Aplikace matematiky

Daniel Mayer

A contribution to the generalized formulation of the matrix methods of mesh currents and node voltages

Aplikace matematiky, Vol. 15 (1970), No. 4, 255–270

Persistent URL: http://dml.cz/dmlcz/103294

A CONTRIBUTION TO THE GENERALIZED FORMULATION OF THE MATRIX METHODS OF MESH CURRENTS AND NODE VOLTAGES

DANIEL MAYER

(Received June 27, 1969)

1. DELIMITATION OF THE PROBLEM

In the basic formulation of the matrix method of mesh currents it is assumed that the circuit to be analyzed contains only voltage sources; if the circuit contains also current sources, they are converted to equivalent voltage sources. (This is described in detail in literature, e.g. [1].) If, however, a branch of the circuit contains an ideal current source (with zero internal admittance), it cannot – exactly speaking – be replaced by an ideal voltage source (with zero internal impedance). Besides, from the viewpoint of numerical computation this procedure is not advantageous since the currents flowing in the branches with ideal current sources are actually known, so that the only aim of the analysis is then the determination of currents in the remaining branches. If we convert the current sources to voltage sources and then analyze the circuit by employing the usual procedure, we do not make use of the known information about branch current values (in branches with current sources) and search for currents in all the branches of the circuit, which is, from the viewpoint of computation, uneconomical.

In the dual case, the situation is quite analogous: if a circuit containing ideal voltage source is analyzed by the method of node voltages, then – exactly speaking – it is not possible, and from the viewpoint of numerical calculation even not advantageous to replace these sources by ideal current sources.

In this article we shall generalize on the one hand the matrix method of mesh currents, and on the other hand the matrix method of node voltages for the case where the circuit contains both kinds of sources. We shall not convert them to a single type of sources (i.e. to voltage sources for the mesh method and to current sources for the node method) and maximum use will be made of the information about mesh current and node voltage values represented by the presence of these sources.

2.1 Analysis of the circuit containing only voltage sources. Since in literature the matrix methods of circuit analysis are presented in a quite diverse manner we shall acquaint the reader with the terminology and basic relations used in the following text: we shall describe a practical procedure for an application of the mesh current method in the basic formulation.

a) We orientate all the l branches of the circuit analyzed, introduce a complete oriented system of n independent meshes and establish the second incidence matrix $\mathbf{C}(l; n)$; its rows correspond to the branches and its columns to the meshes of the circuit. If the i-th branch is included in the j-th mesh and the orientation of the branch coincides with that of the mesh, then the relevant matrix element has the value $c_{ij} = 1$; if the i-th branch is included in the j-th mesh but their orientation is opposite, then $c_{ij} = -1$ and, finally, if the i-th branch is not included in j-th mesh, then $c_{ij} = 0$. Now we introduce mesh currents into n independent meshes.

b) We introduce the column matrix of the branch voltage sources $\mathbf{E}(l; 1)$ and the square matrix of the branch impedances $\mathbf{Z}(l)$. By their transformation we obtain the column matrix of the mesh voltage sources $\mathbf{E}'(n; 1)$ and the symmetric matrix of the mesh impedances $\mathbf{Z}'(n)$:

$$\mathbf{E}' = {}^t\mathbf{C}\mathbf{E}\,, \quad \mathbf{Z}' = {}^t\mathbf{C}\mathbf{Z}\mathbf{C}\,. \tag{1}$$

c) We determine the mesh admittance matrix by inverting the mesh impedance matrix:

$$\mathbf{Y}' = \mathbf{Z}'^{-1}\,. \tag{2}$$

d) From the so-called mesh equation

$$\mathbf{E}' = \mathbf{Z}'\mathbf{I}'\,. \tag{3a}$$

we calculate the mesh current matrix $\mathbf{I}'(n; 1)$:

$$\mathbf{I}' = \mathbf{Y}'\mathbf{E}'\,. \tag{3b}$$

e) The branch current matrix $\mathbf{I}(l; 1)$ is then

$$\mathbf{I} = \mathbf{C}\mathbf{I}'\,. \tag{4}$$

f) If the branch voltage matrix $\mathbf{U}(l; 1)$ is of interest too, we determine it from the equation

$$\mathbf{U} = \mathbf{Z}\mathbf{I} - \mathbf{E}\,. \tag{5}$$

From the above equation it is evident that the branch current matrix may be expressed by the relation

$$\mathbf{I} = \mathbf{C}\,({}^t\mathbf{C}\mathbf{Z}\mathbf{C})^{-1}\,{}^t\mathbf{C}\mathbf{E} \tag{6}$$

and the branch voltage matrix by the relation

$$\mathbf{U} = \{\mathbf{ZC}\,({}^t\mathbf{CZC})^{-1}\,{}^t\mathbf{C} - \mathbf{J}\}\,\mathbf{E} \tag{7}$$

where $\mathbf{J}$ is the unit matrix.

