Czechoslovak Mathematical Journal

Mirko Navara

State space properties of finite logics

Czechoslovak Mathematical Journal, Vol. 37 (1987), No. 2, 188–196

Persistent URL: http://dml.cz/dmlcz/102148

Czechoslovak Mathematical Journal, 37 (112) 1987, Praha

STATE SPACE PROPERTIES OF FINITE LOGICS

MIRKO NAVARA, Praha

(Received April 30, 1985)

We show that certain hypergraphs can be associated with quantum logics in a state-preserving manner. This result – a generalization of the Greechie correspondence (see [2], [3]) – enables us to construct logics with special state space properties (see the applications in § 3).

1. INTRODUCTION AND PRELIMINARIES

In the foundations of the quantum theory it is often assumed that the "event structure" of a physical system is an orthomodular partially ordered set (the so called logic – see [4], [12] etc.). The states of the system then correspond to the probability measures on a logic. Let us agree to call a probability measure on a logic a state and ask how this "axiomatic" state space may look. In this note we investigate the state space properties of finite logics. We generalize the technique of R. Greechie and utilize the generalization for constructing finite logics with preassigned state space properties. As an application we first add to the list of stateless and almost stateless logics (see [3], [11], [7]), and secondly, we construct finite unital non-Boolean fully embeddable logics and finite nearly Jauch-Piron logics (for definitions and all details, see § 3). The latter constructions extend and complete the results of the papers [1], [8], [9] and [10]. Let us first recall basic definitions.

Definition 1.1. A *logic* is a set L endowed with a partial ordering $\leqq$ and a unary operation $'$ such that

(1) $0, 1 \in L$,

(2) $a \leqq b \Rightarrow b' \leqq a'$ for any $a, b \in L$,

(3) $(a')' = a$ for any $a \in L$,

(4) $a \vee a' = 1$ and $a \wedge a' = 0$ for any $a \in L$ (the symbols $\wedge$, $\vee$ mean the lattice-theoretic operations given by $\leqq$),

(5) $a \vee b$ exists in L whenever $a, b \in L$ and $a \leqq b'$,

(6) $b = a \vee (b \wedge a')$ whenever $a, b \in L$ and $a \leqq b$.

Definition 1.2. A *state* s on a logic L is a non-negative real-valued function such that

(1) $s(1) = 1$,

(2) $s(a \vee b) = s(a) + s(b)$ whenever $a, b \in L$ and $a \leqq b'$.

In this paper we shall restrict ourselves to finite logics. Recall that every finite logic is atomistic, that is, each element of a finite logic can be expressed as a supremum of finitely many atoms. (An element b of a logic is called an *atom* if $0 < a \leqq b$ implies $a = b$.) It is obvious (Zorn's lemma) that each Boolean sublogic of a logic can be enlarged to a maximal one (called a *block*). Thus logic can be viewed as a union of its blocks.

Definition 1.3. A *hypergraph* is a couple $H = (A, X)$, where A is a non-empty set and X is a covering of A by non-empty subsets. We call elements of A *vertices* and elements of X *edges*. An edge of a hypergraph is called *isolated* if it is disjoint to every other edge. A path of length n between vertices a, b is an *n-tuple* of edges $(B_1, \ldots, B_n)$ such that $a \in B_1$, $b \in B_n$ and

$$B_i \cap B_j \neq \emptyset \quad \text{if and only if} \quad |i - j| = 1 \, .$$

A *loop* of length n $(n \geqq 3)$ is an n-tuple of edges $(B_1, \ldots, B_n)$ such that

$$B_i \cap B_j \neq \emptyset \quad \text{if and only if} \quad |i - j| \in \{0, 1, n - 1\} \quad \text{and} \quad B_1 \cap B_2 \cap B_3 = \emptyset \, .$$

The distance of two vertices a, b, denoted by $d(a, b)$, is the minimal length of a path between a and b. If no such path exists, we put $d(a, b) = \infty$. For completeness we set $d(a, a) = 0$. The maximal distance of two vertices is called the *diameter* of the hypergraph.

Definition 1.4. A *state on a hypergraph* (A, X) is a non-negative real-valued function s on A satisfying the condition $\sum_{a \in B} s(a) = 1$ for each edge B.

