Czechoslovak Mathematical Journal

Irena Rachůnková

Existence and uniqueness of solutions of four-point boundary value problems for 2nd order differential equations

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 4, 692–700

Persistent URL: http://dml.cz/dmlcz/102346

Terms of use:

© Institute of Mathematics AS CR, 1989

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FOUR-POINT BOUNDARY VALUE PROBLEMS FOR 2ND ORDER DIFFERENTIAL EQUATIONS

IRENA RACHŮNKOVÁ, Olomouc

(Received December 22, 1987)

The three-point boundary value problems for differential equations of the second order were studied in [1], [2], [9] and [11]. The problem of existence of solutions of the equation

$$u'' = f(t, u)$$

satisfying the conditions

$$u(0) = u(a) = u(2a), \quad a \in (-\infty, +\infty)$$

is solved in [1], [2]. Theorems on existence and uniqueness of solutions of the problem

$$u'' = f(t, u, u'),$$

$$u(a) = c_1, \quad u(b) = u(t_0) + c_2, \quad a, b, t_0, \quad c_1, c_2 \in (-\infty, +\infty),$$

$a < t_0 < b$, are proved in [11] and, for the linear differential equation, in [9].

1. Our paper deals with the problem of existence and uniqueness of solutions of the equation

$$u'' = f(t, u, u') \tag{1.1}$$

defined on an interval $[a, b]$ and satisfying the conditions

$$u(c) - u(a) = A, \quad u(b) - u(d) = B, \tag{1.2}$$

where $A, B \in (-\infty, +\infty)$, $-\infty < a < c < d < b < +\infty$. Sufficient conditions for the existence of solutions of the problem (1.1), (1.2) were found in [12]. Other existence theorems for this problem are proved here. Moreover, the problem of uniqueness is solved.

We shall use the following notation:

$$R = (-\infty, +\infty), \quad R_+ = [0, +\infty), \quad D = [a, b] \times R^2, \quad D_+ = [a, b] \times R_+^2.$$

$$\tau = \begin{cases} \max\{c - a, b - c\} & \text{for} \quad d - a > b - c \\ \max\{d - a, b - d\} & \text{for} \quad d - a \leqq b - c, \end{cases}$$

$$g_0(t) = \alpha t^2 + \beta t + \gamma,$$

where

$$\alpha = (B/(b-d) - A/(c-a))(b-c+d-a)^{-1},$$

$$\beta = (A(b+d)/(c-a) - B(c+a)/(b-d))(b-c+d-a)^{-1}, \quad \gamma \in R,$$

$$r_0 = \max\{|g_0(t)|: a \leqq t \leqq b\}, \quad r_1 = \max\{|g_0'(t)|: a \leqq t \leqq b\}.$$

$AC^1(a, b)$ is the set of all real functions which are absolutely continuous together with their first derivatives on $[a, b]$.

$\mathrm{Car}_{\mathrm{loc}}(D)$ is the set of all real functions satisfying the local Carathéodory conditions on D, i.e. $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ iff

$f(\cdot, x, y)$: $[a, b] \to R$ is measurable for every $(x, y) \in R^2$,
$f(t, \cdot, \cdot)$: $R^2 \to R$ is continuous for almost every $t \in [a, b]$,
$\sup\{|f(\cdot, x, y)|: |x| + |y| \leqq \varrho\} \in L(a, b)$ for any $\varrho \in (0, +\infty)$.

Definition. A function $u \in AC^1(a, b)$ which fulfils (1.1) for almost every $t \in [a, b]$ will be called a *solution of the equation* (1.1). Each solution of (1.1) which satisfies the conditions (1.2) will be called a *solution of the problem* (1.1), (1.2).

In the whole paper we suppose that $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ and $\lambda \in \{-1, 1\}$.