2.2 Analysis of the circuit containing voltage sources and current sources. Topological part. Let the analyzed circuit have current sources in its p branches, and voltage sources, which may assume zero values, in the remaining q branches $(p + q = l)$. Later we shall show that for physically realizable circuits there is always $p \leqq n$. Evidently it will be advantageous to introduce such a complete system of independent meshes that each of the p mesh currents will at the same time be the current in a single branch containing a current source. Then these p mesh currents will be known and the subject of the analysis will be to find the remaining $n - p$ mesh currents and finally the remaining $l - p$ branch currents.

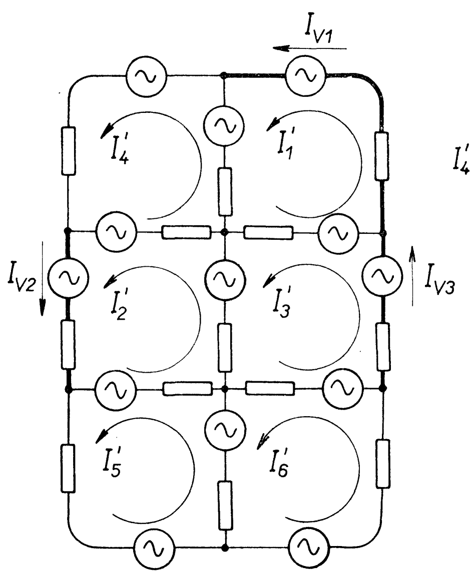


Fig. 1. Circuit with three current sources and with a complete system of independent meshes.

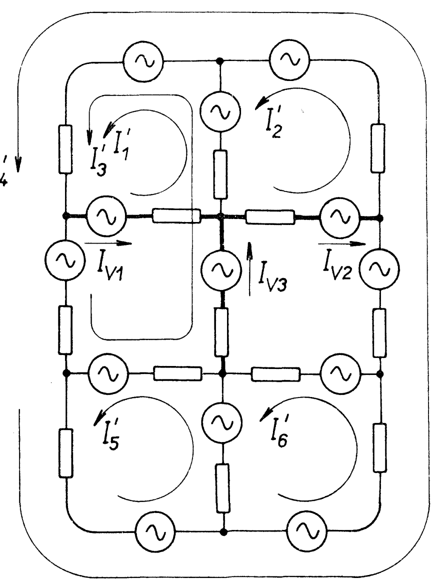


Fig. 2. Circuit with three current sources and with a complete system of independent meshes. Here the choice of meshes shown in Fig. 1 is not advantageous any more.

As an example let us have a circuit with three current sources (I_{v1}, I_{v2}, I_{v3}) as shown in Fig. 1. If we choose the complete system of independent meshes in accordance with Fig. 1, the mesh currents I'_1, I'_2, I'_3 will evidently be known and it remains to determine the mesh currents I'_4, I'_5, I'_6. If, however, the current sources were located in another triplet of branches, then the choice of the complete system of independent meshes might be incorrect. For instance, for the circuit shown in Fig. 2 it is advantageous to choose the meshes in the manner shown in the figure. By introducing the complete system of independent meshes as shown in Fig. 1, the branches with current sources would be incident to two meshes and we could not a priori declare certain mesh currents to be known.

For simple circuits as are those used in the above example, the determination of a suitable system of independent meshes is fairly easy. For circuits with complex and particularly with spatial linear graphs, however, we cannot rely on it that the estimation of a correct solution of the mentioned topological problem will be an easy matter.

For brevity, let us introduce the term "current branches" for branches containing currents sources. (Thus, the currents flowing in the "current branches" are known.) By "voltage branches" we shall understand branches with voltage sources. Thus the linear graph consists of two types of branches: "current branches" and "voltage branches". We shall assume it to be connected and denote it by $\mathcal{G}$.

First we shall prove that the graph $\mathcal{G}$ may always be decomposed to a complete tree containing not even a single "current branch" and a complete system of independent branches that comprises all the "current branches" of the linear graph $\mathcal{G}$. – If we leave out all the "current branches" in the graph $\mathcal{G}$, we obtain the subgraph $\mathcal{G}'$ which has the same set of nodes as the graph $\mathcal{G}$, but its branches are a subset of the set of branches of the graph $\mathcal{G}$. Let us assume for a moment that in the general case the subgraph $\mathcal{G}'$ is not connected: let it consist of μ components $\mathcal{B}_i$ $(i = 1, 2, \ldots \ldots, \mu)$, each of which is connected. (This is to say that the subgraph $\mathcal{G}'$ is a unification of disjoint subgraphs $\mathcal{B}_i$.) It will be seen easily that this case is not physically realizable. The component $\mathcal{B}_i$ is connected with the other components only through the omitted branches (i.e. "current branches"). However, from the generalized Kirchhoff's current law (see [1], p. 80) there follows that the sum of all currents flowing into the component $\mathcal{B}_i$ from the remaining components must be identically equal to zero. In the case consirdered this condition is, of course, not satisfied, for the currents in the "current branches" are given by the values of the current sources, which may have arbitrary magnitudes. The subgraph $\mathcal{G}'$ must therefore be connected. If we omit in the subgraph $\mathcal{G}'$ the minimum number of its "voltage branches" so that there does not remain a single mesh, we obtain a complete tree of the graph $\mathcal{G}$ composed of "voltage branches" only.