In what follows we shall examine the connection between logics and hypergraphs and the corresponding state spaces. This brings rather technical complications in some places. To simplify our task, we will assume that the reader is well acquainted with Greechie's technique of constructing finite logics and the interpretation of diagrams (see [2], [3], [11]). Let us only recall the necessary facts.

Every finite logic can be associated with a hypergraph (called the *diagram* of the logic – see [3]). The atoms of the logic correspond to vertices of the hypergraph, the blocks to edges. Each element of the logic is then described as a subset of some edge. States on a logic and states on its diagram are in a one-to-one correspondence. A state on a logic is obtained by the unique extension of a state (weigth) on its diagram.

Not every hypergraph is a diagram of a logic. Let us recall a sufficient condition given by R. Greechie.

Definition 1.5. The *Greechie hypergraph* is such a hypergraph that
(1) every non-isolated edge has at least 3 vertices,
(2) every isolated edge has at least 2 vertices,
(3) the intersection of any two distinct edges contains at most one vertex,
(4) there is no loop of length less than 4.

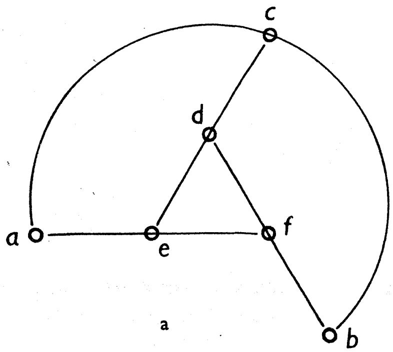
Theorem 1.6. (See [3].) *Every Greechie hypergraph is a diagram of a logic.* (Admitting isolated edges with 2 vertices, we use a mild generalization of [3] already used in [11].)

Let us denote by $S(H)$ the set of all states of a hypergraph $H = (A, X)$. We say that vertices a, b of H are *state-equivalent* if $s(a) = s(b)$ for every state $s \in S(H)$. Let $r(b)$ denote the class of vertices state-equivalent with a vertex b. Let us denote by $r(A)$ the set of all classes of state-equivalent vertices. The state-equivalence of vertices a, b can be expressed either by writing $b \in r(a)$ or $r(a) = r(b)$. The set $S(H)$ is fully determined by the partition $r(A)$ of A within the state equivalence and by the values of all states on these classes.

Definition 1.7. Let $H_1 = (A_1, X_1)$, $H_2 = (A_2, X_2)$ be hypergraphs, let r_1, r_2 be the respective state-equivalence relations. Let f be an injective mapping of $r_1(A_1)$ onto $r_2(A_2)$. The mapping $s_1 \circ f^{-1}$, for $s_1 \in S(H_1)$, is a non-negative function on $r_2(A_2)$. We call f a *state isomorphism* (nad hypergraphs H_1, H_2 *state-isomorphic*) if the following two conditions are satisfied:
(1) for any $s_1 \in S(H_1)$ the mapping $s_1 \circ f^{-1} \circ r_2$ is a state on H_2,
(2) for any $s_2 \in S(H_2)$ the mapping $s_2 \circ f \circ r_1$ is a state on H_1.

The notion of a state isomorphism can be naturally extended to logics and also to state isomorphisms of logics and hypergraphs – instead of a logic we consider its diagram. For the convenience of the reader, let us illustrate the notion of a state isomorphism by the following example. (Observe also that our notion of a state isomorphism is stronger than that of an affine homeomorphism investigated by F. Shultz – see [11].)

Example 1.8. The hypergraph in Fig. 1a is not a Greechie hypergraph. Since it has 3 classes of state-equivalent vertices $\{a, d\}$, $\{b, e\}$, $\{c, f\}$, it is therefore state-isomorphic to the diagram of the Boolean algebra 2^3 (see Fig. 1b).



a

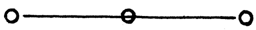


Fig. 1 b

The main result of this paper is Theorem 2.4. Given a relatively general hypergraph, we construct a logic state-isomorphic to it. We shall need the following preliminary result.