Theorem 1. *Let there exist $r \in (0, +\infty)$ such that one the set D the inequalities*

$$\lambda(f(t, x, y) - 2\alpha)\operatorname{sgn} x \geqq 0 \quad \textit{for} \quad |x| > r \tag{1.3}$$

and

$$|f(t, x, y)| \leqq h_1(t)|x| + h_2(t)|y| + \omega(t, |x| + |y|) \tag{1.4}$$

hold, where $h_1, h_2 \in L^2(a, b)$ are non-negative functions satisfying

$$(b-a)^{1/2}\left(\left(\int_a^b h_1^2(t)\,\mathrm{d}t\right)^{1/2} 2(b-a)/\pi + \left(\int_a^b h_2^2(t)\,\mathrm{d}t\right)^{1/2}\right) < 1 \tag{1.5}$$

and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ is a non-negative function, non-decreasing with respect to its second variable and satisfying the condition

$$\lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho)\,\mathrm{d}t = 0. \tag{1.6}$$

Then the problem (1.1), (1.2) *is solvable.*

Theorem 2. *Let $a_1, a_2 \in (0, +\infty)$ satisfy*

$$a_1(2/\pi)^2 \tau(b-a) + a_2(2/\pi)\tau < 1 \tag{1.7}$$

and let there exist $h_1, h_2 \in L(a, b)$ such that

$$0 < \lambda h_1(t) \leqq a_1, \quad |h_2(t)| \leqq a_2 \quad \textit{for} \quad a < t < b \tag{1.8}$$

and on the set D the inequality

$$|f(t, x, y) - h_1(t)x - h_2(t)y| \leqq \omega(t, |x| + |y|) \tag{1.9}$$

is fulfilled, where ω is the function from Theorem 1.

Then the problem (1.1), (1.2) *is solvable.*

Note. The inequality $0 < \lambda h_1(t)$ cannot be replaced by the inequality $0 \leqq \leqq \lambda h_1(t)$ because the problem $u'' = 1$, $u(0) = u(1/2) = u(1)$ has no solution.

Theorem 3. *Let there exist a non-negative function $h \in L(a, b)$ such that on the set D the inequality*

$$(1.10) \quad f(t, x_1, y_1) - f(t, x_2, y_2) + h(t)\,|y_1 - y_2| > 0\,, \quad \textit{where} \quad x_1 > x_2\,,$$

is satisfied.

Then the problem (1.1), (1.2) *does not have more than one solution.*

2. Lemmas. **Lemma 1** ([6], Theorem 256, p. 219). If $f \in AC(t_1, t_2)$, $f' \in L^2(t_1, t_2)$ *and* $f(t_0) = 0$, *where* $-\infty < t_1 < t_2 < +\infty$, $t_0 \in [t_1, t_2]$, *then*

$$\int_{t_1}^{t_2} f^2(t)\,\mathrm{d}t \leqq (2(t_2 - t_1)/\pi)^2 \int_{t_1}^{t_2} f'^2(t)\,\mathrm{d}t\,.$$

Lemma 2. *Let $a_1, a_2 \in (0, +\infty)$ satisfy* (1.7) *and let $h_1, h_2 \in L(a, b)$ satisfy* (1.8). *Then the problem*

$$(2.1) \qquad v'' = h_1(t)\,v + h_2(t)\,v'\,,$$

$$(2.2) \qquad v(c) - v(a) = 0\,, \quad v(b) - v(d) = 0$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.1), (2.2). The equation (2.1) can be written in the form

$$(2.3) \quad \exp\left(\int_a^t h_2(s)\,\mathrm{d}s\right)\left(\exp\left(-\int_a^t h_2(s)\,\mathrm{d}s\right) v'(t)\right)' - h_1(t)\,v(t) = 0 \quad \text{for} \quad a \leqq t \leqq b\,.$$

By (2.2) there exist $t_1 \in (a, c)$, $t_2 \in (d, b)$ such that

$$v'(t_1) = v'(t_2) = 0\,.$$

Consequently, the function $\varphi(t) = \exp\left(-\int_a^t h_2(s)\,\mathrm{d}s\right) v'(t)$ has two zeros on (a, b). Let $v(t) \neq 0$ for $a \leqq t \leqq b$. Then (1.8) and (2.3) imply that φ is strictly monotonous on $[a, b]$ and we get a contradiction. Therefore there exists $t_0 \in (a, b)$ such that $v(t_0) = 0$. By Lemma 1 we have

