

# Aplikace matematiky

---

Gurij Ivanovich Marchuk

Some application of splitting-up methods to the solution of mathematical physics problems

*Aplikace matematiky*, Vol. 13 (1968), No. 2, 103–132

Persistent URL: <http://dml.cz/dmlcz/103142>

## Terms of use:

© Institute of Mathematics AS CR, 1968

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

## SOME APPLICATION OF SPLITTING-UP METHODS TO THE SOLUTION OF MATHEMATICAL PHYSICS PROBLEMS

G. I. MARČUK (Г. И. Марчук)

1. Splitting-up methods are becoming a powerful constructive mathematical tool for solving complicated problems in science and engineering. Originally applied to the solution of the simplest two-dimensional elliptic and parabolic equations, these methods are finding increasing use in solving major problems. Naturally, the application of the splitting-up method to the solution of such problems involves formulation of a number of new theoretical problems which develop at different stages of formulation of a general algorithm.

The most essential thing is to choose a way of splitting a complicated problem into the simplest problems. Splitting of a problem is first of all an attempt to reduce a complicated problem to the simplest ones. Here, certainly, there arise a number of problems, and one of them is convergence of the solution of the splitted problem to the initial exact problem of algorithm optimization. Since we have now some experience in solving complicated mathematical physics problems, we can discuss the question concerning the reduction of the problems. First of all, it should be noted that the most efficient reduction of the problem becomes possible on the basis of profound understanding of the physical process described by the original system of equations with the corresponding boundary conditions and initial data. As a rule, analysis of the studied physical processes on the basis of the dimensionality and similarity theory prompts the most efficient way of splitting. It means that in the studied problem it is very important to find out the most essential relations which determine the fundamental features of the process and their interaction. Attempts to make formal reductions in solving physically complicated problems usually meet with failure.

Another factor of importance of the splitting-up method is connected with implementation of conservation laws inherent in a given physical process. This means that reduction of a problem to the simplest ones must not violate the fundamental conservation laws. If the initial problem is reduced to a system of difference equations, then, also in this case, implementation of conservation laws written in a difference form is necessary.

An essential factor of splitting is the reduction of the initial problem to such

a succession of the simplest problems which allow logically simple ways of realization and optimization of a numerical algorithm both in its components and as a whole. The problem of optimization of a numerical algorithm is a most complicated field of computational theory. It is potentially abundant in new ideas and constantly influences the formation of new algorithms for the solution of the problems. It is an essential feature of the splitting-up methods that they allow introduction of auxiliary parameters, functions as well as operators which may be used for the construction of efficient algorithms of the solution of the problems. It is natural that the problem of numerical algorithm optimization goes far beyond this scope because it is also inseparable from the problems of approximation, stability, and convergence which are fundamental in solving mathematical physics problems.

Modern high-efficiency computers provide investigators with a powerful tool for the solution of a wide class of different problems in science and engineering. Mathematical simulation has become a vital need rather than a fashion in the development of natural sciences above all. Therefore the problems of computational mathematics have attracted attention of a wide circle of researchers and engineers. An amount of scientific information processed by the computers is swiftly growing which inevitably provokes certain tendencies in the construction of new numerical algorithms. As an example we can take either the splitting-up method which is a constructive tool for the solution of complicated mathematical physics problems or the method of weak approximation whose development has been stimulated by practical needs.

A wide search in the field of constructing new numerical algorithms necessarily involves considerable difficulties of their theoretical substantiation. Spectral theory of operators and methods of a priori estimates made it possible to consider, from the general point of view, numerical algorithms for certain classes of problems and to construct efficient numerical algorithms. But, as a rule, these classes of problems do not go beyond linear positively definite operators while, as we know, major scientific and engineering problems necessarily involve operators of a more complicated structure including non-linear operators. In this connection it is necessary to evolve such methods for the construction of algorithms and their optimization which would be based upon a posteriori estimates of the solution or functionals of the solution in the process of realization of the numerical algorithm by the computer. This means that the functionals of the solution of the problem, obtained in the process of the solution, must be used for automatic adjustment or reconstruction of the algorithm for more optimal operating conditions. The tendency to a wide use of the functional analysis and general theory of operators for studying methods of the numerical solution of the problems stands out distinctly against the background of the constructive search for new algorithms. And this is natural because by this time a large amount of rather general results have been accumulated which are successfully used in computational mathematics and applications.

2. Besides the usual requirements of approximation and stability, difference schemes must satisfy some additional conditions such as economy, higher order accuracy, divergence and so forth. Therefore the construction of difference schemes on the basis of the conventional homogeneous approximations met with difficulties which could be overcome by employing the splitting-up method which uses, generally speaking, non-homogeneous approximation. The first results in this connection have been obtained by D. PEACEMAN, H. RACHFORD and J. DOUGLAS who suggested an alternating direction method.

The Soviet authors have based their investigations of the method of splitting the operators into the simplest ones. In this case integration of a given equation reduces to a successive integration of simpler equations, and only at the end the obtained difference schemes must satisfy the conditions of approximation and stability. As a matter of fact, this permits a flexible construction of schemes for all the basic equations of mathematical physics.

Beginning with the Janenko's works, the splitting-up method has been developed in the U. S. S. R. by many scientists. Very important ideas in the splitting-up method are due to JANENKO, SAMARSKIĬ, D'JAKONOV, SAUL'EV and others. The general results connected with the splitting-up method have been used for different applications and for solving some problems of the mathematical physics.

By this time, the splitting-up methods seem to have considerably developed. A range of problems effectively solved by these methods have been formed and simultaneously such problems have taken shape whose solution is of paramount importance for the further development of the theory. It appears that the main problem at present is associated with the choice of optimal parameters, functions and operators occurring in the splitting-up methods. It is hoped that the solution of these questions will be found as a consequence of the use of the best approximation theory, variational principles and probability methods.

Let us now formulate some results of the splitting-up theory.

3. When we solve stationary problems there arises a problem of finding a solution of the equation

$$(1) \quad A\varphi = f,$$

where  $A$  is a differential or an integro-differential operator,  $f$  is a given function, and  $\varphi$  is the solution sought for which satisfies the corresponding boundary conditions. Let us note that  $A$  may be a matrix operator, and  $\varphi$  and  $f$  may be vector functions. In this case (1) is a formal representation of a system of equations.

The solution of (1) in the case of complex operators  $A$  is usually carried out by means of relaxation methods determined as follows:

$$(2) \quad \varphi^{j+1} = \varphi^j - \tau(A\varphi^j - f),$$

where  $\tau$  is an arbitrary parameter and  $j$  is the number of a successive approximation. The parameter  $\tau$  may also be changed with the number of iteration. In this case  $\tau$  must be replaced by  $\tau_j$ . The iterative process of the form (2) may be convergent only under definite conditions, the formulation of the latter being, apparently, the most difficult part in organizing the relaxation process. As a rule, the rate of convergence of the successive approximation method essentially depends on the choice of the parameter  $\tau_j$ . Sometimes it is optimal. Construction of the computational schemes, possessing in a sense the best characteristics of convergence, will be called optimization of a numerical algorithm.

Let us write the relaxation scheme (2) as follows:

$$(3) \quad \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f.$$

In this form the iterative process (2) is a difference analogue of the following non-stationary problem:

$$(4) \quad \frac{\partial \varphi}{\partial t} + A\varphi = f$$

with just the same range of definition of the solution according to geometrical coordinates as in (1). The auxiliary independent variable  $t$  in (4) may be interpreted in different ways. For example, it may be prescribed a sense of time if the dimensions  $A\varphi$  and  $f$ , by multiplying them by the same auxiliary constant, are preliminarily coordinated with that of the time derivative  $\partial\varphi/\partial t$ . In (4) the operator  $A$  and the function  $f$  are certainly independent of  $t$ . Naturally, the iterative process (2) will be of interest only if

$$\lim_{j \rightarrow \infty} \varphi^j = \varphi,$$

where  $\varphi$  is an exact solution of (1). The same is true of the solution of Eq. (4), namely, the solution of (4) is of interest only if, at  $t \rightarrow \infty$ , its limit is a solution of the stationary problem (1). This fact imposes certain requirements on the operator  $A$  which we shall discuss later. Consider now only one important property, that is, a close connection between stationary problems for, at least, a large class of the operators  $A$  which have a unique solution of the form (1) as their asymptotic solution. This means that at least for such problems the stationary and non-stationary problems may be considered in a uniform aspect (4) on the assumption that the solution of stationary problems is a particular case of a general algorithm when  $t \rightarrow \infty$ .

