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A NOTE ON DOMATICALLY CRITICAL AND COCRITICAL GRAPHS

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This paper deals with domatically critical and cocritical graphs. Two problems concerning such graphs are settled.

With minor adaptations, we adopt the terminology of Harary [3].

Let $G = (V(G), E(G))$ be an undirected graph with no loops and multiple edges. A set D of vertices in G is said to be a *dominating set* if every vertex not in D is adjacent to some vertex in D. A set of vertices S is independent if no two vertices in S are adjacent. A domatic partition (D-partition) of G is a partition of $V(G)$ into dominating sets. The maximum order of a D-partition of G is called the *domatic number* of G and is denoted by $d(G)$.

The join of two graphs G, H is the graph $G + H = (V(G) \cup V(H), E)$ where $E = E(G) \cup E(H) \cup \{(u, v) \mid u \in V(G),\ v \in V(H)\}$. We denote by $p(G)$ and $q(G)$ the number of vertices and edges of G, respectively. Finally, $\delta(G)$ will denote the minimum degree among the vertices of G.

A graph G is called *domatically critical*, if $d(G \setminus e) < d(G)$ for each edge e of G [1].

We shall say that the partition $V_1, V_2, \ldots, V_d$ of $V(G)$ possesses property (P), if it satisfies the following conditions:

(i) V_i is an independent set for any $i \in \{1, 2, \ldots, d\}$,

(ii) the subgraph $G_{i,j}$ of G, induced by $V_i \cup V_j$, is a disjoint union of stars (K_1 is not a star) for any $i, j \in \{1, 2, \ldots, d\}$, $i \neq j$.

Conjecture [4]. *Let G be a graph, $d(G) = d$ and let there exist a partition $V_1, V_2, \ldots, V_d$ of $V(G)$ satisfying* (P). *Then G is domatically critical.*

The conjecture is certainly true for all graphs with $d(G) = 1$ or 2. Indeed, if $d(G) = 1$ then (from (P)) G is $\overline{K}_n$; if $d(G) = 2$ then (from (P)) G is a disjoint union of stars without isolated vertices. Both cases give domatically critical graphs. However this does not hold in case $d(G) \geqq 3$.

Theorem 1. *For every integer $d \geqq 3$ there exists a graph G with $d(G) = d$ which has the following properties:*

(i) *there is a partition of $V(G)$ satisfying* (P);

(ii) *G is not domatically critical.*

We shall need the following propositions.

Proposition 1 [2]. *For any graph* G, $d(G) \leqq \delta(G) + 1$.

Proposition 2 [2]. *For any graph* G, $d(G + K_n) = d(G) + n$.

Proposition 3. *A graph* G *is domatically critical with the domatic number* $d(G) = d$, *if and only if any maximum D-partition of* G *satisfies* (P).

Proof. The „only if" part of the proposition follows from definitions.

To prove the sufficiency, consider any maximum D-partition R of G. Since R satisfies (P) the partition R of $G \setminus e$ is not domatic for any e of G.

Obviously $d(G \setminus e) \leqq d(G)$. Assume $d(G \setminus e) = d(G)$ for some edge e of G. Then there exists a D-partition R' of $G \setminus e$ of order $d(G)$. This partition R' is a maximum D-partition of G – a contradiction. Hence $d(G \setminus e) < d(G)$ for any edge e of G and the result follows.

Proposition 4. *Let* $G = H + K_n$. *Then* G *is a domatically critical graph, if and only if* H *is one.*

Proof. Obviously it is sufficient to prove the proposition in case $n = 1$: $G = H + \{v\}$.

Necessity. Assume H is not domatically critical: there exists e of H such that $d(H \setminus e) = d(H) = d(G) - 1$ (using Proposition 2). Consider a D-partition R of $H \setminus e$ of order $d(G) - 1$. Then $R^* = R \cup \{v\}$ is a D-partition of $G \setminus e$ of order $d(G)$ – this contradicts the domatic criticality of G.

Sufficiency. By contradiction. Let G be not domatically critical: there exists e of G such that $d(G \setminus e) = d(G) = d(H) + 1$ (using Proposition 2).

There are two possibilities.