From the above consideration it follows that $p \leqq n$.

Let us note that this property of the linear graph $\mathcal{G}$ follows directly from the physical interpretation of the concept "complete system of independent branches", this being such a set of branches where the currents may be chosen independently on each other (see [1], p. 32, par. 2). Hence it follows that the current sources may exist in n independent branches at most. (Thus, the proof just described confirms once more the known property of a complete system of independent branches.)

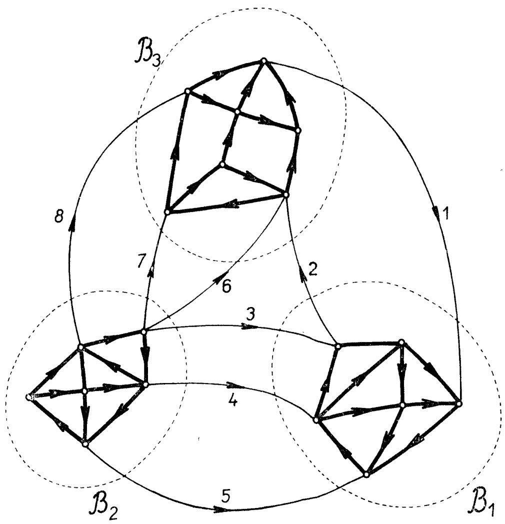


Fig. 3. Linear graph of a circuit with three components ($\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$) composed of "current branches". (This linear graph has no physical significance.)

As an example, Fig. 3 shows a linear graph where the "current branches" are shown by thin lines and the "voltage branches" by heavy lines. A circuit with this linear graph has no physical significance: that is to say, if the "current branches" are omitted we obtain three components $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$. For instance, for the component $\mathscr{B}_1$ there must hold, according to the generalized Kirchhoff's current law, that

$$(*) \qquad I_1 + I_2 + I_3 + I_4 + I_5 = 0\,.$$

However, since these branch currents may have arbitrary values (according to the current sources used) the condition (*) is not generally satisfied, which is a contradiction.

Therefore, when choosing a complete system of independent meshes we proceed as follows: We remove from the linear graph under consideration all "current branches" and such "voltage branches" that a complete tree is obtained. By successive addition of independent branches to the complete tree one can easily introduce such a complete system of independent meshes that just one mesh will be incident to each "current branch" and none of the pairs of these branches will belong to the same mesh.

When numbering the branches we assign numbers first to the p "current branches" and then to the q "voltage branches" (or passive branches). When numbering the meshes, we number first the p meshes, each of them being incident to one of the p "current branches", and then mark the remaining $n - p$ meshes.

The second incidence matrix may then be expressed as a partitioned matrix:

(8) $$\mathbf{C} = \left[\begin{array}{c|c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right]$$

where $\mathbf{C}_{11}(p)$, $\mathbf{C}_{12}(p; n - p)$, $\mathbf{C}_{21}(q; p)$, $\mathbf{C}_{22}(q; n - p)$. Since the first p branches (i.e. "current branches") are incident only to the first p meshes, it is evident that $\mathbf{C}_{12} = \mathbf{0}$. If the orientation of the first p branches and the first p meshes is coincident, then $\mathbf{C}_{11}$ is the unit matrix, $\mathbf{C}_{11} = \mathbf{J}$.

Algebraic part. According to the above agreement on numbering branches and meshes we may express the branch current matrix $\mathbf{I}$ and the mesh current matrix $\mathbf{I}'$ as partitioned matrices:

(9) $$\mathbf{I} = \left[\begin{array}{c} \mathbf{I}_1 \\ \hline \mathbf{I}_2 \end{array}\right], \quad \mathbf{I}' = \left[\begin{array}{c} \mathbf{I}'_1 \\ \hline \mathbf{I}'_2 \end{array}\right]$$

where the submatrix $\mathbf{I}_1(p; 1)$ is known (its elements are the values of current sources, $I_i = \pm I_{vi}$ $(i = 1, 2, \ldots, p)$; the sign is positive if the direction of source current I_{vi} agrees with the orientation of the i-th branch and negative in the opoosite case), so that the submatrix $\mathbf{I}'_1(p; 1)$ is also known because, according to Eq. (4),

(10) $$\mathbf{I}_1 = \mathbf{C}_{11}\mathbf{I}'_1 + \mathbf{C}_{12}\mathbf{I}'_2 = \mathbf{I}'_1 .$$

Suppose for a moment that the current sources I_{vi} are not ideal; let their internal admittance by Y^j_{vii} $(j = 1, i = 1, 2, \ldots, p)$. They may then be replaced by equivalent voltage sources $E_i = I_{vi}Z^j_{vii}$ with the internal impedance $Z^j_{vii} = Y^{-j}_{vii}$. Then the branch impedance matrix of the circuit is

(11) $$\mathbf{Z}(l) = \underbrace{\left[\begin{array}{c|c} \mathbf{Z}_{11} + \mathbf{Z}_v & \mathbf{Z}_{12} \\ \hline \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{array}\right]}_{p \qquad\qquad q} \begin{array}{l} \}p \\ \}q \end{array}$$

where $\mathbf{Z}_v(p)$ is a diagonal matrix whose diagonal elements are $Y_{v_{ii}}^{-j}$ ($j = 1$, $i = 1, 2, \ldots, p$). The matrix of branch voltage sources is