We may note, however, that the relaxation process (2) is not a most general and effective form of organizing the iterative process. Some authors in the works concerning linear algebra problems suggested more general universal iterative processes which, if properly chosen, essentially improved the convergence of the iterative process. By analogy with them it is possible to organize the following generalized re-

laxation process:

$$(5) \quad \varphi^{j+1} = \varphi^j - \tau A(A\varphi^j - f),$$

where  $A$  is an operator chosen from the condition of the speed of the iterative process convergence. In the case when  $A = E$  is a unit operator we come to the classical relaxation process (2). Let us consider another limiting case when  $A = A^{-1}$ . Here, whatever is the function  $\varphi^0$ , the first approximation yields an accurate solution of (1)

$$(6) \quad \varphi = A^{-1}f.$$

The generalized relaxation process (5) may be rewritten in the form

$$(7) \quad B \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f,$$

where  $B = A^{-1}$ . This scheme may be considered as realization of a more general non-stationary equation

$$(8) \quad B \frac{\partial \varphi}{\partial t} + A\varphi = f.$$

All the above considerations with regard to (4) are valid in this more general case as well. One must remember only that there is a chance here to choose the operator  $B$  so as to make the algorithm (5) converge more rapidly. If one chooses  $A$  to stand for  $B$ , the problem will not be simplified because it again reduces to the calculation of the inverse operator  $A^{-1}$ . Yet, it can be shown that for a wide class of the operators  $A$  it is possible to construct the operators  $B$  which approximate the operator  $A$  sufficiently well and which are effective in realization.

Consider now a more general problem

$$(9) \quad A\varphi = f,$$

where  $A$  is a linear operator which can be either differential or integral, or a matrix. The functions  $\varphi$  and  $f$  satisfy certain conditions which make possible the realization of the operator  $A$  of  $\varphi$  and the expansion of  $\varphi$  and  $f$  into the Fourier series.

Let us now introduce into consideration the scalar product  $(a, b)$ , and a conjugate, in the Lagrangian sense, operator  $A^*$  by means of the identity

$$(10) \quad (\varphi^*, A\varphi) = (\varphi, A^*\varphi^*).$$

Here, for the sake of simplicity, the operators  $A, A^*$  and the functions  $\varphi, \varphi^*$  are supposed to be real.

Consider now two eigenvalue problems

$$(11) \quad A\psi = \lambda\psi,$$

$$(12) \quad A^*\psi^* = \lambda\psi^*.$$

Note that in the case when  $A$  is a differential operator and Eq. (9) determines a boundary value problem with homogeneous boundary conditions, the class of the functions  $\psi$  from which the solutions of Eqs. (9) and (11) are chosen, satisfies the given boundary conditions. As for the boundary conditions for the function  $\psi^*$  defined by Eq. (12), they are obtained from the requirement to satisfy the conditions (10).

Let us suppose that Eqs. (11), (12) determine two complete systems of the eigenfunctions  $\{\psi_n\}$  and  $\{\psi_n^*\}$  orthonormalized in the following way:

$$(13) \quad \psi_n \psi_m = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Represent the functions  $f$  and  $\varphi$  in the form of Fourier series

$$(14) \quad f(r) = \sum_n f_n \psi_n(r), \quad \varphi(r) = \sum_n \varphi_n \psi_n(r),$$

where  $f_n, \varphi_n$  are Fourier coefficients determined by

$$(15) \quad f_n = (f, \psi_n^*), \quad \varphi_n = (\varphi, \varphi_n^*).$$

Thus the problem is reduced to finding the unknown Fourier coefficients for the function  $\{\varphi\}$ . Let us substitute the series (14) into Eq. (9) and scalarly multiply the result by  $\psi_m^*$ , and use the following equalities:

$$A\psi_n = \lambda_n \psi_n$$

resulting from (13) and (10).

Then, for the Fourier coefficients  $\varphi_n$ , we obtain the equality system

$$(16) \quad \lambda_m \varphi_m = f_m.$$

Hence

$$(17) \quad \varphi_m = \lambda_m^{-1} f_m.$$

In accordance with the second relation (14) we get

$$(18) \quad \varphi(r) = \sum_n \lambda_n^{-1} f_n \psi_n(r).$$

Further, let us substitute the stationary problem (9) by the following non-stationary problem

$$(19) \quad \frac{\partial \varphi}{\partial t} + A\varphi = f, \quad \varphi(r, 0) = 0$$

with the same boundary conditions as in (9). The solution of (19) will be sought in the form of Fourier expansion

$$(20) \quad f(r) = \sum_n f_n \psi_n(r), \quad \varphi(r, t) = \sum_n \varphi_n(t) \psi_n(r).$$

Substitute (20) into (19) and scalarly multiply the result by  $\psi_m^*$ . Then we are led to a system of equations with the initial data

$$(21) \quad \frac{d\varphi_m}{dt} + \lambda_m \varphi_m = f_m, \quad \varphi_m(0) = 0.$$

Since  $f_m$  does not depend on  $t$ , the solution of (21) is obtained trivially

$$(22) \quad \varphi_m(t) = (1 - e^{-\lambda_m t}) \lambda_m^{-1} f_m.$$

Analysis of the relation (22) reveals that, when  $t \rightarrow \infty$ , Fourier coefficients  $\varphi_m(t)$  will tend to the solution (17) if the spectrum  $\{\lambda\}$  of the problems (11), (12) is positive, i.e. all

$$\lambda_n > 0.$$

Let us substitute now Eq. (22) for the Fourier coefficient into the second relation of (20). Then we get

$$(23) \quad \varphi(r, t) = \sum_n (1 - e^{-\lambda_n t}) \lambda_n^{-1} f_n \psi_n(r).$$

If  $A$  is a positively definite and symmetrical operator, then, at  $t \rightarrow \infty$ , the solution (23) becomes

$$(24) \quad \varphi(r, \infty) = \sum_n \lambda_n^{-1} f_n \psi_n(r),$$

which coincides with the solution of the stationary problem (18). Thus we are led to the conditions under which the limiting element of a non-stationary problem yields a solution of a stationary problem.

It can be easily shown that the difference approximations of (19) lead to an accurate solution of a stationary problem when  $j \rightarrow \infty$ . However, in this case the conditions of positivity of the spectrum  $\{\lambda_k\}$  can prove insufficient. In fact, let us consider an explicit scheme of solving the problem (19) in the form

$$(25) \quad \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f.$$

Let us assume then that the solution of this problem may be sought in the form of a Fourier series

$$(26) \quad \varphi^j(r) = \sum_n \varphi_n^j \psi_n(r)$$

on condition that the function  $f(r)$  may be represented in the form

$$(27) \quad f(r) = \sum_n f_n \psi_n(r).$$

Here  $\psi_n(r)$  are eigenfunctions of (11). Let us substitute the series (26) and (27) into Eq. (25) and scalarly multiply the result by  $\psi_m^*(r)$ . Then we come to the difference equations for the Fourier coefficients

$$(28) \quad \frac{\varphi_n^{j+1} - \varphi_n^j}{\tau} + \lambda_n \varphi_n^j = f_n$$

if

$$(29) \quad \varphi_n^0 = 0.$$

Represent now (28), (29) in a more suitable form

$$(30) \quad \varphi_n^{j+1} = (1 - \tau\lambda_n) \varphi_n^j + \tau f_n, \quad \varphi_n^0 = 0.$$

By successive elimination of the unknown quantities, we come to the geometrical progression

$$(31) \quad \varphi_n^{j+1} = (1 + q_n + q_n^2 + \dots + q_n^j) \tau f_n,$$

where

$$q_n = 1 - \tau\lambda_n$$

is a common ratio of the progression.

By summing the series (31), we get

$$(32) \quad \varphi_n^{j+1} = \frac{1 - q_n^{j+1}}{1 - q_n} \tau f_n$$

or, using the notations for  $q_n$ ,

$$(33) \quad \varphi_n^{j+1} = [1 - (1 - \tau\lambda_n)^{j+1}] \lambda_n^{-1} f_n.$$

From this relation it follows that the limiting element, at  $j \rightarrow \infty$ , will coincide with the Fourier coefficient of the corresponding stationary problem provided that

$$(34) \quad |1 - \tau\lambda_n| < 1$$

or, which is the same,

$$(35) \quad 0 < \tau\lambda_n < 2.$$

Thus in this case instead of the condition

$$\lambda_n > 0$$

it is required to satisfy (35). Though, there is one more arbitrary parameter  $\tau$  here. However, if the spectrum  $\lambda_n$  is fixed, the condition (35) imposes some additional requirements limiting the choice of the time step  $\tau$ . One can note that (35) is a condition of the calculating stability of the difference scheme (25) corresponding to the non-stationary problem (19). At the same time it is the criterion of convergence of the relaxation process for solving a stationary problem. Thus the approach considered establishes a close relationship between stability and convergence in solving mathematical physics problems.