Proposition 2.1. *Let 2^3 denote the Boolean algebra with atoms a, b, c. Then for any positive integers N, D we can find a Greechie hypergraph H with the following properties*:

(1) *H is state-isomorphic to 2^3 under a state-isomorphism f,*

(2) *for suitably chosen N-element sets A_1, A_2, A_3 with $A_1 \subset f^{-1}(\{a\})$, $A_2 \subset f^{-1}(\{b\})$, $A_3 \subset f^{-1}(\{c\})$, the distance of each two vertices of $A_1 \cup A_2 \cup A_3$ is greater than D.*

Proof. We shall start with an auxiliary construction. Let us consider the hypergraph $G_1 = (\{a_0, \ldots, a_{17}\}, X)$, where X consists of all sets of the form

$$\{a_{2n}, a_{2n+1}, a_{2n+2}\}, \qquad n = 0, \ldots, 8,$$

or

$$\{a_{2n-5}, a_{2n}, a_{2n+5}\}, \qquad n = 0, \ldots, 8$$

(all indices are taken modulo 18). The hypergraph G_1 is a Greechie hypergraph. Let s be a state on G_1. Put $x_i = s(a_i)$ $(i = 0, \ldots, 17)$. Then we have the following equations (we close in brackets the sums over all vertices of an edge, which, of course, equal 1):

$$\begin{aligned} 3 &= (x_1 + x_6 + x_{11}) + (x_7 + x_{12} + x_{17}) + (x_8 + x_9 + x_{10}) = \\ &= (x_6 + x_7 + x_8) + (x_{10} + x_{11} + x_{12}) + x_1 + x_9 + x_{17} = \\ &= 2 + x_1 + x_9 + x_{17}. \end{aligned}$$

Hence we have

$$x_1 + x_9 + x_{17} = 1.$$

Comparing this equality with

$$(x_{17} + x_4 + x_9) = 1 \quad \text{and} \quad (x_9 + x_{14} + x_1) = 1,$$

we obtain

$$x_1 = x_4 \quad \text{and} \quad x_{14} = x_{17}.$$

Adding 2 (modulo 18) to each index, we obtain the same logic. Hence the same equations can be derived for the indices which differ from the above ones by $2n$. We thus obtain more general equations

$$x_{2n-3} = x_{2n} \quad \text{and} \quad x_{2n} = x_{2n+3}$$

for all $n = 0, \ldots, 8$ (the indices being again taken modulo 18). For $i = 0, 1, 2$, we have

$$x_{3k+i} = s(a_{3k+i}) = x_i \quad \text{for} \quad k = 0, \ldots, 5.$$

There are 3 classes C_{a_i} $(i = 0, 1, 2)$ of state-equivalent vertices in G_1:

$$C_{a_i} = \{a_{3k+i}: k = 0, \dots, 5\} .$$

If we choose non-negative x_0, x_1, x_2 with $x_0 + x_1 + x_2 = 1$ then the formula

$$s(C_{a_i}) = x_i \quad (i = 0, 1, 2)$$

defines a state on G_1 and, conversely, all states on G_1 are of this form. So G_1 is state-isomorphic to the Boolean algebra 2^3.

Let us return to the proof of Proposition 2.1. We shall construct a sequence of hypergraphs G_n such that each G_n is state-isomorphic to 2^3 and such that the diameter of G_n is greater than n. We take n copies $H^1, \dots, H^n$ of the hypergraph G_1. Denote their vertices by $a_0^1, \dots, a_{17}^1;\ a_0^2, \dots, a_{17}^2;\ \dots;\ a_0^n, \dots, a_{17}^n$. For $k = 1, \dots$ $\dots, n - 1$, let us identify vertices a_{10}^k with a_0^{k+1}, a_{11}^k with a_1^{k+1} and a_{12}^k with a_2^{k+1}. This leads to identifying one edge of H^k with one edge of H^{k+1}. A state defined on a single edge of H^{k+1} extends uniquely to the whole of H^{k+1}. Every state on the hypergraph G_n obtained by our construction is a unique extension of a state on H^1 and therefore G_n is state-isomorphic to 2^3.