(12)
$$\mathbf{E}(l; 1) = \begin{bmatrix} \mathbf{E}_1 \\ \hline \mathbf{E}_2 \end{bmatrix}$$

where the submatrix $\mathbf{E}_1(p; 1)$ has the elements E_i ($i = 1, 2, \ldots, p$) and the elements of the submatrix $\mathbf{E}_2(q; 1)$ are the given values of voltage sources contained in q "voltage branches". Thus the circuit is converted to one containing voltage sources only; its analysis is described in Section 2.1. The mesh equation (3a) of this circuit where Eq. (1) holds for matrices $\mathbf{E}'$ and $\mathbf{Z}'$, is expressed by using partitioned matrices:

(13)
$$\begin{bmatrix} \mathbf{E}'_1 \\ \hline \mathbf{E}'_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}'_{11} & \mathbf{Z}'_{12} \\ \hline \mathbf{Z}'_{21} & \mathbf{Z}'_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}'_1 \\ \hline \mathbf{I}'_2 \end{bmatrix} .$$

This equation may also be expressed as a system of two matrix equations

(14)
$$\begin{aligned} \mathbf{E}'_1 &= \mathbf{Z}'_{11}\mathbf{I}'_1 + \mathbf{Z}'_{12}\mathbf{I}'_2 , \\ \mathbf{E}'_2 &= \mathbf{Z}'_{21}\mathbf{I}'_1 + \mathbf{Z}'_{22}\mathbf{I}'_2 . \end{aligned}$$

Since there are ideal current sources in the circuit, we express $\mathbf{E}'_1$ and $\mathbf{Z}'_{11}$ for $\lim_{j \to +\infty} Z_{v_{ii}}^j$. This limiting process, however, does not concern the second of Eqs. (14) which is fully sufficient for the determination of the mesh current submatrix $\mathbf{I}'_2$:

(15)
$$\mathbf{I}'_2 = \mathbf{Z}'^{-1}_{22}(\mathbf{E}'_2 - \mathbf{Z}'_{21}\mathbf{I}'_1)$$

where

(16)
$$\begin{aligned} \mathbf{Z}'_{22} &= {}^t\mathbf{C}_{22}\mathbf{Z}_{22}\mathbf{C}_{22} , \\ \mathbf{Z}'_{21} &= {}^t\mathbf{C}_{22}\mathbf{Z}_{21} + {}^t\mathbf{C}_{22}\mathbf{Z}_{22}\mathbf{C}_{21} , \\ \mathbf{E}'_2 &= {}^t\mathbf{C}_{22}\mathbf{E}_2 . \end{aligned}$$

The branch current matrix is determined according to Eq. (4): the submatrix $\mathbf{I}_1$ whose elements are the currents in the first p branches (i.e. "current branches") is given by Eq. (10),

$$\mathbf{I}_1 = \mathbf{I}'_1$$

and the submatrix $\mathbf{I}_2$ whose elements are the searched currents in the remaining $l - p$ branches is determined, according to Eq. (4), from the relation

$$\mathbf{I}_2 = \mathbf{C}_{21}\mathbf{I}'_1 + \mathbf{C}_{22}\mathbf{I}'_2 . \tag{17}$$

Substituting for $\mathbf{I}'_2$ from Eq. (15) we have

$$\mathbf{I}_2 = (\mathbf{C}_{21} - \mathbf{C}_{22}\mathbf{Z}'^{-1}_{22}\mathbf{Z}_{21})\,\mathbf{I}'_1 + \mathbf{C}_{22}\mathbf{Z}'^{-1}_{22}\mathbf{E}'_2 \tag{18}$$

where the matrices $\mathbf{Z}'_{22}$ and $\mathbf{Z}'_{21}$ are determined from Eq. (16).

3. GENERALIZATION OF THE NODE VOLTAGE METHOD

3.1 Analysis of the circuit containing only current sources. Mathematical operations involved in circuit analysis using the node voltage method do not differ from those of the mesh current method described in Section 2.1, but in their physical interpretation there appear dual quantities. We shall briefly describe the algorithm of the calculation and confine it, for simplicity, to a circuit without inductive coupling. (Matrix analysis of a circuit with inductive couplings using the method of node voltages is described in [2]).

a) First the circuit is orientated and one node in each of its separate parts is chosen as the reference node. In each separate part we introduce oriented paths between the remaining nodes – we call them independent nodes – and the reference node. Let the circuit considered have k nodes and s separate parts. As we know, the number of orientated paths is then $m = k - s$. We orientate them in such a way as to be always directed towards the reference node. Now we establish the first reduced incidence matrix $\mathbf{K}_r(l; m)$ with rows and columns corresponding to the branches and the independent nodes of the circuit, respectively. If the i-th branch is incident to the j-th independent node, this being its initial node, then $k_{ij} = 1$; if the i-th branch is incident to the j-th node, this being its terminal node, then $k_{ij} = -1$ and, finally, if the i-th branch is not incident to the j-th node, then $k_{ij} = 0$.