In conclusion it should be pointed out that the condition of the type (35), when using a universal algorithm, can be made much easier by a proper choice of the stabilizing operator  $B$ . Consider now the stationing theory from a more general point of view. For this purpose, let us consider the universal algorithm

$$(36) \quad B \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f,$$

where  $B$  is an arbitrary operator for the present. Eq. (36) will be solved with respect to  $\varphi^{j+1}$ , then we obtain

$$(37) \quad \varphi^{j+1} = (E - \tau B^{-1}A) \varphi^j + \tau B^{-1}f.$$

Assume that

$$(38) \quad \varphi^0 = 0.$$

Let us analyse (37) and (38) in more detail. Supposing that at  $j \rightarrow \infty$  the successive approximation method is convergent, then the limiting element of the problem satisfies the equation

$$\varphi^\infty = (E - \tau B^{-1}A) \varphi^\infty + \tau B^{-1}f.$$

From this it follows that

$$A\varphi^\infty = f,$$

i.e. the limiting element is a solution of the stationary problem. Here we assume  $B$  to be a nonsingular operator. Let us find further the conditions under which the successive approximation method (37) is convergent. For this purpose the recurrence relation (37) will be rewritten as

$$(39) \quad \varphi^{j+1} = P\varphi^j + F,$$

where

$$(40) \quad \begin{aligned} P &= E - \tau B^{-1}A, \\ F &= \tau B^{-1}f. \end{aligned}$$

Using the recurrence relation (39) under the condition (37) we get

$$(41) \quad \varphi^{j+1} = (E + P + P^2 + \dots + P^j) F$$

by successive elimination of the unknowns. Let us consider further the spectral problem

$$(42) \quad P\psi = \mu\psi$$

and the conjugate problem

$$(43) \quad P^*\psi^* = \mu\psi^* .$$

Let (42), (43) determine the complete biorthogonal and normalized systems of the functions  $\{\psi_k\}$  and  $\{\psi_k^*\}$ . Then we can make the following expansion

$$(44) \quad F = \sum_k F_k \psi_k(r) ,$$

where

$$F_k = (F, \psi_k^*) .$$

Let us represent the function  $\varphi^{j+1}$  in the form of a Fourier series

$$(45) \quad \varphi^{j+1} = \sum_k \varphi_k^{j+1} \psi_k(r) .$$

Then substitute the Fourier series (44) and (45) into Eq. (41) and scalarly multiply the result, term by term, by  $\psi_m^*(r)$ . Then, applying biorthogonality and normalization of the functions  $\psi_n$  and  $\psi_m^*$ , we get the system of equalities

$$(46) \quad \varphi_m^{j+1} = (1 + \mu_m + \mu_m^2 + \dots + \mu_m^j) F_m , \quad (m = 1, 2, \dots) .$$

Summing the geometrical progression (46), we have

$$(47) \quad \varphi_m^{j+1} = \frac{1 - \mu_m^{j+1}}{1 - \mu_m} F_m .$$

Now, substituting (47) into (45), we are led to the solution of the problem in the form

$$(48) \quad \varphi^{j+1} = \sum_m \frac{1 - \mu_m^{j+1}}{1 - \mu_m} F_m \psi_m(r) .$$

Analysis of (48) shows that the series converges at  $j \rightarrow \infty$  provided that

$$(49) \quad |\mu_m| < 1$$

is satisfied. Thus the condition is found under which the universal algorithm leads to the solution of the stationary problem.

4. In solving stationary problems we often deal with the complex operators which are not readily realized on the computers. Particularly it concerns multi-dimensional equations of mathematical physics. On the other hand the modern computers permit an enormous amount of information to be handled in no time. This new quality of the computers ought to be used in constructing efficient algorithms for solving the problems. Such methods are associated with a peculiar organization of the universal algorithm based on a generalized relaxation process. Some authors give the following formulation of the numerical algorithm for the problems with complex operators. Let us solve the equations

$$(50) \quad A\varphi = f,$$

where

$$(51) \quad A = \sum_{\alpha} A_{\alpha},$$

and  $A_{\alpha}$  are operators of a simple structure. The function  $f$  is supposed to be given and  $\varphi$  is the solution sought for from the corresponding class of functions satisfying, probably, certain boundary conditions. For simplicity, these conditions are considered to be homogeneous. Let us formulate then the successive approximation method according to

$$(52) \quad \prod_{\alpha} \left( E + \frac{\tau}{2} A_{\alpha} \right) \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f,$$

where  $\tau$  is an arbitrary relaxation parameter. The difference equation of the form (52) was formulated by D'jakonov, Douglas Jr. and GUNN, and Samarskiĭ.

The author and N. N. JANENKO considered (52) from the point of view of the scheme for the construction of different universal realization methods. This means that the scheme (52) has become a constructive apparatus for constructing difference schemes by the splitting-up method. This idea is considered here rather systematically and in detail. Obviously, Eq. (52) is a particular case of the general algorithm

$$(53) \quad B \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f.$$

Here the stabilizing operator is of the form

$$(54) \quad B = \prod_{\alpha} \left( E + \frac{\tau}{2} A_{\alpha} \right).$$

Represent the realization scheme corresponding to every step of the iterative

process (52) as

$$\begin{aligned}
 (55) \quad & \left(E + \frac{\tau}{2} A_1\right) \xi^{j+1/n} = -\tau F^j, \\
 & \left(E + \frac{\tau}{2} A_2\right) \xi^{j+2/n} = \xi^{j+1/n}, \\
 & \dots\dots\dots \\
 & \left(E + \frac{\tau}{2} A_n\right) \xi^{j+1} = \xi^{j+(n-1)/n}, \\
 & \varphi^{j+1} = \varphi^j + \xi^{j+1},
 \end{aligned}$$

where  $F^j$  is the discrepancy of the relaxation process determined by the equation

$$(56) \quad F^j = A\varphi^j - f.$$

Analysis of the equation system (55) shows that each equation has a simple structure as it is connected with realization of the elementary operators of which the operator  $A$  is made up.

Discuss now the formulation of the generalized splitting-up scheme on the basis of the universal algorithm. In some works it has been shown that (52) may be substituted by a more general scheme

$$(57) \quad \prod_{\alpha} \left(E + \frac{\tau_{\alpha}}{2} A_{\alpha}\right) \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j - f = 0,$$

which has an advantage over (52) since it contains  $n + 1$  arbitrary parameters  $\tau_{\alpha}$ ,  $\tau$  which may be used for constructing an optimal numerical algorithm. In this case the realization scheme has the form

$$\begin{aligned}
 (58) \quad & \left(E + \frac{\tau_1}{2} A_1\right) \xi^{j+1/n} = -\tau F^j, \\
 & \left(E + \frac{\tau_2}{2} A_2\right) \xi^{j+2/n} = \xi^{j+1/n}, \\
 & \dots\dots\dots \\
 & \left(E + \frac{\tau_n}{2} A_n\right) \xi^{j+1} = \xi^{j+(n-1)/n}, \\
 & \varphi^{j+1} = \varphi^j + \xi^{j+1}.
 \end{aligned}$$

Here  $F^j$  is the discrepancy of the relaxation process determined by

$$(59) \quad F^j = A\varphi^j - f.$$

Let us consider theoretically the generalized universal algorithm in the case when the operators  $A_\alpha$  commute and have a common basis. In this case we consider  $n$  particular spectral problems

$$(60) \quad A_\alpha u_\alpha = \lambda^{(\alpha)} u_\alpha \quad (\alpha = 1, 2, \dots, n)$$

and, similarly,

$$(61) \quad A_\alpha^* u_\alpha^* = \lambda^{(\alpha)} u_\alpha^* \quad (\alpha = 1, 2, \dots, n).$$

Now, let the spectral problems (60) and (61) form  $n$  complete orthogonal and normalized bases of the functions  $\{u_{\alpha k}\}$ ,  $\{u_\alpha^*\}$ . Then the solution of Eq. (57) is sought in the form of a series

$$(62) \quad \varphi^j = \sum_{k=1}^{\infty} \varphi_k^j u_{1k} u_{2k} \dots u_{nk}.$$

Here we assume that the function  $f$  may also be extended into a series according to the eigenfunctions of the problem

$$(63) \quad f = \sum_{k=1}^{\infty} f_k u_{1k} u_{2k} \dots u_{nk}.$$

Substitute the series (62), (63) into Eq. (57) and scalarly multiply the result by  $u_{1l}^* u_{2l}^* \dots u_{nl}^*$ . Then, for the Fourier coefficients  $\varphi_k^j$ , we come to the simplest equation

$$(64) \quad \prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right) \frac{\varphi_k^{j+1} - \varphi_k^j}{\tau} + \lambda_k \varphi_k^j - f_k = 0,$$

where

$$(65) \quad \lambda_k = \sum_{\alpha} \lambda_k^{(\alpha)}.$$

Solve Eq. (64) with respect to the unknown  $\varphi_k^{j+1}$ . Then we obtain

$$(66) \quad \varphi_k^{j+1} = \frac{\prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right) - \tau \lambda_k}{\prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right)} \varphi_k^j + \frac{\tau f_k}{\prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right)}.$$

Assume the spectra  $\{\lambda_k^{(\alpha)}\}$  to be positive, then it is easy to formulate the criterion of convergence of the iterative process (66). To this end, consider a common ration of the progression

$$(67) \quad q_k = \frac{\prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right) - \tau \lambda_k}{\prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right)}.$$

It is easily seen that the convergence of the iterative process ( $q_k < 1$ ) takes place when the inequality

$$(68) \quad \prod_{\alpha} \left( 1 + \frac{\tau_{\alpha}}{2} \lambda_k^{(\alpha)} \right) > \frac{\tau \lambda_k^{(\alpha)}}{2}$$

is satisfied.