$$\left(\int_a^b v'^2(t)\,\mathrm{d}t\right)^{1/2} \leqq (2\tau/\pi)\left(\int_a^b v''^2(t)\,\mathrm{d}t\right)^{1/2}$$

and

$$\left(\int_a^b v^2(t)\,\mathrm{d}t\right)^{1/2} \leqq (2/\pi)^2\,\tau(b - a)\left(\int_a^b v''^2(t)\,\mathrm{d}t\right)^{1/2},$$

and by virtue of (2.1), the inequality

$$\left(\int_a^b v''^2(t)\,\mathrm{d}t\right)^{1/2} \leqq \left(a_1(2/\pi)^2\,\tau(b - a) + a_2\,2\tau/\pi\right)\left(\int_a^b v''^2(t)\,\mathrm{d}t\right)^{1/2}$$

is true. Since (1.7), we get $\left(\int_a^b v''^2(t)\,\mathrm{d}t\right)^{1/2} = 0$ and thus $\left(\int_a^b v^2(t)\,\mathrm{d}t\right)^{1/2} = 0$.

Lemma 3. *Let $g \in \mathrm{Car}_{\mathrm{loc}}(D)$, $h_1, h_2 \in L(a, b)$ and let the problem* (2.1), (2.2) *have only the trivial solution. If there exists $g^* \in L(a, b)$ such that*

$$|g(t, x, y)| \leqq g^*(t) \quad \textit{on} \quad D\,,$$

then the problem

$$v'' = h_1(t)\, v + h_2(t)\, v' + g(t, v, v')\,, \quad (2.2)$$

is solvable

Proof. See [8], Theorem 2.4, p. 25.

Lemma 4. *Let $a_1, a_2, b_1, b_2 \in L(a, b)$ and for any $h_1, h_2 \in L(a, b)$ satisfying*

$$a_i(t) \leqq h_i(t) \leqq b_i(t) \quad \text{for} \quad a \leqq t \leqq b\,, \quad i = 1, 2\,, \tag{2.4}$$

let the problem (2.1), (2.2) *have only the trivial solution.*

Then there exists $\gamma \in (0, +\infty)$ such that for any $h_1, h_2 \in L(a, b)$ satisfying (2.4) *the inequality*

$$\left|\frac{\partial G(t, s)}{\partial t}\right| + |G(t, s)| \leqq \gamma\,, \quad a \leqq t\,, \quad s \leqq b \tag{2.5}$$

is fulfilled, where G is Green's function of the problem (2.1), (2.2).

Proof. See [8], Lemma 2.2, p. 12.

3. Lemmas for a priori estimates. Lemma 5. *Let $r \in (0, +\infty)$,* let $h_1, h_2 \in L^2(a, b)$ *be non-negative functions satisfying* (1.5) *and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ a non-negative function, non-decreasing with respect to its second variable and satisfying* (1.6). *Then there exists $r^* \in (r, +\infty)$ such that for any function $v \in AC^1(a, b)$ the conditions*

$$v(a) = v(c)\,, \quad v(d) = v(b)\,, \tag{3.1}$$

$$\lambda\, v''(t) \operatorname{sgn} v(t) > 0 \quad \text{for} \quad |v(t)| > r\,, \quad t \in [a, b]\,, \tag{3.2}$$

$$|v''(t)| \leqq h_1(t)\, |v(t)| + h_2(t)\, |v'(t)| + \omega(t, |v| + |v'|)\,, \quad a \leqq t \leqq b \tag{3.3}$$

imply the estimate

$$|v(t)| + |v'(t)| \leqq r^* \quad \text{for} \quad a \leqq t \leqq b\,. \tag{3.4}$$

