Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 573–577

Persistent URL: http://dml.cz/dmlcz/102252

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CZECHOSLOVAK MATHEMATICAL JOURNAL

Mathematical Institute of the Czechoslovak Academy of Sciences

V. *38* (*113*), PRAHA 30. 12. 1988, No. 4

SOME PROPERTIES OF LATTICE ORDERED REES MATRIX SEMIGROUPS

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(Received April 5, 1986)

The purpose of this note is to give some properties of lattice ordered completely simple Rees matrix Semigroups and to construct such semigroups.

1. INTRODUCTION

As in our previous papers [3], [4], by a lattice ordered semigroup, we mean a semigroup S, on which we can define an order relation $\leqq$ such that

– $(S, \leqq)$ is distributive lattice; $\vee$ and $\wedge$ are the least upper bound and the greatest lower bound.

– $\forall a\ \forall b\ \forall c\ a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)\, a = ba \wedge ca$

– $\forall a\ \forall b\ \forall c\ a(b \vee c) = ab \vee ac$ and $(b \vee c)\, a = ba \vee ca$.

In matter of Rees matrix semigroups, we keep the notations of [1]. In particular, if G is a group, I and Λ are two sets non voids, P denote a fixed matrix, a $\Lambda \times I$ matrix over G, $P = (p_{\lambda i})$. Recall that by a Rees $I \times \Lambda$ matrix over G°, we mean an $I \times \Lambda$ matrix over G° having at most one non zero element. If $a \in G$, $i \in I$ and $\lambda \in \Lambda$, then $(a)_{i\lambda}$ will denote the Rees $I \times \Lambda$ matrix on G° having "a" in the ith row and λth column, its remaining entries being 0. The set of all Rees matrix over G° is a semigroup for the product

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a p_{\lambda j} b)_{i\mu}\,.$$

Moreover, if P is a matrix over G, P regular, then $S = \mathscr{M}^\circ(G; I, \Lambda, P)$ is a completely 0 simple semigroup.

2. PROPERTIES OF LATTICE ORDERED COMPLETELY SIMPLE REES MATRIX SEMIGROUPS

In this paragraph, we consider $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ a Rees matrix semigroup, completely 0-simple, P being a $\Lambda \times I$ matrix over G. Moreover we suppose that this semigroup is lattice ordered, for the order relation $\leqq$.

Proposition 1. *If* $(a)_{i\mu} \leqq (a)_{j\mu}$, *then* $(x)_{i\mu} \leqq (x)_{j\mu}$ *for all* x *of* G. *If* $(a)_{i\lambda} \leqq (a)_{i\mu}$, *then* $(y)_{i\lambda} \leqq (y)_{i\mu}$ *for all* y *of* G.

For the proof, we use the following identities

(1) $(a)_{i\mu} = (ab^{-1}p_{\lambda j}^{-1})_{i\lambda} \circ (b)_{j\mu}$ for all λ,

(2) $(a)_{i\lambda} = (b)_{i\mu} \circ (p_{\mu j}^{-1} b^{-1} a)_{j\lambda}$ for all μ.

Suppose $(a)_{i\mu} \leqq (a)_{j\mu}$: then $(a)_{i\mu} \vee (a)_{j\mu} = (a)_{j\mu}$. Using (1), we can write

$$(a)_{i\mu} = (p_{\lambda j}^{-1})_{i\lambda} \circ (a)_{j\mu}\,, \quad (a)_{j\mu} = (p_{\lambda j}^{-1})_{j\lambda} \circ (a)_{j\mu}\,,$$

and $(a)_{i\mu} \vee (a)_{j\mu} = [(p_{\lambda j}^{-1})_{i\lambda} \vee (p_j^{-1})_{j\lambda}] \circ (a)_{j\mu} = (a)_{j\mu}$. The element between the brackets is necessarly of the form $(c)_{j\nu}$ and depends only of i and j. So, we have $(c)_{j\nu} \circ (a)_{j\mu} = (a)_{j\mu}$, and $cp_{\nu j}a = a$. Consequently, $c = p_{\nu j}^{-1}$ and $(p_{\lambda j}^{-1})_{i\lambda} \vee (j_{\lambda j}^{-1})_{j\lambda} = (p_{\nu j}^{-1})_{j\nu}$. Therefore, if now, we calculate $(x)_{i\mu} \vee (x)_{j\mu}$, we find:

$$(x)_{i\mu} \vee (x)_{j\mu} = (p_{\nu j}^{-1})_{j\nu} \circ (x)_{j\mu} = (x)_{j\mu}\,. \quad \text{Q.E.D.}$$

Proposition 2. *Each* $\mathscr{H}$*-class of* S *is a lattice ordered group, subsemigroup of* S *and sublattice of* S. *Moreover, all* $\mathscr{H}$*-classes are isomorphic and* G *can be considered as a subgroup and a sublattice ordered group of* $S = \mathscr{M}^\circ(G; I, \Lambda; P)$.