Obviously,

$$(69) \quad \tau_{\alpha} \geq \tau > 0$$

is a sufficient condition to solve the inequality. Thus we come to the criterion of the choice of the parameters for the relaxation process. It is understood that the condition (69) allows us to conduct a wide search for the optimal parameters of the relaxation process  $\tau_{\alpha}, \tau$ . This problem is always solved independently starting from the concrete structure of the operators of the problem. If the solution of the problem is repeated, such parameters can be chosen once and for all.

In conclusion let us formulate a more general iterative process which can be applied to the solution of the stationary problems

$$(70) \quad \prod_{\alpha} \left( E + \frac{\tau_{\alpha}}{2} A_{\alpha} \right) \frac{\varphi^{j+1} - \varphi_j}{\tau} + A \varphi^j = f.$$

Here  $A_{\alpha}$  are arbitrary operators. The structure of the operators  $A_{\alpha}$  is chosen so that the realization scheme of the iterative process might be as simple as possible and could be effectively realized on computers.

5. Now we shall consider non-stationary problems. It is necessary to formulate the splitting-up method applicable to the problems of such kind. First of all we shall start from the requirement that the numerical algorithm must be effective in realization and absolutely stable. The problem of approximation is of essential importance because it is closely connected with economy of the numerical algorithm. Actually, if the schemes of the first order accuracy are used for the approximate solution of a non-stationary equation, then, for the required accuracy of the result, it is necessary to make calculations with small time and space steps. However, in many cases, it is reasonable to use schemes of the second and higher order accuracy. In this case the time and space steps can be taken large, which can considerably save the calculation time. Though this fact is obvious, the application of the schemes of the second order accuracy leads sometimes to insuperable difficulties since, in many cases, the difference schemes of the second order accuracy prove unstable, which makes calculation by them impracticable. When calculation is stable, for example, if centred implicit schemes are used, the method appears to be algorithmically ineffective because no suitable and economical algorithms of realization are available for it. Thus the choice of

efficient algorithms for the solution of non-stationary problems is one of the central problems in numerical mathematics.

Let us consider the non-stationary equation

$$(71) \quad \frac{\partial \varphi}{\partial t} + A\varphi = f, \quad \varphi(0, r) = g(r).$$

Break up the interval  $0 \leq t \leq T$  into the partial intervals  $\tau = \Delta t$  and substitute (71) by the difference equation

$$(72) \quad \prod_{\alpha} \left( E + \frac{\tau}{2} A_{\alpha} \right) \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f^{j+1/2}.$$

Here  $A_{\alpha}$  are parts of the operator  $A$ ,  $\sum_{\alpha} A_{\alpha} = A$ . For the sake of simplicity, we assume that  $A$  and  $A_{\alpha}$  are independent of  $\tau$ , but this assumption is of no importance to us. The realization scheme for Eq. (72) coincides with (55). Let us show that Eq. (71) is approximated by (72).

In fact, let us consider Taylor's series

$$(73) \quad \begin{aligned} \varphi^{j+1} &= \varphi^{j+1/2} + \varphi_t^{j+1/2} \frac{\tau}{2} + \varphi_{tt}^{j+1/2} \frac{\tau^2}{8} + \dots \\ \varphi^j &= \varphi^{j+1/2} - \varphi_t^{j+1/2} \frac{\tau}{2} + \varphi_{tt}^{j+1/2} \frac{\tau^2}{8} - \dots \end{aligned}$$

If (73) is substituted into (72), we obtain

$$(74) \quad \frac{\partial \varphi}{\partial t} + A\varphi = f + O(\tau^2) \quad \text{at} \quad t = t_{j+1/2}.$$

Hence, the difference scheme (72) approximates Eq. (71) with the order  $\tau^2$ .

For stability we must choose  $\tau$  from the condition

$$\max_n |\lambda_n^{(T)}| \leq 1,$$

where  $T$  is the operator considered in the case of stationary problems.

Consider now another scheme of the splitting-up method, considered by E. G. D'jakonov, which is also a scheme of the second order accuracy

$$(75) \quad \prod_{\alpha} \left( E + \frac{\tau}{2} A_{\alpha} \right) \varphi^{j+1} = \prod_{\alpha} \left( E - \frac{\tau}{2} A_{\alpha} \right) \varphi^{j+1/2}.$$

If we substitute Taylor's series (73) into (75), we obtain (74) as the result. The scheme

of realization is as follows:

$$(76) \quad \begin{aligned} \varphi^{j+\alpha/2n} &= \left(E - \frac{\tau}{2} \Lambda_n\right) \varphi^{j+(\alpha-1)/n}, \\ \varphi^{j+1/2} &= \left(E - \frac{\tau}{2} \Lambda_1\right) \varphi^{j+(n-1)/2n} + \frac{\tau}{2} f^{j+1/2}, \\ \left(E + \frac{\tau}{2} \Lambda_1\right) \varphi^{j+2/n} &= \varphi^{j+1/n} + \frac{\tau}{2} f^{j+1/2}, \\ \left(E + \frac{\tau}{2} \Lambda_\alpha\right) \varphi^{j+\alpha/n} &= \varphi^{j+(\alpha-1)/n}, \end{aligned}$$

where

$$\alpha = 1, 2, \dots, n-1.$$

In order to test the stability of Eq. (75), we shall consider the homogeneous equation

$$(77) \quad \prod_{\alpha} \left(E + \frac{\tau}{2} \Lambda_{\alpha}\right) \varepsilon^{j+1} = \prod_{\alpha} \left(E - \frac{\tau}{2} \Lambda_{\alpha}\right) \varepsilon^j,$$

where  $\varepsilon^0$  is an error in the function  $\varphi^0$  given at  $t = 0$ . Find the solution of the equation

$$(78) \quad \varepsilon^j = \eta^j \psi_n(r),$$

where  $\eta = \text{const.}$ ,  $j$  is an exponent,  $q_n(r)$  is a solution of the homogeneous problem

$$(79) \quad \Lambda \psi_n = \lambda_n \psi_n.$$

Taking into account that  $\Lambda$  and  $\Lambda_\alpha$  have a common basis, we obtain

$$(80) \quad \Lambda_\alpha \psi_n = \lambda_n^{(\alpha)} \psi_n.$$

Let us substitute (78) into (77) and take account of (79) and (80). As a result we obtain

$$(81) \quad \eta_n = \prod_{\alpha} \frac{1 - \tau \lambda_n^{(\alpha)}}{1 + \tau \lambda_n^{(\alpha)}}.$$

If  $\lambda_n^{(\alpha)} \geq 0$ , then  $|\eta_n| \leq 1$  for all the values of  $n$ . Therefore, in this case, the difference scheme (75) is stable.