We introduce the node voltages into m oriented paths.

b) We establish a column matrix of branch source currents $\mathbf{I}_v(l; 1)$ and a square diagonal matrix of branch admittances $\mathbf{Y}(l)$. By their transformation we obtain the column matrix of node source currents $\bar{\mathbf{I}}_v(m; 1)$ and a symmetrical matrix of node admittances $\bar{\mathbf{Y}}(m)$:

$$\bar{\mathbf{I}}_v = {}^t\mathbf{K}_r\mathbf{I}_v , \quad \bar{\mathbf{Y}} = {}^t\mathbf{K}_r\mathbf{Y}\mathbf{K}_r . \tag{19}$$

c) By inverting the node admittance matrix we determine the node impedance matrix:

$$\bar{\mathbf{Z}} = \bar{\mathbf{Y}}^{-1} . \tag{20}$$

d) From the so-called node equation

(21a) $$\overline{\mathbf{I}}_v = -\overline{\mathbf{Y}}\overline{\mathbf{U}}$$

we calculate the node voltage matrix $\overline{\mathbf{U}}(m; 1)$:

(21b) $$\overline{\mathbf{U}} = -\overline{\mathbf{Z}}\overline{\mathbf{I}}_v .$$

e) The branch voltage matrix $\mathbf{U}(l; 1)$ is determined from the equation

(22) $$\mathbf{U} = \mathbf{K}_r\overline{\mathbf{U}} .$$

f) If the branch current matrix $\mathbf{I}(l; 1)$ is of interest too, we determine it from the equation

(23) $$\mathbf{I} = \mathbf{Y}\mathbf{U} + \mathbf{I}_v .$$

Note that the node voltage matrix may be determined directly by the equation

(24) $$\mathbf{U} = -\mathbf{K}_r({}^t\mathbf{K}_r\mathbf{Y}\mathbf{K}_r)^{-1}\,{}^t\mathbf{K}_r\mathbf{I}_v$$

and the branch current matrix by the expression

(25) $$\mathbf{I} = \{\mathbf{J} - \mathbf{Y}\mathbf{K}_r({}^t\mathbf{K}_r\mathbf{Y}\mathbf{K}_r)^{-1}\,{}^t\mathbf{K}_r\}\,\mathbf{I}_v$$

where $\mathbf{J}$ is the unit matrix.

3.2 Analysis of the circuit containing current sources and voltage sources. Topological part. If the circuit under consideration contains a voltage source with an impedance connected in series we shall consider this arrangement as two branches in series: one branch contains only the voltage source – we shall call it "pure voltage branch" – and the other comprises only the impedance. In other words, as nodes of the circuit we consider on the one hand nodes of the second order, which are incident to the "pure voltage branches" and to the branches containing only impedance, and on the other hand, of course, all the nodes of the third and higher orders (Fig. 4).

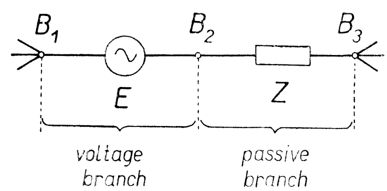


Fig. 4. In the circuit analysis by the method of node voltages, the node B_2 is one of the nodes of the analyzed circuit.

Suppose again that the linear graph $\mathscr{G}$ of the circuit is connected. We denote the number of its "pure voltage branches" by v and the number of all "current branches" and passive branches by w $(v + w = l)$. In comparison with the method of mesh currents this situation is more complicated, for in the general case the reference node incident to all v "pure voltage branches" evidently can not be chosen. Let the "pure voltage branches" of the circuit be formed by ν connected subgraphs: $\mathscr{K}_1$, $\mathscr{K}_2, \ldots, \mathscr{K}_\nu$. Each subgraph $\mathscr{K}_i$ $(i = 1, 2, \ldots, \nu)$ must be a tree. If any of them contained a mesh, this mesh would be incident only to the "pure voltage branches" and the Kirchhoff's voltage law would not hold for it; thus the case would not be physically realizable. This is the case dual to that analyzed in more detail in Section 2.2 where the component $\mathscr{B}_i$ is connected with the remaining components only through "current branches".

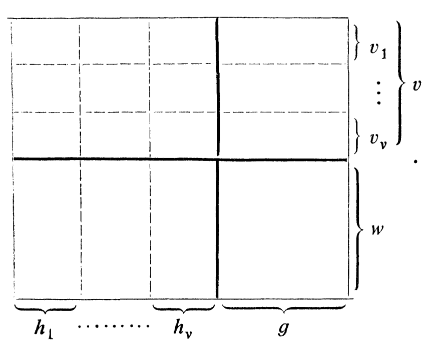
The fact that each subgraph $\mathscr{K}_i$ $(i = 1, 2, \ldots, v)$ must be a tree is directly evident from the physical interpretation of the concept "complete tree", which is a system of the minimum number of branches where the voltages between an arbitrarily chosen (reference) node and the remaining (independent) nodes of the circuit uniquely determine the voltages of the individual branches (see [1], p. 31). From this it follows that if a circuit is to be uniquely determined the "pure voltage branches" must form a tree or a complete tree at most.

