonempty digraph D, the digraph K_p^* is D-decomposable for infinitely many p.

Along these lines we have:

Corollary. *For every nonempty digraph D, there exists a connected regular digraph F such that $F = F_0 \oplus F_1 \oplus \ldots \oplus F_{n-1}$ for some $n \geqq 1$ and each part is an induced subdigraph of F isomorphic to D.*

Proof. Let H be the underlying graph of the digraph D. According to Corollary 2, there exists a connected circulant G such that G is rotationally H-decomposable and every part of such a decomposition is an induced subgraph of G. With the aid of G, we will now construct the desired digraph F so that G is, in fact, the underlying graph of F.

Let f be the labeling of H which gave origin to the decomposition

$$G = G_0 \oplus G_1 \oplus \ldots \oplus G_{n-1}\,, \quad \text{where} \quad V(G_0) = \{v_{f(x)} \mid x \in V(H)\}\,.$$

For each part G_i in the H-decomposition of G, let φ^i be the ith power of the cycle $(v_0 v_1 \ldots v_{n-1})$ such that φ^i is the isomorphism from G_0 to G_i (φ^i preserves the length of each edge of G_0). Now, for each part G_i and each edge $v_s v_t$ of G_i, we employ the following procedure.

There are unique vertices $a, b \in V(D) = V(H)$ such that $v_s = \varphi^i(v_{f(a)})$ and $v_t = \varphi^i(v_{f(f)})$. Let $E(ab)$ be the set of arcs of D joining the vertices a and b of D. Replace the edge $v_s v_t$ of G by the set of arcs $\{(v_s, v_t), (v_t, v_s)\}$, $\{(v_s, v_t)\}$ or $\{(v_t, v_s)\}$ according to whether $E(ab)$ is $\{(a, b), (b, a)\}$, $\{(a, b)\}$ or $\{(b, a)\}$, respectively. Then the digraph F so obtained is conected, regular and has a D-decomposition where each part is induced.

We note, in closing this chapter, that a similar proof can be applied to multigraphs and multidigraphs.

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