Consider the particular case when  $\Lambda = \Lambda_1 + \Lambda_2$ . We have

$$(82) \quad \left(E + \frac{\tau}{2} \Lambda_1\right) \left(E + \frac{\tau}{2} \Lambda_2\right) \varepsilon^{j+1} = \left(E - \frac{\tau}{2} \Lambda_1\right) \left(E - \frac{\tau}{2} \Lambda_2\right) \varepsilon^j.$$

Solving (82) with respect to  $\varepsilon^{j+1}$ , we get

$$(83) \quad \varepsilon^{j+1} = \left(E + \frac{\tau}{2}A_2\right)^{-1} \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E - \frac{\tau}{2}A_1\right) \left(E - \frac{\tau}{2}A_2\right) \varepsilon^j.$$

Let

$$\Theta^j = \left(E + \frac{\tau}{2}A_2\right) \varepsilon^j.$$

In this case Eq. (83) becomes

$$(84) \quad \Theta^{j+1} = \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E - \frac{\tau}{2}A_1\right) \left(E - \frac{\tau}{2}A_2\right) \left(E + \frac{\tau}{2}A_2\right)^{-1} \Theta^j.$$

In accordance with the property of a Hilbert norm, we get

$$(85) \quad \|\Theta^{j+1}\| \leq \left\| \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E - \frac{\tau}{2}A_1\right) \right\| \left\| \left(E - \frac{\tau}{2}A_2\right) \left(E + \frac{\tau}{2}A_2\right)^{-1} \right\| \|\Theta^j\|$$

and, on the basis of Kellogg's lemma, assuming the positivity of the operators  $A_1$  and  $A_2$  we determine the computing stability of the difference scheme

$$\|\Theta^{j+1}\| \leq \|\Theta^j\|.$$

Let us consider the non-stationary problem

$$(86) \quad \frac{\partial \varphi}{\partial t} + A\varphi = f, \quad \varphi = g \quad \text{at} \quad t = 0.$$

First of all, let us find the solution of (86) in the interval  $t_j \leq t \leq t_{j+1/2}$  where  $t_{j+1/2} = t_j + \frac{1}{2}\tau$  by the splitting-up method

$$(87) \quad \begin{aligned} \frac{\varphi^{j+1/2n} - \varphi^j}{\tau/2} + A_1\varphi^{j+1/2n} &= f^{j+1/2}, \\ \frac{\varphi^{j+1/2n} - \varphi^{j+1/2n}}{\tau/2} + A_2\varphi^{j+2n} &= 0, \\ &\dots\dots\dots \\ \frac{\varphi^{j+1/2} - \varphi^{j+n-1/2n}}{\tau/2} + A_n\varphi^{j+1/2} &= 0. \end{aligned}$$

If the solution of (87) is obtained, then we find the solution of Eq. (86) for  $t = t_{j+1}$  using the difference scheme of the second order approximation

$$(88) \quad \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^{j+1/2} = f^{j+1/2}.$$

The scheme of realization of the system (87) is

$$(89) \quad \left(E + \frac{\tau}{2} A_1\right) \left(E + \frac{\tau}{2} A_2\right) \dots \left(E + \frac{\tau}{2} A_n\right) \varphi^{j+1/2} = \varphi^j + \frac{\tau}{2} f.$$

Eq. (89) is a scheme of the first order approximation for (86) in the interval  $t_j \leq t \leq t_{j+1/2}$  for a class of smooth solutions. In fact,

$$(90) \quad \left(E + \frac{\tau}{2} A_1\right) \left(E + \frac{\tau}{2} A_2\right) \dots \left(E + \frac{\tau}{2} A_n\right) = E + \frac{\tau}{2} A + \frac{\tau^2}{4} R,$$

where

$$R = (A_1 A_2 + \dots) + \frac{\tau}{2} (A_1 A_2 A_3 + \dots) + \dots + \left(\frac{\tau}{2}\right)^n A_1 A_2 \dots A_n.$$

From (89) and (90), we obtain

$$(91) \quad \frac{\varphi^{j+1/2} - \varphi^j}{\tau/2} + A\varphi^{j+1} = f^{j+1/2} + O(\tau).$$

Then, correct the solution of (86) by means of Eq. (88). We have

$$(92) \quad \varphi^{j+1} = \varphi^j - \tau(A\varphi^{j+1/2} - f^{j+1/2}).$$

Substitute  $\varphi^{j+1/2}$  from (91) into (92). Then we arrive at the conclusion that the scheme (92) has the second order approximation. In order to find the condition of stability for (87) and (88), let us consider the homogeneous equations (89) and (90) and assume that  $\varphi = \varepsilon$  is the error. As a result we get

$$(93) \quad \left(E + \frac{\tau}{2} A_1\right) \left(E + \frac{\tau}{2} A_2\right) \dots \left(E + \frac{\tau}{2} A_n\right) \varepsilon^{j+1/2} = \varepsilon^j,$$

$$\varepsilon^{j+1} = \varepsilon^j - \tau A \varepsilon^{j+1/2}.$$

If we substitute  $\varepsilon^{j+1/2}$  from (93), we have

$$(94) \quad \varepsilon^{j+1} = \Pi \varepsilon^j,$$

where

$$\Pi = E - \tau A B^{-1}$$

and

$$B = \prod_{\alpha} \left(E - \frac{\tau}{2} A_{\alpha}\right).$$

Now we have the condition of stability

$$(95) \quad \max_n |\lambda_n^{(n)}| < 1,$$

where  $\lambda_n^{(l)}$  are eigenvalues of the spectral problem

$$(96) \quad \Pi \psi_n = \lambda_n^{(l)} \psi_n.$$

If we consider the non-homogeneous problem (87), we are led to the scheme

$$(97) \quad \varphi^{j+1} = \Pi \varphi^j + G,$$

where

$$(98) \quad G = \tau \left( E - \frac{\tau}{2} \Lambda B^{-1} \right) F^{j+1/2}.$$

Discuss now a more general problem of the splitting-up method. Let us consider a non-stationary problem of the solution of the linear equations

$$(99) \quad \frac{\partial \varphi}{\partial t} + \sum_{\alpha=1}^n \Lambda_{\alpha} \varphi = 0, \quad \varphi(0) = f.$$

In the interval  $t_j \leq t \leq t_{j+1}$  this problem yields

$$(100) \quad \begin{aligned} \frac{\partial \varphi_1}{\partial t} + \Lambda_1 \varphi_1 &= 0, & \varphi_1(t_j) &= \varphi^j, \\ \frac{\partial \varphi_2}{\partial t} + \Lambda_2 \varphi_2 &= 0, & \varphi_2(t_j) &= \varphi_1(t_{j+1}), \\ & \dots & & \dots \\ \frac{\partial \varphi_n}{\partial t} + \Lambda_n \varphi_n &= 0, & \varphi_n(t_j) &= \varphi_{n-1}(t_{j+1}). \end{aligned}$$

Such kind of splitting is considered in general in the works by A. A. Samarskiĭ and N. N. Janenko.

Let us assume now that the problem considered is periodical for all geometrical variables  $\Lambda_{\alpha}$  ( $\alpha = 1, \dots, n$ ), and all the operators  $\Lambda_{\alpha}$  have a common basis. In this case we find the solution of (100) in the form of a series

$$\varphi_{\alpha} = \sum_{\nu} \varphi_{\alpha}^{\nu} \psi_{\nu}(r),$$

where  $\psi_{\nu}(r)$  is the solution of the homogeneous problem

$$\Lambda \psi = \lambda \psi.$$

Consider a conjugate problem

$$\Lambda^* \psi^* = \lambda \psi^*.$$

Since  $A_\alpha$  have a common basis we get

$$A_\alpha \psi = \lambda^{(\alpha)} \psi \quad (\alpha = 1, 2, \dots, n).$$

The function

$$\varphi = \sum_v \varphi^v \psi_v(r),$$

is a solution of (100) in the interval  $t_j \leq t \leq t_{j+1}$  where  $\varphi^v$  are solutions of the simplest problems

$$\frac{d\varphi^v}{dt} + A_v \varphi^v = 0, \quad \varphi^v(t_j) = \varphi^v(t_j).$$

The solution of these problems is

$$(101) \quad \varphi^v(t) = \varphi^v(t_j) e^{-\lambda_v(t-t_j)} \quad (t \geq t_j).$$

Let us consider the system (100). Again, analogously to the foregoing, we obtain

$$(102) \quad \begin{aligned} \frac{d\varphi_1^v}{dt} + \lambda_v^{(1)} \varphi_1^v &= 0, \quad \varphi_1^v(t_j) = \varphi^{vj}, \\ \frac{d\varphi_2^v}{dt} + \lambda_v^{(2)} \varphi_2^v &= 0, \quad \varphi_2^v(t_j) = \varphi_1^v(t_{j+1}), \\ &\dots\dots\dots \\ \frac{d\varphi_n^v}{dt} + \lambda_v^{(n)} \varphi_n^v &= 0, \quad \varphi_n^v(t_j) = \varphi_{n-1}^v(t_{j+1}). \end{aligned}$$

Solving the problem (102), we have

$$(103) \quad \begin{aligned} \varphi_1^v &= \varphi_1^v(t_j) e^{-\lambda_v^{(1)}(t-t_j)}, \\ \varphi_2^v &= \varphi_1^v(t_{j+1}) e^{-\lambda_v^{(2)}(t-t_j)}, \\ &\dots\dots\dots \\ \varphi_n^v &= \varphi_{n-1}^v(t_{j+1}) e^{-\lambda_v^{(n)}(t-t_j)}. \end{aligned}$$

By successive substitutions, we obtain the solution of the problem at the moment  $t_{j+1}$

$$(104) \quad \varphi_n^v(t_{j+1}) = \varphi^v(t_j) e^{-\lambda_v \tau},$$

where

$$\lambda_v = \sum_\alpha \lambda_v^{(\alpha)}, \quad \tau = \Delta t.$$

Consider the exact solution (101) and put  $t = t_{j+1}$ . Then we obtain

$$(105) \quad \varphi^v(t_{j+1}) = \varphi^v(t_j) e^{-\lambda_v \tau}.$$

Comparing the results, we have

$$\varphi_n^y(t_{j+1}) = \varphi^y(t_{j+1}).$$

This means that the system (100) provides for the exact solution of (99) when  $t = t_j, t_{j+1}, \dots$ . Certainly, the equivalence of the problems takes place provided that the commutative operators  $A_x$  have a common basis. Generally, when the operators are non-commutative, there are some important theorems proving the convergence of the approximate solution to the exact one (refer to Demidov-Janenko's theorem).