For the analysis of the circuit we shall use the principle of superposition: first we analyze the partial circuit containing only the subgraph $\mathscr{K}_1$ ($\mathscr{K}_2, \mathscr{K}_3, \ldots, \mathscr{K}_v$ are removed), then the partial circuit containing only the subgraph $\mathscr{K}_2$ ($\mathscr{K}_1, \mathscr{K}_3, \ldots$ $\ldots, \mathscr{K}_v$ are removed) etc. and superpose the results. The general i-th partial circuit contains of the subgraphs $\mathscr{K}_1$ to $\mathscr{K}_v$ only the subgraph $\mathscr{K}_i$; let this subgraph have v_i branches $(v_1 + \ldots + v_v = v)$. As the reference node we choose the node which is incident to the subgraph $\mathscr{K}_i$ by maximum of its branches; let its degree with respect to the branches of the subgraph be r_i.

When numbering the l circuit branches we number first the v_1 branches of the subgraph $\mathscr{K}_1$, then the v_2 branches of the subgraph $\mathscr{K}_2$, etc. up to the v_v branches of the subgraph $\mathscr{K}_v$ and finally the remaining w branches (i.e. the "current branches" and the passive branches). When numbering the k circuit nodes we number first all h_1 nodes of the subgraph $\mathscr{K}_1$ etc. up to the h_v nodes of the subgraph $\mathscr{K}_v$ and then the remaining g nodes $(h_1 + h_2 + \ldots + h_v + g = k)$.

Thus the first incidence matrix has the form

(26) $\mathbf{K}(l; k) =$



From this matrix we derive the first reduced incidence matrix for the i-th partial circuit $(\mathbf{K}_{ri})$ as follows:

– We determine the reference node by finding a column containing the maximum (r_i) of non-zero elements in the group of v_i rows and h_i columns (belonging to branches and nodes of subgraph $\mathscr{H}_i$). This column corresponds to the node which we choose as the reference node and therefore we remove it from the matrix $\mathbf{K}(l; k)$.

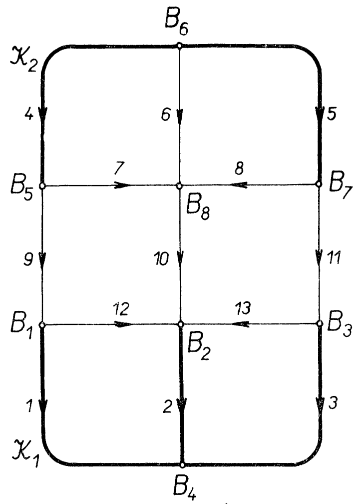


Fig. 5. Linear graph containing two subgraphs ($\mathscr{H}_1$ and $\mathscr{H}_2$) composed of "pure voltage branches".

– We replace the remaining "pure voltage branches" (i.e. the branches of subgraphs $\mathscr{H}_1, \ldots, \mathscr{H}_{i-1}, \mathscr{H}_{i+1}, \ldots, \mathscr{H}_v$) by the internal impedances of their sources, i.e. we short-circuit them. Consequently we omit the rows corresponding to the branch groups $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_v$; we add up the columns corresponding to each of the groups $h_1, h_2, \ldots, h_{i-1}, h_{i+1}, \ldots, h_v$ and replace them by a single column (the "sum column").

– We arrange the sequence of columns as follows: first we put the $h_i - 1$ columns (corresponding to $h_i - 1$ independent nodes of the "pure voltage branches"), then the $v - 1$ "sum columns" follow and finally the g columns (corresponding to g independent nodes of the "current branches") are placed.

Therefore, the first reduced incidence matrix of the i-th partial circuit $\mathbf{K}_{ri}$ is of the type $(v_i + w; m_i)$ where

$$m_i = g + v + h_i - 2 . \tag{27}$$

As an example, Fig. 5 shows an oriented linear graph of a circuit where $l = 13$, $s = 1$, $k = 8$, $m = k - s = 7$. The "pure voltage branches" (drawn by heavy lines) form two subgraphs: $\mathscr{H}_1$ $(v_1 = 3,\ h_1 = 4)$ and $\mathscr{H}_2$ $(v_2 = 2,\ h_2 = 3)$. Evidently, $w = 8$, $g = 1$. The first incidence matrix is

$\mathbf{K} =$

	1	2	3	4	5	6	7	8	
1	1			−1					$v_1 \ldots\ldots \mathscr{K}_1$
2		1		−1					
3			1	−1					
4					−1	1			$v_2 \ldots\ldots \mathscr{K}_2$
5						1	−1		
6						1		−1	w
7					1			−1	
8							1	−1	
9	−1				1				
10		−1						1	
11			−1				1		
12	1	−1							
13		−1	1						
	$h_1 \ldots \mathscr{K}_1$				$h_2 \ldots \mathscr{K}_2$			g	