Proof. The condition (3.1) implies the existence of $t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$. If $|v(t)| > r$ on (a, b), then by (3.2), v' has to be strictly monotonous on (a, b) and we get a contradiction. Therefore there exists $t_0 \in (a, b)$ such that $v(t_0) = c_0$, where $|c_0| \leqq r$. Put $y(t) = v(t) - c_0$ for $a \leqq t \leqq b$. By virtue of (3.3), $|y''(t)| \leqq h_1(t)\, |y(t)| + h_2(t)\, |y'(t)| + \omega(t, |v| + |v'|) + h_1(t)\, r$ for $a \leqq t \leqq b$. Integrating the last inequality from t to t_1 and applying the Hölder inequality, we get

$$|y'(t)| \leqq \left(\int_a^b h_1^2(s)\, \mathrm{d}s\right)^{1/2} \left(\int_a^b y^2(s)\, \mathrm{d}s\right)^{1/2} + \left(\int_a^b h_2^2(s)\, \mathrm{d}s\right)^{1/2} \left(\int_a^b y'^2(s)\, \mathrm{d}s\right)^{1/2} + \\ + \int_a^b (\omega(s, |v| + |v'|) + h_1(s)\, r)\, \mathrm{d}s\,, \quad a \leqq t \leqq b\,.$$

Put $\varrho_0 = \max\{|y'(t)|: a \leqq t \leqq b\}$. Then $|y(t)| \leqq (b - a)\, \varrho_0$ and we get

$$\varrho_0 \leqq \left(\int_a^b h_1^2(s)\, \mathrm{d}s\right)^{1/2} \left(\int_a^b y^2(s)\, \mathrm{d}s\right)^{1/2} + \left(\int_a^b h_2^2(s)\, \mathrm{d}s\right)^{1/2} \left(\int_a^b y'^2(s)\, \mathrm{d}s\right)^{1/2} + \\ + \int_a^b (\omega(s, r + \varrho_0(1 + b - a)) + h_1(s)\, r)\, \mathrm{d}s\,.$$

By Lemma 1, we obtain

$$(\int_a^b y^2(s)\,ds)^{1/2} \leqq (2(b-a)/\pi)\,(\int_a^b y'^2(s)\,ds)^{1/2}$$

and since

$$(\int_a^b y'^2(s)\,ds)^{1/2} \leqq \varrho_0(b-a)^{1/2}\,,$$

we have

$$\varrho_0 \leqq [(\int_a^b h_1^2(s)\,ds)^{1/2}\,2(b-a)^{3/2}/\pi + (\int_a^b h_2^2(s)\,ds)^{1/2}\,(b-a)^{1/2}]\,\varrho_0 + \\ + \int_a^b (\omega(s, r + \varrho_0(1+b-a)) + h_1(s)\,r)\,ds\,.$$

In view of (1.5) and (1.6) there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$[(\int_a^b h_1^2(s)\,ds)^{1/2}\,2(b-a)^{3/2}/\pi + (\int_a^b h_2^2(s)\,ds)^{1/2}\,(b-a)^{1/2}]\,\varrho + \\ + \int_a^b (\omega(s, r + \varrho(1+b-a)) + h_1(s)\,r)\,ds < \varrho$$

is satisfied. Consequently $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + \varrho^*(b-a+1)\,,$$

we get the estimate (3.4).

Lemma 6. *Let $a_1, a_2 \in (0, +\infty)$ satisfy* (1.7), *let $h_1, h_2 \in L(a, b)$ satisfy* (1.8) *and let $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ be a non-negative function, non-decreasing with respect to its second variable and satisfying* (1.6). *Then there exists $r^* \in (r, +\infty)$ such that for any function $v \in AC^1(a, b)$ the conditions* (3.1) *and*

$$|v'' - h_1(t)\,v - h_2(t)\,v'| \leqq \omega(t, |v| + |v'|)\,, \quad a \leqq t \leqq b \tag{3.5}$$

imply the estimate (3.4).