To conclude with, we shall make some remarks on a weak convergence of the schemes for the solution of stationary problems.

Let us consider a more general problem

$$(106) \quad A\varphi = f,$$

where the operator  $A$  is positively definite.

To solve this problem, let us consider a non-stationary problem

$$(107) \quad \frac{\partial^n \psi}{\partial t^n} + A\psi = f, \quad \psi = \psi_t = \dots = \psi_t^{(n-1)} = 0.$$

Here  $A$  and  $f$  are the functions independent of  $t$ . Let us assume that the non-stationary problem is solved and the solution of (107) belongs to the space  $\{\psi\}$ . Let us suppose further that all the functions of  $\{\psi\}$  have continuous and bounded derivatives up to the order  $n$  inclusive and satisfy

$$|\psi| < M = \text{const},$$

where  $M$  does not depend on  $t$ . In this case the solution of (106) may be obtained as a functional of the solution of (107).

Actually, let us consider

$$(108) \quad \varphi^{(T)} = \frac{n!}{T^n} \int_0^T dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \psi(r, t_n) \equiv S^{(T)}\psi,$$

where  $T$  is a parameter connected with the interval  $0 \leq t \leq T$ . Integrate Eq. (107). We get

$$(109) \quad S^{(T)} \frac{\partial^n \psi}{\partial t^n} + A\varphi^{(T)} = f.$$

Consider the relation

$$(110) \quad S^{(T)} \frac{\partial^n \psi}{\partial t^n} = \frac{n!}{T^n} \int_0^T dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} \frac{\partial^n \psi}{\partial t_n^n} dt_n = \frac{n!}{T^n} \psi(r, T).$$

Use here the first condition from (107). As a result (109) is rewritten as

$$(111) \quad \Lambda\varphi^{(T)} = f - \frac{n!}{T^n} \psi(r, T).$$

Since we let  $|\psi(r, T)| < M$ , we obtain

$$(112) \quad \Lambda\varphi^{(\infty)} = f,$$

when

$$T \rightarrow \infty.$$

Therefore, on this assumption, the solution of the stationary problem is obtained as a result of the solution of the non-stationary problem. In the case  $n = 2$ , some suggestions in this connection are given in Saul'ev's work.

6. By this time a great number of methods have been developed for the solution of the Boltzmann linear kinetic equation. The well-known of them, namely, the method of a Gaussian quadrature, spherical-harmonic method, Vladimirov's method of characteristics, Carleson's  $S_n$ -method, are widely used in solving various problems of nuclear physics, astrophysics, meteorology and so on.

At present, owing to the intensive development of computational techniques and creation of high-efficiency computers as well as to the progress in computational mathematics, it became possible to work out new methods based on splitting of the complex operators of the problems into the simplest ones. Different aspects of the splitting-up method have been considered in many investigations. The author and N. N. Janenko in their work have discussed basic points of the construction of various splitting-up schemes for the solution of non-stationary and stationary transport equations. The formulated algorithms are being used for the solution of multi-dimensional kinetic equations in the region with complex boundaries. This study has been a starting point for the research connected with the splitting-up method in the field of the kinetic equations. A stationary problem has been considered in some studies by the author, V. V. PENENKO, U. M. SULTANGAZIN and in A. A. SAMARSKIĭ's work, and by B. I. GOLDIN.

Let us consider a mathematical formulation of a problem. Let  $G$  be an open connected set in an  $n$ -dimensional Euclidean space.

Let the boundary  $\Gamma$  of the set  $G$  be convex and piecewise smooth (i.e. let a normal  $n$  be in every point of the boundary except for the finite set of points). Let  $\Omega \subset R_n$  be a set of all unit vectors of the directions  $\Omega$ .

Consider a one-velocity equation describing the transport of neutrons in a medium with isotropic scattering

$$(113) \quad \frac{1}{v} \frac{\partial \varphi}{\partial t} + \Omega \varphi + \sigma \varphi = \frac{\sigma_s}{4\pi} \int d\Omega \varphi(r, \Omega, t) + f(r, \Omega, t).$$

For Eq. (113) we pose a Cauchy problem with boundary conditions

$$(114) \quad \begin{aligned} \varphi(r, \Omega, t) &= 0; \quad (\Omega, n) < 0; \quad r, \Omega, t \in \gamma; \\ \varphi(r, \Omega, 0) &= \Phi(r, \Omega); \quad r, \Omega \in \Omega \times G \end{aligned}$$

in the cylindric range  $\Pi = \Omega \times G \times T$  with the basis  $\Omega \times G$ , and the side boundary  $\gamma = \Gamma \times T$ . Together with the non-stationary problem (113)–(114), we shall consider the stationary problem

$$(115) \quad \begin{aligned} \Omega \nabla \varphi + \sigma \varphi &= \frac{\sigma_s}{4\pi} \int_{\Omega} d\Omega \varphi(r, \Omega) + (r, \Omega); \\ \varphi(r, \Omega) &= 0; \quad (\Omega, n) < 0; \quad r, \Omega \in \Gamma. \end{aligned}$$

Let the functions  $\sigma = \sigma(r)$ ,  $\sigma_s = \sigma_s(r)$  characterizing the properties of the medium be measurable, positive, piecewise continuous and bounded on  $G$ . As for the functions  $f(r, \Omega, t)$  or  $f(r, \Omega)$  which characterize the radiative source strength, we let them be piecewise continuous in the region  $\Pi$  or  $\Omega \times G$ . For further consideration, it is convenient to introduce the operators

$$(116) \quad A_1 = \Omega \nabla, \quad A_2 = \sigma E - \frac{\sigma_s}{4\pi} \int_{\Omega} d\Omega.$$

Following V. S. VLADIMIROV, introduce the variable  $\xi$  by the formula

$$(117) \quad r' = r - \xi \Omega.$$

Then, along the ray  $\Omega$ , the kinetic equation and the boundary conditions are written accordingly, as follows:

$$(118) \quad \begin{aligned} \frac{\partial}{\partial \xi} \varphi(r' + \xi \Omega, \Omega) + \sigma \varphi(r' + \xi \Omega, \Omega) \\ = \frac{\sigma_s}{4\pi} \int \varphi(r' + \xi \Omega, \Omega) d\Omega + f(r, \Omega), \quad \varphi(r' + \xi_0 \Omega, \Omega) = 0, \end{aligned}$$

where  $\xi_0$  is the value of  $\xi$  at which the radius vector  $r$  crosses the boundary of the region  $G$ . The formula (117) yields, at each  $\Omega$ , a representation of the set  $G$  as a Cartesian product of the two-dimensional set  $\pi_{\Omega}$  and one-dimensional set  $\pi_{\Omega, r'}$ .

$$G = \pi_{\Omega} \times \pi_{\Omega, r'}.$$

The solution of (118) will be sought in the class  $D$  of the real functions. The function  $\varphi$  is said to belong to the class  $D$ , if it possesses the following properties.

1. At almost all values  $(r', \Omega)$  from  $\Omega \times \pi_{\Omega}$  the function  $\varphi(r' + \xi \Omega, \Omega)$  is absolutely continuous in the closed set  $\pi_{\Omega, r'}$ .

2. At almost all values  $(r', \Omega)$  from  $\Omega \times \pi_\Omega$  the function  $\varphi$  satisfies the condition

$$\varphi(r' + \xi_0 \Omega, \Omega) = 0.$$

3. The function  $\varphi$  must be such that

$$L\varphi = \frac{\partial}{\partial \xi} \varphi(r + \xi \Omega, \Omega) + \sigma_s \varphi(r' + \xi \Omega, \Omega) \in L_2(\Omega \times G) \quad \text{for } \sigma \geq 0.$$

The set  $D$  of functions is compact in  $L_2(\Omega \times G)$ . The conditions 1)–3) define the class  $D$  of the generalized solutions of Eq. (118). In this case, any function of  $D$ , satisfying Eq. (118) almost everywhere, will be called a generalized solution of Eq. (118).