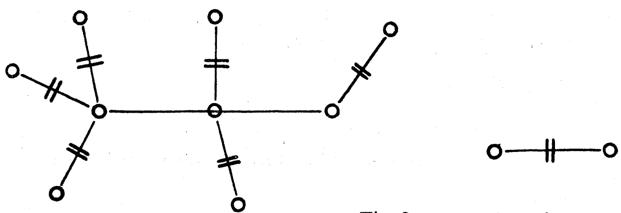
One checks easily that G_n is again a Greechie hypergraph (with $15n + 3$ vertices). Moreover, the distance of each two vertices $a_i^k \in H^k$, $a_j^{k+2} \in H^{k+2}$ is greater than 1 and, more generally, $d(a_i^k, a_j^{k+m}) \geqq m - 1$ for $m \geqq 2$ $(k = 1, \dots, n - m;\ i, j = 0, \dots, 17)$. Given N and D, the hypergraphs G_n fulfil all the conditions of Proposition 2.1 for a sufficiently large n and the proof is complete.

Before stating our main result, let us make a few observations. The first one may be stated as follows.

Proposition 2.2. *Let every two vertices taken from the set* $\{b_1, \dots, b_k\}$ $(k \geqq 3)$ *of a Greechie hypergraph* (A, X) *have distance at least* 3. *Then* $(A, X \cup \{\{b_1, \dots, b_k\}\})$ *is also a Greechie hypergraph.*

Proof of Proposition 2.2 is straightforward.

In our construction, we shall make use of sub-hypergraphs isomorphic to G_n. Our intention is to obtain sets of state-equivalent vertices with mutual distance at least 3. Any subhypergraph isomorphic to G_n may be represented similarly as portrayed in Fig. 2a. Following the language of [11], we express the state equivalence of two vertices a, b in the way indicated in Fig. 2b. For the sake of simplicity we shall



a Fig. 2 b

use this convention only for the state equivalence forced by the structure of G_n, not for the possibly larger state equivalence given by the remaining part of the hypergraph in question. Further, we assume that the distance of each two vertices connected in the manner of Fig. 2b is at least 3.

To reinforce the intuition of the reader, let us make a final technical remark concerning the proof of the next theorem. Let us consider Fig. 3a. This figure gives an example of a non-Greechie hypergraph. We transform it to a state-isomorphic Greechie hypergraph, portrayed in Fig. 3b, by adding two sub-hypergraphs G_p, G_q (p, q being taken sufficiently large). Suppose that $\{a_1, c_1, d\}$ is an edge of G_p and $\{b_1, c_2, e\}$ is an edge of G_q. The vertex a was replaced by three state-equivalent vertices a_1, a_2, a_3, and so were vertices b and c. Either c_1 or c_2 belongs to both G_p and G_q. The hypergraph obtained is state-isomorphic to the hypergraph from Fig. 3a. Another example is the hypergraph from Fig. 3c, which is state-isomorphic to the hypergraph from Fig. 1a.