We know [1], that the $\mathscr{H}$-class are of the form $H_{i,\lambda} = \{(a)_{i,\lambda};\ a \in G\}$ and that they are isomorphic to the group G. They are sublattices of G. Effectively, if we calculate $(a)_{i\lambda} \vee (b)_{i\lambda}$, we have from (1)

$$(b)_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda} \circ b_{i\lambda} \quad \text{and} \quad (a)_{i\lambda} = (ab^{-1}p_{\lambda i}^{-1})_{i\lambda} \circ (b)_{i\lambda}\,.$$

Therefore $(a)_{i\lambda} \vee (b)_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda} \vee (ab^{-1}p_{\lambda i}^{-1})_{i\lambda}] \circ (b)_{i\lambda}$ and so $(a)_{i\lambda} \vee (b)_{i\lambda}$ is equivalent mod $\mathscr{L}$ to $(b)_{i\lambda}$. Symmetrically, we can prove that $(a)_{i\lambda} \vee (b)_{i\lambda}$ is equivalent mod $\mathscr{R}$ to $(b)_{i\lambda}$ and $(a)_{i\lambda} \vee (b)_{i\lambda}$ belongs to the same $\mathscr{H}$-class, $H_{i\lambda}$. Finally, as we go from H to H' by multiplications, because $H' = u \circ H \circ v$, $H = u' \circ H' \circ v'$ and as the products keeps the inequalities, all the $\mathscr{H}$-clases are isomorphic as lattices. We can restrict the order of S to G and we can put, in G

$$\begin{aligned} a \leqq b &\Leftrightarrow \exists i \in I\,,\ \ \exists \lambda \in \Lambda \quad (a)_{i\lambda} \leqq (b)_{i\lambda} \\ &\Leftrightarrow \forall i \in I\,,\ \ \forall \lambda \in \Lambda \quad (a)_{i\lambda} \leqq (b)_{i\lambda}\,. \end{aligned}$$

In the sequel, G well be ordered by this relation.

Proposition 3. *If* S *is lattice ordered, we have*

$$\begin{cases} (a)_{i\mu} \leqq (a)_{j\mu} \Rightarrow p_{\nu i} \leqq p_{\nu j} & \textit{in } G \textit{ for all } \nu \textit{ of } \Lambda \\ (a)_{i\mu} \leqq (a)_{i\lambda} \Rightarrow p_{\mu j} \leqq p_{\lambda j} & \textit{in } G \textit{ for all } j \textit{ of } I\,. \end{cases}$$

We prove only the first implication. From

$$(a)_{i\mu} \leqq (a)_{j\mu}\,, \quad \text{we can deduce that for all } (c)_{k,\nu}$$

we have $(c)_{k\nu} \circ (a)_{i\mu} \leqq (c)_{k\nu} \circ (a)_{j\mu}$ i.e.

$(cp_{\nu i}a)_{k\mu} \leqq (cp_{\nu j}a)_{k\mu}$. As the order relation on S is an extension of this on G, (Prop. 2), we have

$$cp_{\nu i}a \leqq cp_{\nu j}a \text{ in } G, \text{ and so we have}$$

$p_{\nu i} \leqq p_{\nu j}$ by simplification in G.

Proposition 4. *If S is lattice ordered, we have: for all λ of Λ, $(a)_{i\lambda} \vee (a)_{j\lambda}$ is an element of the form $(b)_{k\lambda}$, and for all i of I, $(a)_{i\mu} \vee (a)_{i\nu}$ is of the form $(c)_{i\varrho}$.*

We have seen, in the proof of Proposition 1, that $(a)_{i\lambda} \vee (a)_{j\lambda} = [(p_{\lambda j}^{-1})_{i\lambda} \vee \vee (p_{\lambda j}^{-1})_{j\lambda}] \circ (a)_{j\lambda}$, and necessarly this least element is of the form $(b)_{k\lambda}$. Q.E.D.

We know that all the Rees matrix semigroups can be constructed from a normalized sandwich matrix [1]. (A normalized matrix is a matrix in which all the elements of the λ_0th row and of the i_0th column are equal to 1, if 1 is the neutral element of G.

Now, we note $(*)$ the following condition

$(*)$ $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ where P is normalized.