Proof. Put $h_0(t) = v''(t) - h_1(t)\,v(t) - h_2(t)\,v'(t)$ for $a \leqq t \leqq b$ and consider the equation

$$v'' = h_1(t)\,v + h_2(t)\,v' + h_0(t)\,. \tag{3.6}$$

Since h_1, h_2 satisfy the conditions of Lemma 2, the problem (2.1), (2.2) has only the trivial solution. Consequently, by Lemma 4, there exists $\gamma \in (0, +\infty)$ such that Green's function G for the problem (2.1), (2.2) fulfils the estimate (2.5). Therefore the solution

$$v(t) = \int_a^b G(t, s)\,h_0(s)\,ds$$

of the problem (3.6), (2.2) satisfies the estimate

$$|v(t)| + |v'(t)| \leqq \gamma \int_a^b \omega(s, |v| + |v'|)\,ds \quad \text{for} \quad a \leqq t \leqq b\,.$$

Putting $\max\{|v(t)| + |v'(t)|: a \leqq t \leqq b\} = \varrho_0$, we get

$$\varrho_0 \leqq \int_a^b \omega(t, \varrho_0)\,dt\,.$$

It follows from (1.6) that there exists $r^* > 0$ such that

$$\gamma \int_a^b \omega(t, \varrho)\,dt < \varrho \quad \text{for any} \quad \varrho > r^*\,.$$

Therefore $\varrho_0 \leqq r^*$ and Lemma 6 is proved.

4. Proofs of Theorems.

Proof of Theorem 1. Let $\varepsilon_0 \in (0, +\infty)$ satisfy

$$(4.1) \qquad (b-a)^{1/2}\left[\left(\int_a^b (h_1(t)+\varepsilon_0)^2\,dt\right)^{1/2} 2(b-a)/\pi + \left(\int_a^b h_2^2(t)\,dt\right)^{1/2}\right] < 1$$

and let r^* be the constant constructed by means of Lemma 5 for the functions $h_1(t) + \varepsilon_0$, $h_2(t)$ and $\tilde{\omega}(t, s) = \omega(t, s + r_0 + r_1) + h_1(t)\, r_0 + h_2(t)\, r_1 + 2|\alpha|$ and for the constant $\tilde{r} = r + r_0$. Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq r^* \\ 2 - s/r^* & \text{for } r^* < s < 2r^* \\ 0 & \text{for } s \geqq 2r^*, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t),\ y + g_0'(t)) - 2\alpha\,,$$

$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$(4.2) \qquad v'' = \lambda\varepsilon v + \tilde{g}(t, v, v')\,, \quad \varepsilon \in (0, \varepsilon_0]\,.$$

Since ε satisfies the assumptions of Lemma 2, the problem

$$v'' = \lambda\varepsilon v\,, \quad (2.2)$$

has only the trivial solution. Consequently, by Lemma 3, the problem (4.2), (2.2) has a solution v.

Clearly v satisfies (3.1). Now, let $v(t) > \tilde{r}$ for some $t \in [a, b]$. Then $v(t) + g_0(t) > r$ and

$$\lambda\, v''(t) = \lambda\chi(r^*, |v| + |v'|)\,(f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha) + \varepsilon\, v(t) > 0\,.$$

Analogously, if $v(t) < -\tilde{r}$, then $v(t) + g_0(t) < -r$ and $\lambda\, v''(t) < 0$. Consequently, v satisfies (3.2) with the constant $\tilde{r}$. Further,

$$\begin{aligned} |v''(t)| &\leqq |f(t, v + g_0, v' + g_0') - 2\alpha| + \varepsilon|v| \leqq \\ &\leqq h_1(t)\,(|v| + r_0) + h_2(t)\,(|v'| + r_1) + \\ &\quad + 2|\alpha| + \varepsilon_0|v| + \omega(t, |v| + r_0 + |v'| + r_1) = \\ &= (h_1(t) + \varepsilon_0)\,|v| + h_2(t)\,|v'| + \tilde{\omega}(t, |v| + |v'|) \quad \text{for} \quad a \leqq t \leqq b\,. \end{aligned}$$

It follows from (4.1) that the functions $h_1 + \varepsilon_0$, h_2 satisfy (1.5). Since ω satisfies (1.6), there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the conditions $r_0 + r_1 + \varrho \leqq 2\varrho$ and

$$\lim_{\varrho \to +\infty} \frac{1}{2\varrho} \int_a^b (\omega(t, 2\varrho) + h_1(t)\, r_0 + h_2(t)\, r_1 + 2|\alpha|)\,dt = 0$$

are fulfilled. Therefore $\tilde{\omega}$ satisfies (1.6) and, by Lemma 5, the estimate (3.4) is valid. Thus v is a solution of the equation

$$v'' = \lambda\varepsilon v + g(t, v, v')$$

and $u = v + g_0$ is a solution of the equation

$$u'' = \lambda\varepsilon(u - g_0(t)) + f(t, u, u') \tag{4.3}$$

and satisfies the conditions (1.2).