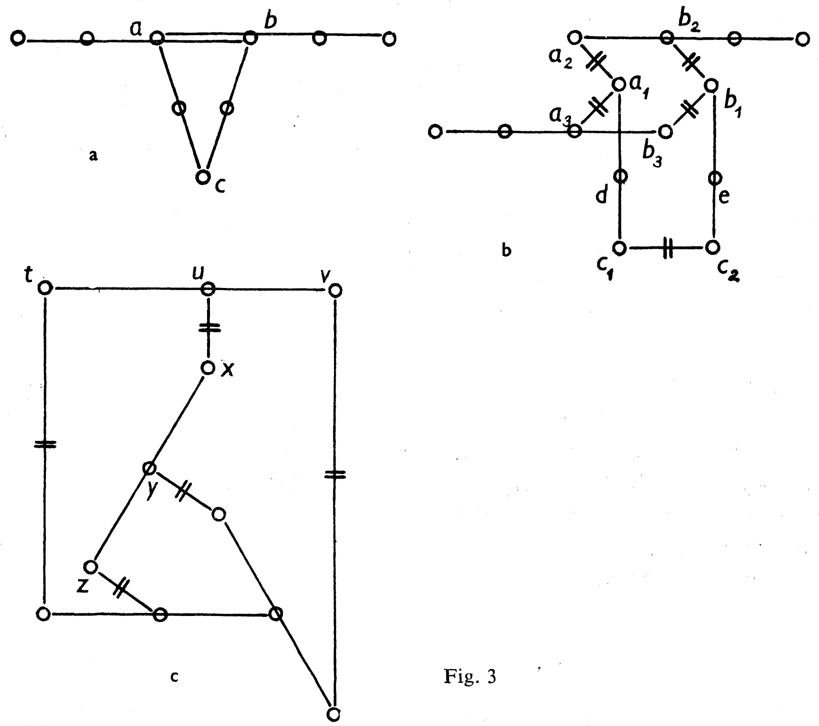


Fig. 3

Definition 2.3. Let $H = (A, X)$ be a hypergraph and let C be a subset of A. By the hypergraph obtained by excluding the vertices of C from H we mean the hypergraph $(A - C, X_1)$, where $X_1 = \{E - C \colon E \in X, E - C \neq \emptyset\}$.

Theorem 2.4. *Let $H = (A, X)$ be a finite hypergraph satisfying the following conditions:*

(1) *every non-isolated edge has at least* 3 *vertices,*

(2) *every isolated edge has at least* 2 *vertices,*

and the sub-hypergraph obtained by excluding all vertices contained in 3*-element edges satisfies*

(3′) *the intersection of any two distinct edges contains at most one vertex,*

(4′) *there is no loop of length less than* 4.

Then there exists a Greechie hypergraph (and therefore a logic) which is state-isomorphic to H. Particularly, if A is covered with 3*-element edges and H satisfies the conditions* (1) *and* (2) *then there exists a logic state-isomorphic to H.*

Proof. Let us consider an edge $\{a, b, c\}$ of H. Let m_a, m_b, m_c be the numbers of edges containing a, b, c respectively. Put $m = \max\{m_a, m_b, m_c\}$. Now replace the edge $\{a, b, c\}$ by a sub-hypergraph G_n with an edge $\{a_0, b_0, c_0\}$, possessing sets of state-equivalent vertices $\{a_0, \ldots, a_m\}$, $\{b_0, \ldots, b_m\}$, $\{c_0, \ldots, c_m\}$ such that their mutual distance it at least 3. Let b be included in edges $B_1, \ldots, B_{m_b} \in X$. Substitute b by b_i in the edge B_i $(i = 1, \ldots, m_b)$ and proceed analogously for a and c. We obtain a hypergraph state-isomorphic to H. Repeating this procedure for all 3-element edges, we obtain a hypergraph G which is state-isomorphic to H. It is a routine verification that if H satisfies the assumption of Theorem 2.4 then G is a Greechie hypergraph. The proof is complete.

Theorem 2.4 generalizes the result of R. Greechie (see [3]). Apparently, there is still a chance for further generalizing. Since the questions we wanted to answer are resolved by a direct application of Theorem 2.4, we do not search here for ultimate generalizations. On the other hand, there is a limit of generalizations as the following example shows:

Example 2.5. There is a hypergraph which does not allow a state-isomorphism to a diagram of a logic. Put $A_2 = \{1, 2, 3\}$. Suppose that X_2 consists of all two-element subsets of A_2. Assume that L_2 is a logic which is state-isomorphic to $H_2 = (A_2, X_2)$. The only state on L_2 has to attain the value $1/2$ on each atom and therefore every block must contain exactly two atoms. If an atom belongs to two blocks, it has two distinct complements. So every atom belongs to exactly one block. Such a logic cannot have only one state – a contradiction.

3. LOGICS WITH SPECIAL STATE SPACES

In this part we apply Theorem 2.4 to exhibit examples of logics with preassigned state space properties. Let us start with certain peculiarities.

Example 3.1. There is a logic whose state space is empty.

Let $A_3 = \{1, 2, 3, 4, 5, 6\}$, $X_3 = \{\{1, 2, 3\}, \{4, 5, 6\}, A_3\}$. According to Theorem

2.4, the hypergraph $H_3 = (A_3, X_3)$ is state-isomorphic to a diagram of a logic. Obviously, H_3 admits no states. Note that, following our definition, all logics admitting no states are state-equivalent. (There are constructions of logics with void state space – see [3]. Our example is apparently the simplest one.)

Example 3.2. There is a nontrivial logic whose state space is a singleton. Moreover, we can ensure that the only state on the logic vanishes only at the zero element.