Proposition 5. *If S is lattice ordered, and if the condition $(*)$ is satisfied, we have*

$$\forall i \ \forall j \ \exists k \text{ such that } \forall a \ \forall \mu \ (a)_{i\mu} \vee (a)_{j\mu} = (a)_{k\mu}$$

$$\forall \mu \ \forall \nu \ \exists \lambda \text{ such that } \forall a \ \forall i \ (a)_{i\mu} \vee (a)_{i\nu} = (a)_{i\lambda}$$

(so k depends only of i and j, and similarly λ depends only of μ and ν).

Proposition 5'. *Proposition Analogous to Proposition 5, where we replace $\vee$ by $\wedge$.*

Proof of Prop. 5. First, we show that

$$(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{k\lambda_0}\,.$$

From the Proposition 4, we deduce that $(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (c)_{k\lambda_0}$; we multiply this last equality on the left by $(1)_{k\lambda_0}$. We obtain:

$$(1)_{k\lambda_0} \circ [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] = (1p_{\lambda_0 i}1)_{k\lambda_0} \vee (1p_{\lambda_0 j}1)_{k\lambda_0} =$$
$$= (1)_{k\lambda_0} \ [\text{because } p_{\lambda_0 i} = p_{\lambda_0 j} = 1] =$$
$$= (1)k_{\lambda_0} \circ (c)_{k\lambda_0} = (1p_{\lambda_0 k}c)_{k\lambda_0} = (c)_{k\lambda_0}\,(p_{\lambda_0 k} = 1)\,.$$

Therefore $c = 1$.

Now, we calculate $a_{i\mu} \vee a_{j\mu}$; we can write this element (see Proposition 1) $[(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda i}^{-1})_{j\lambda}] \circ a_{i\mu}$ for all λ of Λ, and in particular for λ_0. So we have

$$(a)_{i\mu} \vee (a)_{j\mu} = [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] \circ a_{i\mu} = (1)_{k\lambda_0} \circ (a)_{i\mu} = (a)_{k\mu}\,.$$

We prove the second equality similarly by using the fact that $p_{\lambda_i} = 1$ for all λ of Λ.

Proposition 6. *If S is lattice ordered, and if the condition $(*)$ is satisfied, I and Λ are distributive lattice ordered sets.*

We prove this proposition only for the set I. We define on I two binary relations:

$$I \times I \to^f I$$

$$(i,j) \mapsto f(i,j) \quad \text{such that} \quad (1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0}$$

and

$$I \times I \to^g I$$

$$(i,j) \mapsto g(i,j) \quad \text{such that} \quad (1)_{i\lambda_0} \wedge (1)_{j\lambda_0} = (1)_{g(i,j),\lambda_0}\,.$$

We know (Prop. 5 and 5′) that this two binary relations are well defined from $I \times I$ on I.

It is trivial to see that $f(i,j) = f(j,i), f(i,i) = i$. Now, we prove the associativity:

$$\text{if} \quad (1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0} \quad \text{then}$$

$$[(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] \vee (1)_{k\lambda_0} = (1)_{f(i,j),\lambda_0} \vee (1)_{k\lambda_0} = (1)_{f[f(i,j),k],\lambda_0} =$$

$$= (1)_{i\lambda_0} \vee [(1)_{j\lambda_0} \vee (1)_{k\lambda_0}] = (1)_{i\lambda_0} \vee (1)_{f(i,j),\lambda_0} = (1)_{f[i,f(j,k)],\lambda_0}\,.$$

And finally, $f[f(i,j),k] = f[i,f(j,k)]$. Evidently, we have the same properties for g.

There is absorption between f and g. By example, we show that $g(i,f(i,j)) = i$:

$$(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0}$$

$$(1)_{i\lambda_0} \wedge [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] = (1)_{i\lambda_0} \wedge (1)_{f(i,j),\lambda_0} = (1)_{g[i,f(i,j)],\lambda_0} = (1)_{i\lambda_0}\,.$$

And so we have the asked result. Similarly, $f(i,g(i,j)) = i$. From the preceding results, we can affirm that I and J are lattice ordered sets and that if we note, as usually, $\leqq$ the order relation on I and S we have

P_1 $i \leqq j \Leftrightarrow f(i,j) = j \Leftrightarrow g(i,j) = i$,
P_2 $i \leqq j \Leftrightarrow (1)_{i\lambda_0} \leqq (1)_{j\lambda_0}$,
P_3 $i \leqq j \Leftrightarrow (a)_{i\mu} \leqq (a)_{j\mu}$ for one a of G and one μ of Λ,
P'_3 $i \leqq j \Leftrightarrow (a)_{i\mu} \leqq (a)_{j\mu}$ for any a of G and any μ of Λ,
P_4 $\lambda \leqq \mu \Leftrightarrow (1)_{i_0\lambda} \leqq (1)_{i_0\mu}$,
P_5 $\lambda \leqq \mu \Leftrightarrow (a)_{i\lambda} \leqq (a)_{i\mu}$ for one a of G and one i of I,
P'_5 $\lambda \leqq \mu \Leftrightarrow (a)_{i\lambda} \leqq (a)_{i\mu}$ for any a of G and any i of I.