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ there exists a solution u_ε of the problem (4.3), (1.2) satisfying the estimate

$$|u_\varepsilon| + |u'_\varepsilon| \leqq r^* + r_0 + r_1 \quad \text{for} \quad a \leqq t \leqq b\,.$$

It follows that all functions of the set $\{u_\varepsilon\colon \varepsilon \in (0, \varepsilon_0]\}$ are uniformly bounded together with their derivatives and so also equi-continuous on $[a, b]$. Therefore, by the Arzelà-Ascoli lemma there exist a sequence $(\varepsilon_k)_{k=1}^\infty$, $\varepsilon_k \to 0$ for $k \to \infty$, and a sequence $(u_{\varepsilon_k})_{k=1}^\infty$ uniformly convering together with $(u'_{\varepsilon_k})_{k=1}^\infty$ on $[a, b]$, such that $u_0(t) = \lim_{k\to+\infty} u_{\varepsilon_k}(t)$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 2. Let r^* be the constant constructed by means of Lemma 6 for the function

$$\tilde{\omega}(t, s) = \omega(t, s + r_0 + r_1) + |h_1(t)|\, r_0 + |h_2(t)|\, r_1 + 2|\alpha|\,.$$

Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq r^* \\ 2 - s/r^* & \text{for} \quad r^* < s < 2r^* \\ 0 & \text{for} \quad s \geqq 2r^*\,, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t), y + g'_0(t)) - 2\alpha - h_1(t)\, x - h_2(t)\, y\,,$$

$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$v'' = h_1(t)\, v + h_2(t)\, v' + \tilde{g}(t, v, v')\,. \tag{4.4}$$

By Lemma 2, the problem (2.1), (2.2) has only the trivial solution. Consequently, by Lemma 3, the problem (4.4), (4.2) has a solution v. Now (1.9) implies

$$\begin{aligned} |v'' - h_1(t)\, v - h_2(t)\, v'| &\leqq |f(t, v + g_0, v' + g'_0) - h_1(t)\, v - h_2(t)\, v' - 2\alpha| \leqq \\ &\leqq |f(t, v + g_0, v' + g'_0) - h_1(t)\,(v + g_0(t)) - h_2(t)\,(v' + g'_0)| + \\ &+ |h_1(t)|\, r_0 + |h_2(t)|\, r_1 + 2|\alpha| \leqq \omega(t, |v + g_0| + |v' + g'_0|) + |h_1(t)|\, r_0 + \\ &+ |h_2(t)|\, r_1 + 2|\alpha| \leqq \tilde{\omega}(t, |v| + |v'|) \quad \text{for} \quad a \leqq t \leqq b\,. \end{aligned}$$

In the same way as in the proof of Theorem 1 we can show that $\tilde{\omega}$ satisfies (1.6). Consequently, by Lemma 6, the estimate (3.4) is valid and v is a solution of the equation

$$v'' = h_1(t)\, v + h_2(t)\, v' + g(t, v, v')\,.$$

Therefore $u = v + g_0$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 3. Let us assume that the problem (1.1), (1.2) has two solu-

tions u_1, u_2. Put $v = u_1 - u_2$ on $[a, b]$. Then

$$v(c) - v(a) = 0\,, \quad v(b) - v(d) = 0 \tag{4.5}$$

and thus there exist $t_1 \in (a, c)$, $t_2 \in (d, b)$ such that $v'(t_1) = v'(t_2) = 0$.