The assumptions regarding the functions  $\sigma(r)$ ,  $\sigma_s(r)$  and  $f(r', \Omega)$  being made, we can establish some properties of the operators  $A_1$  and  $A_2$  considered above in (116) in the space  $L_2(\Omega \times G)$ .

**Lemma 1.** *The operator  $A_2$  at  $\sigma_s > 0$  is positively definite.*

**Lemma 2.** *The operator  $A_1$  is positive.*

**Lemma 3.** *The operator  $(E + \frac{1}{2}\tau A_1)^{-1}$  exists and is bounded from  $L_2(\Omega \times G)$  in  $D$ , whereas*

$$\left\| \left( E + \frac{\tau}{2} A_1 \right)^{-1} F \right\| \leq (1 - e^{-2d/\tau}) \|F\|, \quad F \in L_2,$$

where  $d$  is the diameter of the region  $G$ .

In the linear set  $D$ , introduce the norm setting

$$\|\varphi\|_{D_2} = \left\| \left( E + \frac{\tau}{2} A_1 \right) \varphi \right\|, \quad \varphi \in D \quad (\varphi \in D_2)$$

and point out the inequality

$$\|\varphi\| \leq (1 - e^{-2d/\tau}) \|\varphi\|_{D_2}$$

which takes place for all values  $\varphi \in D$ .

Hence, the convergence in  $D_2$  follows, accordingly, from the convergence in  $L_2$ . Setting that the range of definition of the operator  $A_1$  constitutes the functions of the set  $D$ , we can write the boundary value problem (118) in the operator form

$$(119) \quad (A_1 + A_2) \varphi = f, \quad A_1 + A_2 = A.$$

For the solution of (119), let us construct the iterative process representing the

universal algorithm

$$(120) \quad B \frac{\varphi^{j+1} - \varphi^j}{\tau} + A\varphi^j = f,$$

where  $B$  is an operator chosen as follows:

$$B = \left(E + \frac{\tau}{2} A_1\right) \left(E + \frac{\tau}{2} A_2\right),$$

$\tau$  is a time step or the iterative parameter. Let us prove the convergence of the iterative process (120) to a unique solution of the stationary boundary value problem (119). Uniqueness of the solution of the boundary value problem for the transport equation follows from the maximum principle proved by T. A. GERMOGENOVA. The proof of convergence of the iterative process (120) for the case  $(\sigma_s > 0)$  can be found in a number of studies.

In our further discussion we shall not go beyond this limitation. In order to prove convergence, we shall need only bounded, positively definite or semi-definite operators  $A_1$  and  $A_2$  in  $L_2$ . Consider representation of an arbitrary linear operator as a sum of the two operators

$$A = C + D,$$

where

$$C = \frac{1}{2}(A + A^*), \quad D = \frac{1}{2}(A - A^*).$$

Here  $A^*$  denotes a conjugate operator with respect to  $A$ . Let us introduce the scalar product as usual and assume that

$$\|\varphi\| = \sqrt{(\varphi, \varphi)}, \quad \|A\|^2 = \sup_{\|\varphi\|=1} \frac{(A\varphi, A\varphi)}{(\varphi, \varphi)}$$

are the norms in  $L_2$  of the function  $\varphi$  and the operator  $A$  respectively, and  $N(A)$  is the zero space of the operator  $A$ .

By virtue of the relation

$$(A\varphi, \varphi) = (C\varphi, \varphi)$$

in the real vector space with a scalar product, instead of the operator  $A$ , one can consider the operator  $C$  which is self-adjoint in contrast to  $A$ .

**Lemma 4.** *If  $0 < \alpha < \beta < \infty$  and the operator  $A$  satisfies, in  $L_2$ , the condition  $(A\varphi, \varphi) \geq 0$  for any  $\varphi$ , then there exists  $\lambda$  ( $0 < \lambda \leq 1$ ) such that for any  $\alpha \leq a \leq b \leq \beta$  the operator  $(E + bA)$  has the inverse operator, and*

$$\|(E + aA)(E + bA)^{-1}\| \leq \lambda.$$

*If  $(A\varphi, \varphi) > 0$  for  $\varphi \neq 0$ , then  $\lambda < 1$ .*

The proof is based on Kellogg's lemma.

**Lemma 5.** Let  $0 < \alpha < \beta < \infty$  and  $A, A_1$  be linear operators in  $L_2$  such that the operators  $C$  and  $C_1$  satisfy the conditions  $(C\varphi, \varphi) \geq 0$ ,  $(C_1\varphi, \varphi) \geq 0$  and  $N(C) \cap N(C_1) = \emptyset$ , then there exists a positive number  $\gamma < 1$  such that for each  $\alpha < a < b < \beta$ ,  $\|(E - aA)(E + bA)^{-1}(E - aA_1)(E + bA_1)^{-1}\| < \gamma$ . This lemma is a generalization of the Douglas-Pierce lemma.

**Theorem 1.** For any initial approximation  $\varphi^0 \in D$  the iterative process (120) converges in  $L_2$  to the unique solution of the problem (119).

Proof. By identical transformations, we rewrite Eq. (120)

$$\varphi^{j+1} = T\varphi^j + g,$$

where

$$(121) \quad T = \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E + \frac{\tau}{2}A_2\right)^{-1} \left(E - \frac{\tau}{2}A_2\right) \left(E - \frac{\tau}{2}A_1\right),$$

$$g = \tau \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E + \frac{\tau}{2}A_2\right)^{-1} f.$$

Obviously,  $T$  is a linear transformation and maps the set  $D$  into itself.

Let  $\varphi \in D$ , then

$$\begin{aligned} \|T\varphi\|_{D_2} &= \left\| \left(E + \frac{\tau}{2}A_1\right) T\varphi \right\| = \\ &= \left\| \left(E + \frac{\tau}{2}A_2\right)^{-1} \left(E - \frac{\tau}{2}A_2\right) \left(E - \frac{\tau}{2}A_1\right) \left(E + \frac{\tau}{2}A_1\right)^{-1} \left(E + \frac{\tau}{2}A_1\right) \varphi \right\| \leq \\ &\leq \|P\| \left\| \left(E + \frac{\tau}{2}A_1\right) \varphi \right\| = \|P\| \|\varphi\|_{D_2}, \end{aligned}$$

where

$$P = \left(E + \frac{\tau}{2}A_2\right)^{-1} \left(E - \frac{\tau}{2}A_2\right) \left(E - \frac{\tau}{2}A_1\right) \left(E + \frac{\tau}{2}A_1\right)^{-1}.$$

By virtue of lemma 5, we get

$$(122) \quad \|T\|_{D_2} \leq \|P\| < 1.$$

Convergence follows from the inequalities (122), and uniqueness of the solution follows from the maximum principle. The result of Theorem 1 holds directly, in consequence of lemma 5, for the case of non-increasing sequence of the iterative parameters.

**Theorem 2.** Let  $\alpha \leq \beta$  be certain positive numbers, then, for any sequences of the relaxation parameters  $\tau_j$ ,  $s = \{\beta = \tau_1 \geq \tau_2 \geq \dots \geq \tau_{n_0} = \alpha, \tau_j = \tau_j \pmod{n_0}\}$  for  $j > n\}$ , the iterative process (120) converges to the unique solution of the problem (119).

Setting  $v \Delta t = \tau$ , we see that the algorithm (120) is applicable to the solution of the non-stationary problem (113), whereas  $T$  (121) is an amplification operator. The range of definition of the operator  $T$  is compact in  $L_2(\Omega \times G)$ . The fulfilment of the inequality

$$\|T\| < 1 \quad \text{at any } \tau > 0$$

provides that the family of operators  $T^n$ , for all  $n > 0$ , is uniformly bounded. Besides, after the fulfilment of the conditions of lemma 5, the inequality

$$\left\| \left( E + \frac{\tau}{2} A_1 \right)^{-1} \left( E + \frac{\tau}{2} A_2 \right)^{-1} \right\| < 1 \quad \text{at any } \tau > 0$$

holds. The facts mentioned above provide for the stability of computation. A direct test reveals that there is an approximation in  $\tau$ . By a corresponding choice of the finite-difference schemes we provide for approximation in spatial variables. Then, according to the Lax equivalence theorem, we obtain convergence of the solution of the approximate problem (119) to that of the initial non-stationary problem (113) and (114).