Let A_4 be a 4-element set and X_4 the set of all 3-element subsets of A_4. The only state on the hypergraph $H_4 = (A_4, X_4)$ attains 1/3 on each vertex. There exists a logic L_4 state-isomorphic to H_4. From the properties of the state-isomorphism we immediately obtain that L_4 admits exactly one state. This state attains the value 1/3 on each atom. We note that another example with the required properties is given in Fig. 4. (There are examples of logics admitting exactly one state. In [11], [7] a logic with a single two-valued state is constructed, in [2], the authors construct an example possessing our properties. The construction we offer here seems to be the simplest known.)

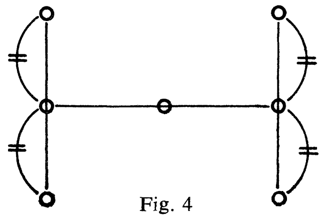


Fig. 4

Other examples are less exotic. The need of such examples has actually been the basic stimulation for our investigations. Recall that a logic is *Jauch-Piron* provided every state s on L fulfils the following condition: If $s(a) = s(b) = 1$ for some $a, b \in L$, then there exists an element $c \in L$ satisfying $c \leqq a, c \leqq b$ and $s(c) = 1$ (cf. [5], [6]). First observe that L_4 is an example of a non-Boolean finite Jauch-Piron logic with a nonvoid state space. Our technique appeared useful also in constructing logics with "rich" state spaces, especially in connection with the questions of the so called unital logics. (A logic is called *unital* if for every non-zero element there is a state evaluating it to 1.) In the course of investigating the Jauch-Piron logics (see [1]) the following question remained open for some time: Does any finite unital logic have to be Boolean if each atom has exactly one state evaluating it to 1? It follows from Proposition 2.1 that this need not be the case. It suffices to take the logic which is state-isomorphic to G_1. (This logic has 18 atoms.)

Let us end this note by exhibiting an example extending the paper [8] and completing the papers [9] and [10]:

Example 3.3. There is a finite unital fully embeddable logic which is not Boolean. Recall that a logic K is called *fully embeddable* if each state on K can be extended over each unital logic L which contains K. For instance, each logic K_n corresponding to the hypergraph G_n used in the proof of Proposition 2.1 has all the properties required in Example 3.3. As we have seen, the logic K_n has the following property: If B is a block of K_n then each state on K_n is completely determined by its values on the atoms of B. This property obviously guarantees that K_n is fully embeddable.

Acknowledgement. I wish to express my gratitude to Pavel Pták and Vladimír Rogalewicz for their valuable suggestions and obliging help during the preparation of this paper.

References

[1] *Bunce, L. J., Navara, M., Pták, P., Wright, J. D. M.:* Quantum logics with Jauch-Piron states, Oxford Quarterly Journal, 1985 (to appear).

[2] *Greechie, R. J., Miller, F. R.:* On structures related to states on an empirical logic I. Weights on finite spaces. Technical Report 14, Dept. of Mathematics, Kansas State University, Manhattan, Kansas, 1970.

[3] *Greechie, R. J.:* Orthomodular lattices admitting no states, J. of Combinatorial Theory *10*, 119—132, 1971.

[4] *Gudder, S. P.:* Stochastic Methods in Quantum Mechanics, North Holland, New York, 1979.

[5] *Jauch, M.:* Foundations of Quantum Mechanics, Addison-Wesley, 1968.

[6] *Piron, C.:* Foundations of Quantum Physics, Benjamin, Reading (Mass.), 1976.

[7] *Pták, P.:* Exotic logics, Colloquium Math., 1985 (to appear).

[8] *Pták, P.:* Extensions of states on logics, to appear.

[9] *Pulmanová, S.:* A note on the extensibility of states, Math. Slovaca *30*, 177—181, 1981.

[10] *Rüttimann, G. T.:* Jauch-Piron states, J. Math. Phys. *18*, 189—193, 1977.

[11] *Shultz, F. W.:* A characterization of state spaces of orthomodular lattices, J. of Combinatorial Theory (A) *17*, 317—328, 1974.

[12] *Varadarajan, V. S.:* Geometry of Quantum Theory, Vol. I, Van Nostrand, Princeton, N.J., 1969.

Author's address: 166 27 Praha 6, Suchbátarova 2, Czechoslovakia (elektrotechnická fakulta ČVUT).