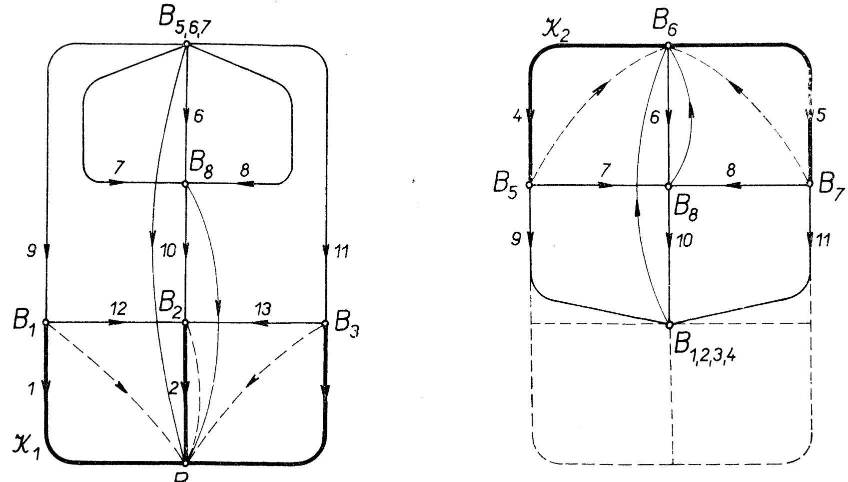


Fig. 6. Two linear graphs which are obtained from the linear graph in Fig. 5 where the subgraphs $\mathscr{K}_1$ or $\mathscr{K}_2$ are replaced by a "short circuit".

The nodes B_4 and B_6 are the reference nodes. The corresponding reduced first incidence matrices are:

$\mathbf{K}_{r1} =$

	1	2	3	5 + 6 + 7	8
1	1				
2		1			
3			1		
6				1	−1
7				1	−1
8				1	−1
9	−1			1	
10		−1			1
11			−1	1	
12	1	−1			
13		−1	1		

, $\mathbf{K}_{r2} =$

	1 + 2 + 3 + 4	5	7	8
4		−1		
5			−1	
6				−1
7		1		−1
8			1	−1
9	−1	1		
10	−1			1
11	−1		1	
12				
13				

.

The linear graphs corresponding to these matrices are shown in Fig. 6a,b.

We express the reduced first incidence matrix of the i-th partial circuit as a partitioned matrix:

(28)
$$\mathbf{K}_{ri} = \underbrace{\begin{bmatrix} \mathbf{K}_{ri_{11}} & \mathbf{K}_{ri_{12}} \\ \mathbf{K}_{ri_{21}} & \mathbf{K}_{ri_{22}} \end{bmatrix}}_{v_i \quad\quad m'_i} \begin{matrix} \}\, v_i \\ \}\, m'_i \end{matrix}$$

where

(29)
$$m'_i = m_i - v_i = g + v + h_i - v_i - 2\,.$$

Since the new nodes corresponding to the "sum columns" and the g nodes of the "current branches" are not incident to the "pure voltage branches" of the subgraph $\mathscr{K}_i$, we have $\mathbf{K}_{ri_{12}} = \mathbf{0}$.

Algebraic part. For the i-th partial circuit we express the node voltage matrix $\overline{\mathbf{U}}_i(m_i; 1)$ as a partitioned matrix

(30)
$$\overline{\mathbf{U}}_i = \begin{bmatrix} \overline{\mathbf{U}}_{i1} \\ \overline{\mathbf{U}}_{i2} \end{bmatrix}$$

where the submatrix $\overline{\mathbf{U}}_{i1}(v_i; 1)$ is known (its elements are the values of voltage sources of the individual paths between the independent nodes of the subgraphs $\mathscr{K}_i$ and its reference node) and $\overline{\mathbf{U}}_{i2}(m'_i; 1)$ is the submatrix to be determined. We proceed in a way analogous to that used in the method of mesh currents described in Section 2.2 (but with dual quantities): let us admit for a moment that the voltage sources (E_i) are not ideal (i.e. their internal impedance $Z^j_{v_{ii}} \neq 0$) and convert them to equivalent current sources $I_{v_i} = E_i Z^{-j}_{v_{ii}}$ of internal admittance $Y^j_{v_{ii}} = Z^{-j}_{v_{ii}}$ $(j = 1, i = 1, 2, \ldots \ldots, v_i)$. The branch admittance matrix $\mathbf{Y}_i$ and the matrix of branch current sources $\mathbf{I}_{v_i}$ are then expressed as partitioned matrices:

$$\mathbf{Y}_i(l_i) = \underbrace{\begin{bmatrix} \mathbf{Y}_{i_{11}} + \mathbf{Y}_{vi} & \mathbf{Y}_{i_{12}} \\ \mathbf{Y}_{i_{21}} & \mathbf{Y}_{i_{22}} \end{bmatrix}}_{v_i \quad\quad w} \begin{matrix} \}\, v_i \\ \}\, w \end{matrix}, \qquad \mathbf{I}_{v_i}(l_i; 1) = \begin{bmatrix} \mathbf{I}_{vi_1} \\ \mathbf{I}_{vi_2} \end{bmatrix}$$

and by their transformation we obtain, according to Eq. (19):