There are various schemes and their modifications for a practical realization of the algorithms of the numerical solution of the multi-dimensional kinetic equation by means of the splitting-up method. Above we have considered a general philosophy of the construction of such schemes and discussed a question of formulating boundary conditions for the schemes of complete splitting. We shall construct realization schemes on the basis of the formula of a universal algorithm. Rewrite Eq. (120), carrying out obvious transformations, as

$$(123) \quad \left( E + \frac{\tau}{2} A_2 \right) \left( E + \frac{\tau}{2} A_1 \right) (\varphi^{j+1} - \varphi^j) = \tau(A\varphi^j - f).$$

Introduce the subsidiary functions  $\varphi_1$  and  $\varphi_2$  and represent Eq. (123) as a system of equations

$$(124) \quad \begin{aligned} \left( E + \frac{\tau}{2} A_2 \right) \varphi_1^{j+1/2} &= F^j, \\ \left( E + \frac{\tau}{2} A_1 \right) \varphi_2^{j+1} &= \varphi_1^{j+1/2}, \\ \varphi^{j+1} &= \varphi^j + \tau \varphi_2^{j+1}, \end{aligned}$$

where  $F^j = A\varphi^j - f$  is the discrepancy of the iterative process.

7. The mathematical apparatus, discussed in the previous section, has been used for the solution of a non-linear system of hydrodynamical equations. Since the system of hydrodynamical equations is quasilinear, it is internally closely connected with linear equations with variable coefficients obtained from the first time linearization at each time interval. The theory of a system of partial differential equations with variable coefficients has been thoroughly studied and can be used for substantiation of the numerical algorithms which are being developed.

Consider a system of prognostic equations in the form

$$(125) \quad \begin{aligned} \frac{du}{dt} &= -\frac{\partial H}{\partial x} + lv, \\ \frac{dv}{dt} &= -\frac{\partial H}{\partial y} - lu, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \tau}{\partial p} &= 0, \\ \frac{\partial T}{\partial t} - \frac{\gamma_a - \gamma}{g} RT \frac{\tau}{p} &= 0, \\ T &= -\frac{p}{R} \frac{\partial H}{\partial p} \end{aligned}$$

with the boundary conditions

$$(126) \quad \tau = \frac{p}{RT} \frac{\partial H}{\partial t} \quad \text{at } p = p_0, \quad \tau = \frac{p}{RT} \frac{\partial H}{\partial t} \quad \text{at } p = 0.$$

Here  $u, v, \tau$  are components of the velocity vector  $u$  in the  $x, y, p$  coordinate axes,  $H$  is the height of the isobaric surface  $p = \text{const}$ ,  $T$  is temperature,  $p = 0$  is the "upper boundary" of the atmosphere,  $p = p_0$  is the isobaric surface corresponding to the level of the day surface of the Earth.

The initial data of the problem will be chosen in the form

$$(127) \quad u = u^0, \quad v = v^0, \quad T = T^0 \quad \text{at } t = 0.$$

In order to split the formulated problem into the simplest ones, we must imagine the fundamental factors determining evolution of the fields of meteorological elements. In our opinion, there are at least two such factors, namely, transfer of meteorological substances along particle trajectories and adjustment of the fields of meteorological elements. One can represent the following simplified scheme of evolution of meteorological fields. Consider an elementary time interval  $\Delta t$ . For this time interval, meteorological substances, fixed at the initial moment with respect to this interval, will shift along trajectories so that the initial vector of the position of the particles,  $r_0$ , will change to  $r = r_0 + u \Delta t$  with velocity  $u$ . Naturally, this will cause a certain redistribution of the meteorological fields which will disturb the dynamic adjustment of the fields described by the system of the basic equations (125).

It may be assumed that the mentioned adjustment can be fulfilled if we solve the equation of adaptation in the same interval by taking, as the initial state of the fields of meteorological elements, the obtained fields after their shift along trajectories. In this case we can consider the adjective terms in the corresponding equations (125) to be missing, since the shifting of the fields has been taken into account at the first stage of consideration. At the second stage adjustment of the fields will take place after their shifting along trajectories. Mechanisms allowing such adjustment are wave processes which asymptotically develop into gravitational and sonic waves. Through the wave processes, discrepancies of the adjustment will be distributed in the fields of meteorological elements and will be corrected in accordance with the laws of dynamics. So, instead of the continuous interaction of all the factors in the process of evolution of meteorological fields, we consider a simplified scheme of a discrete representation of each mechanism separately on condition that each time we can assume the fields of meteorological elements to be additive with respect to the effect of the two factors considered. Naturally, this is a certain simplification of the nature of dynamics of the atmospheric processes. However, such a model roughly represents the fundamental constantly operating factors the better the less is the time interval  $\Delta t$ .

According to the adopted model, let us formulate a splitting-up scheme of the dynamics equations (125) in the time interval  $(t_j, t_{j+1})$ . At the first stage, along trajectories of the air particles,

$$(128) \quad \begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= 0, \quad u = u^j \quad \text{at} \quad t = t_j; \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= 0, \quad v = v^j \quad \text{at} \quad t = t^j; \\ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= 0, \quad T = T^j \quad \text{at} \quad t = t_j. \end{aligned}$$

After the solution of (128) is found in the interval  $t_j \leq t \leq t_{j+1}$ , let us solve the problem

$$(129) \quad \begin{aligned} \frac{\partial u}{\partial t} - lv &= -\frac{\partial H}{\partial x}, \\ \frac{\partial v}{\partial t} + lu &= -\frac{\partial H}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \tau}{\partial p} &= 0, \\ \frac{\partial T}{\partial t} &= \frac{\gamma_a - \gamma}{g} RT \frac{\tau}{p}, \\ T &= -\frac{p}{R} \frac{\partial H}{\partial p} \end{aligned}$$

with the boundary conditions

$$(130) \quad \tau = \frac{p}{RT} \frac{\partial H}{\partial t} \quad \text{at } p = p_0, \quad \tau = \frac{p}{RT} \frac{\partial H}{\partial x} \quad \text{at } p = 0,$$

the solution of the problem (128) at the time interval  $t_j \leq t \leq t_{j+1}$  being taken as the initial data for the solution of the system (129). Thus, the problem of weather prediction has been reduced to the two simpler problems which are solved by the splitting-up method described in detail in the author's monograph "Numerical methods in weather prediction". G. V. DEMIDOV has substantiated this splitted algorithm theoretically. On the basis of the Demidov-Janenko theorem on a weak approximation and by introducing a special norm, he has found out that, if the initial data belong to Sobolev's  $W_2^{m,k_0}$  class, i.e. to a set of functions having generalized derivatives up to the order  $m \geq 3$  in  $x, y$  and  $k_0 \geq 2$  in  $p$ , then the solution of the problem (128)–(130) exists for small values of  $t$ , and tends, by the norm, to the solution of the initial problem (125)–(127) when  $\Delta t \rightarrow 0$ .

Similarly, we can solve other complicated problems of mathematical physics: those of hydrodynamics, theory of elasticity, filtration, ocean dynamics, etc.

For references concerning application of the splitting-up method to the solution of mathematical physics problems, see papers and monographs. Some of them are mentioned in the present study.

#### References

- [1] D. Peaceman, H. Rachford: The numerical solution of parabolic and elliptic differential equations. J. Soc. Ind. Appl. Math., 3, 1955.
- [2] J. Douglas, H. Rachford: On the numerical solution of heat conduction problems in two and three space variables, Trans. Amer. Math. Soc., 82, 1956.  
Aplikace matematiky, 10, 1965.
- [3] Н. Н. Яненко: Метод дробных шагов, Новосибирск, 1966.
- [4] А. А. Самарский: О разностных схемах для многомерных дифференциальных уравнений математической физики, Апликасе математикy, 10, 1965.
- [5] А. А. Самарский: Некоторые вопросы теории разностных схем, Ж. вычисл. матем. и матем. физ., 6, 1966, № 4.
- [6] Е. Г. Дьяконов: Разностные схемы второго порядка точности с расщепляющимся оператором для многомерных параболических уравнений с переменными коэффициентами, Вычислительные методы и программирование, III изд., МГУ, 1965.
- [7] Е. Г. Дьяконов: О некоторых итерационных методах решения систем разностных уравнений, возникающих при решении методом сеток уравнений в частных производных эллиптического типа, Вычислительные методы и программирование, III изд., МГУ, 1965.
- [8] J. Douglas, Jr., J. Gunn: A general formulation of alternating direction methods, Part I. Parabolic and hyperbolic problems, Numer. Math., 6, No. 5, 1964.
- [9] Г. И. Марчук: Численные методы в прогнозе погоды, Гидрометеоздат, 1967.
- [10] Г. И. Марчук, Н. Н. Яненко: Применение метода расщепления (дробных шагов) для решения задач математической физики, доклад на конгрессе, Нью-Йорк, 1965.

G. I. Marčuk (Г. И. Марчук), Новосибирск 90, СССР.