P_1 is true by definition of $f(i,j)$ and of $g(i,j)$ which define the least upper bound and the greatest lower bound in I.

P_2 is true by definition of $f(i,j)$ and of $g(i,j)$. From the propositions 5 and 5′ we deduce easily P_3 and P'_3; $\mathrm{P}_4, \mathrm{P}_5, \mathrm{P}'_5$ are the analogous (for Λ) of $\mathrm{P}_2, \mathrm{P}_3, \mathrm{P}'_3$.

Finally, it is trivial to see that under this order relation I and Λ are distributive lattices.

Theorem. *If $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ is a lattice ordered completely simple semigroup, where P is a normalized sandwich matrix, then I and Λ are distributive lattice ordered sets and we can construct an isotone application φ of $\Lambda \times I$ in G, defined by $\varphi(\lambda, i) = p_{\lambda_i}$.*

$\Lambda \times I$ is a lattice ordered set under the relation $(\lambda, i) \leqq (\mu, j) \Leftrightarrow \lambda \leqq \mu$ in Λ and $i \leqq j$ in I, since Λ and I are lattice ordered by proposition 6. Therefore, if $(\lambda, i) \leqq \leqq (\mu, j)$, we have $\lambda \leqq \mu$ and $i \leqq j$. Hence $(a)_{k\lambda} \leqq (a)_{k\mu}$ for any k of I and $(a)_{i\nu} \leqq \leqq (a)_{j\nu}$ for any ν of Λ and this for any a of G. Let us now use Proposition 3; we obtain

$$p_{\nu i} \leqq p_{\nu j} \quad \text{for any } \nu \text{ of } \Lambda \text{ and} \quad p_\lambda k \leqq p_\mu k \quad \text{for any } k \text{ of } I\,.$$

Finally (putting $k = i$, $\nu = \mu$) we obtain $p_{\lambda i} \leqq p_{\mu j}$ et so φ is isotone.

CONSTRUCTION OF A LATTICE ORDERED REES MATRIX SEMIGROUPS

Let G be a lattice ordered group; let Λ and I be two distributive lattices. Then we can construct a Rees matrix semigroup $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ where P is a regular sandwich matrix whose entries belong to G and are all equal to 1, neutral element of G: $p_{\lambda i} = 1$ for all λ of Λ and all i of J.

We define $\leqq$ an order relation on S by

$$(a)_{i\lambda} \leqq (b)_{j\mu} \Leftrightarrow \lambda \leqq \mu \text{ in } \Lambda\,, \quad i \leqq j \text{ in } I \text{ and } a \leqq b \text{ in } G$$

(product order of I, Λ and G).

Therefore, S is a lattice, a distributive lattice as I, Λ and G. (Recall that a lattice ordered group is always a distributive lattice [2]).

But, S is also a lattice ordered semigroup. We have

$$(a)_{i\lambda} \vee (b)_{j\mu} = (a \vee b)_{i \vee j, \lambda \vee \mu} \quad \text{and} \quad (a)_{i\lambda} \circ (b)_{j\mu} = (ab)_{i\mu}\,.$$

So

$$(x)_{k\nu} \circ [a_{i\lambda} \vee b_{j\mu}] = x_{k\nu} \circ [(a \vee b)_{i \vee j, \lambda \vee \mu}] = [x(a \vee b)]_{k, \lambda \vee \mu}\,,$$

$$(x)_{k\nu} \circ (a_i\lambda) \vee (a)_{k\nu} \circ (b_{j\mu}) = (xa)_{k\lambda} \vee (xb)_{k\mu} = x(a \vee b)_{k, \lambda \vee \mu}\,.$$

By an argument of symmetry, we have also

$$(a_{i\lambda} \vee b_{j\mu}) \circ (x)_{k\nu} = (a_{i\lambda} \circ x_{k\nu}) \vee (b_{j\mu} \circ x_{k\nu})$$

and also the same equalities by replacing $\vee$ by $\wedge$.

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