(a) The edge e is non-incident to v. Consider a maximum D-partition $R = \{V_1, V_2, \ldots, V_{d+1}\}$ of $G \setminus e$, where $d = d(H) \geqq 1$, and assume (without loss of generality) that $v \in V_1$. Then $R^* = \{V_2 \cup (V_1 \setminus \{v\}), V_3, \ldots, V_{d+1}\} \neq \emptyset$ is a D-partition of $H \setminus e$ of order $d(H)$ – this is impossible, as H is domatically critical.

(b) The edge e is incident to v. Consider the partitions R, R^* constructed above. Clearly R^* is a D-partition of H of order $d(H)$. Since $V_1 \setminus \{v\} \neq \emptyset$ (v is not dominating in $G \setminus e$) and V_2 is dominating in $G \setminus e$ (V_2 exists, as $d(H) + 1 \geqq 2$) the set $V_2 \cup (V_1 \setminus \{v\})$ is dependent. Hence the maximum D-partition R^* of H does not satisfy (P). By Proposition 3, H is not domatically critical. Again, we arrive at a contradiction. Thus all cases have been considered and the proof is complete.

Proof of Theorem 1. Let H be the graph in Figure 1. On the one hand, $d(H) \leqq \delta(H) + 1 = 3$ (using Proposition 1). On the other hand, the sets $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2, z_3\}$ form a D-partition of H. Hence $d(H) = 3$. The D-partition $\{\{x_1, y_3, z_2\}, \{x_2, y_1, y_2\}, \{x_3, z_1, z_3\}\}$ of H does not satisfy (P), as the set $\{x_2, y_1, y_2\}$ is dependent. By Proposition 3, the graph H is not domatically critical.

Now we shall prove that the graph $G = H + K_n$, $n \geqq 0$ has the properties (i), (ii) of Theorem 1. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By Propositions 2, 4, $d(G) = d(H) + n \geqq 3$ and G is not domatically critical (as H is not so). Obviously the D-partition

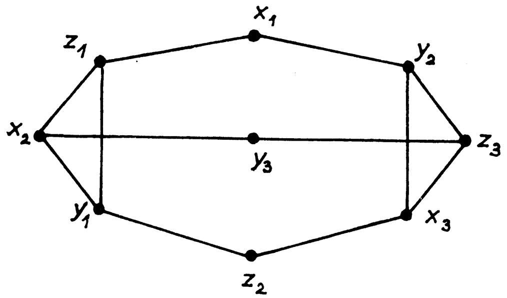

Figure 1

$\{X, Y, Z\}$ of H satisfies (P), therefore the D-partition $\{X, Y, Z, \{v_i\}, i = \overline{1 \,.\, n}\}$ of G satisfies (P), too. This completes the proof.

A graph G is called *domatically cocritical*, if for every pair of its non-adjacent vertices u, v the inequality $d(G \cup (u, v)) > d(G)$ holds.

Problem [5]. *Does there exist a domatically cocritical graph G whose complement $\bar{G}$ has more than $p(G) - d(G)$ edges?*

The answer is affirmative.

Theorem 2. *For every positive integer k there exists a domatically cocritical graph G for which*

$$q(\bar{G}) = k + p(G) - d(G)\,.$$

Proof. Consider the graph G_k whose complement $\bar{G}_k$ is shown in Figure 2.

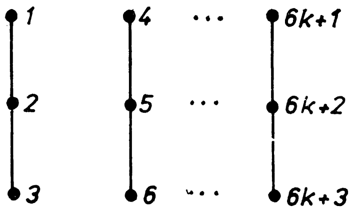

Figure 2

Clearly $p(G_k) = 6k + 3$ and $q(\bar{G}_k) = 4k + 2$. Each dominating set of G_k contains at least two vertices. Hence $d(G_k) \leqq [p(G_k)/2] = 3k + 1$. Let $I = \{r + 6t \mid r = \overline{1 \,.\, 3},\ t = \overline{0 \,.\, k - 1}\}$. The sets $\{i, i + 3\}$, $i \in I$, $\{6k + 1, 6k + 2, 6k + 3\}$ form a D-partition of G_k with $3k + 1$ classes, therefore $d(G_k) = 3k + 1$.

It is not difficult to see that $d(G_k \cup e) > d(G_k)$ for each edge e of $\bar{G}_k$. Thus the graph G_k is domatically cocritical and $q(\bar{G}_k) - p(G_k) + d(G_k) = (4k + 2) - (6k + 3) + (3k + 1) = k$, as required.

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