First, let us suppose that

$$v(t_0) \neq 0 \quad \text{for some} \quad t_0 \in (t_1, t_2)\,.$$

Without loss of generality we may consider that $v(t_0) > 0$. Then there exist $t_*, t^* \in \in [t_1, t_2]$ such that

$$v(t) > 0 \quad \text{for} \quad t \in (t_*, t^*) \quad \text{and} \quad v'(t_*) \geqq 0\,, \quad v'(t^*) \leqq 0\,. \tag{4.6}$$

From (1.10) we get

$$v''(t) + \tilde{h}(t)\, v'(t) > 0 \quad \text{for} \quad t_* \leqq t \leqq t^*\,, \quad \text{where} \quad \tilde{h}(t) = h(t) \operatorname{sgn} v'(t)\,,$$

and thus the inequality

$$(\exp(\textstyle\int_a^t \tilde{h}(s)\,\mathrm{d}s)\, v'(t))' > 0 \quad \text{for} \quad t_* \leqq t \leqq t^* \tag{4.7}$$

is satisfied. Integrating (4.7) from t_* to t^*, we obtain, by (4.6), that

$$0 \geqq \exp(\textstyle\int_a^{t^*} \tilde{h}(s)\,\mathrm{d}s)\, v'(t^*) - \exp(\textstyle\int_a^{t_*} \tilde{h}(s)\,\mathrm{d}s)\, v'(t_*)) > 0\,. \tag{4.8}$$

The contradiction (4.8) implies $v(t) = 0$ for $t_1 \leqq t \leqq t_2$.

From this, according to (4.5), we get

$$v(a) = v(c) = v(d) = v(b) = 0\,. \tag{4.9}$$

Now, let us suppose that

$$v(t_0) > 0 \quad \text{for some} \quad t_0 \in (a, t_1)\ [t_0 \in (t_2, b)]\,.$$

On the basis of (4.9) we can find $t_*, t^* \in [a, t_1]$ $[\in [t_2, b]]$ such that the conditions (4.6) are fulfilled. Therefore we obtain the contradiction (4.8) in the same way as in the first part of this proof. Thus $v(t) = 0$ for $a \leqq t \leqq b$.

References

[1] *J. Andres:* On a possible modification of Levinson's operator, Proceed. 11th ICNO held in Budapest, 1987.

[2] *J. Andres:* On some modification of the Levinson operator and its application to a three-point boundary value problem, Acta UPO 91, Math. *29* (1990).

[3] *P. B. Bailey, L. F. Shampine, P. E. Waltman:* Nonlinear two-point boundary value problems, Acad. Press., New York, 1968.

[4] *R. Conti:* Equazioni differenziali ordinarie quasilineari con condizioni lineari, Ann. Mat. Pura ed Appl., *57* (1962), 49—61.

[5] *V. V. Gudkov, J. A. Klokov, A. J. Lepin, V. D. Ponomarjev:* Dvuchtočečnyje krajevyje zadači dlja obyknověnnych differencialnych uravněnij, Zinatne, Riga, 1973.

[6] *G. H. Hardy, J. E. Littlewood, G. Pólya:* Inequalities (in Russian), IL Moscow, 1948.

[7] *P. Hartman:* Ordinary differential equations (in Russian), Mir, Moscow, 1970.

[8] *I. T. Kirguradze:* Nekotoryje singuljarnyje krajevyje zadači dlja obyknověnnych differencialnych uravněnij, ITU, Tbilisi, 1975.
[9] *I. T. Kiguradze, A. G. Lomtatidze:* On certain boundary value problems for second-order linear ordinary differential equations, J. Math. Anal. and Appl., *101* (1984), 325—347.
[10] *I. T. Kiguradze:* K teorii nelinějnych dvuchtočečnych krajevych zadač, Summer school on ordinary diff. eq. Difford 74, Czechoslovakia 1974.
[11] *A. G. Lomtatidze:* Ob odnoj singuljarnoj trechtočečnoj krajevoj zadače, Trudy IPM, Tbilisi *17* (1986), 122—134.
[12] *I. Rachůnková:* A four-point problem for differential equations of the second order, Arch. Math. Brno, Vol. *25*, No 4 (1989).
[13] *N. I. Vasiljev, J. A. Klokov:* Osnovy teorii krajevych zadač obyknověnnych differencialnych uravněnij, Zinatne, Riga, 1978.

Author's address: 771 46 Olomouc, Gottwaldova 15, Czechoslovakia (Přírodovědecká fakulta PU).