(31)
$$\overline{\mathbf{Y}}_i = {}^t\mathbf{K}_{ri}\mathbf{Y}_i\mathbf{K}_{ri} \quad \text{and} \quad \overline{\mathbf{I}}_{vi} = {}^t\mathbf{K}_{ri}\mathbf{I}_{vi}\,.$$

The node equation (21a) and the mesh equation (13) are written by means of partitioned matrices whence we obtain a system of two matrix equations:

(32)
$$\overline{\mathbf{I}}_{vi_1} = -\overline{\mathbf{Y}}_{i_{11}}\overline{\mathbf{U}}_{i_1} - \overline{\mathbf{Y}}_{i_{12}}\overline{\mathbf{U}}_{i_2}\,,$$
$$\overline{\mathbf{I}}_{vi_2} = -\overline{\mathbf{Y}}_{i_{21}}\overline{\mathbf{U}}_{i_1} - \overline{\mathbf{Y}}_{i_{22}}\overline{\mathbf{U}}_{i_2}\,.$$

For an actual circuit (i.e. for one with ideal sources) these equations hold for $\lim\limits_{j \to +\infty} Y^j_{v_{ii}}$. However, it is not necessary to calculate this limit, as it does not concern the second equation (32), whence

(33)
$$\overline{\mathbf{U}}_{i_2} = \overline{\mathbf{Y}}^{-1}_{i_{22}}(-\overline{\mathbf{Y}}_{i_{22}}\overline{\mathbf{U}}_{i_1} - \overline{\mathbf{I}}_{vi_2})$$

where

$$\overline{\mathbf{Y}}_{i22} = {}^t\mathbf{K}_{ri22}\overline{\mathbf{Y}}_{i22}\mathbf{K}_{ri22}\,, \tag{34}$$
$$\overline{\mathbf{Y}}_{i21} = {}^t\mathbf{K}_{ri22}\overline{\mathbf{Y}}_{i22}\mathbf{K}_{ri22}\,,$$
$$\mathbf{I}_{vi2} = {}^t\mathbf{K}_{ri22}\mathbf{I}_{vi2}\,.$$

By suitable orientation of branches it is possible to make $\mathbf{K}_{ri11} = \mathbf{J}$.

The branch voltage matrix of the i-th partial circuit is

$$\mathbf{U}_i = \begin{array}{|c|} \hline \mathbf{U}_{i_1} \\ \hline \mathbf{U}_{i_2} \\ \hline \end{array} \tag{35}$$

where, according to Eq. (22),

$$\mathbf{U}_{i_1} = \mathbf{K}_{ri11}\mathbf{U}_{i_1}\,, \tag{36}$$

$$\mathbf{U}_{i_2} = \mathbf{K}_{ri21}\mathbf{U}_{i_1} + \mathbf{K}_{ri22}\mathbf{U}_{i_2}\,. \tag{37}$$

Then the branch voltage matrix of the circuit under consideration is, according to the principle of superposition,

$$\mathbf{U} = \sum_{i=1}^{v} \mathbf{U}_i\,. \tag{38}$$

Acknowledgement. The author is pleased to express his thanks to Professor J. Schmidtmayer, CSc. and Mr. J. Kůs for their careful inspection of the article.

Bibliography

[1] *D. Mayer:* Analýza elektrických obvodů maticovým počtem (Analysis of Electric Circuits by Matrix Calculus), Academia, Praha 1966.

[2] *D. Mayer:* Analysis of Electric Circuits with Magnetic Coupling by the Node Voltage Method. Acta Technica ČSAV, 1970, No. 1, pp 16—29.

Souhrn

PŘÍSPĚVEK KE ZOBECNĚNÉ FORMULACI MATICOVÉ METODY SMYČKOVÝCH PROUDŮ A MATICOVÉ METODY UZLOVÝCH NAPĚTÍ

DANIEL MAYER

Obsahem článku je odvození algoritmu analýzy lineárního elektrického obvodu, při čemž se vychází ze základních maticových metod, tj. metody smyčkových proudů a metody uzlových napětí. Dosavadní známé maticové metody řešení (jsou popsány např. v [1]) se tím zobecňují i na obvody, které mohou současně obsahovat oba typy zdrojů (tj. zdroje elektromotorického napětí a zdroje vnitřního proudu) a tyto zdroje mohou být též ideální.

Nejprve jsou stručně zrekapitulovány základní postupy a vztahy maticové metody smyčkových proudů a maticové metody uzlových napětí, jestliže řešený obvod obsahuje pouze zdroje jednoho typu. Pak následuje popis obou zobecněných metod, jenž má dvě části: topologickou a algebraickou. V topologické části je zaveden takový způsob indexování větví, uzlů, popřípadě smyček řešeného obvodu, aby při jeho kompletní analýze byly v maximální míře využity známé údaje obvodu, tj. proudy ve větvích, jestliže tyto větve obsahují zdroje vnitřního proudu a napětí větví, jestliže tyto obsahují pouze zdroje elektromotorického napětí; pro tento způsob značení jsou pak obecně formulovány incidenční matice. V algebraické části jsou odvozeny maticové operace, jimiž je obecně provedena kompletní analýza elektrického obvodu.

Význačnými vlastnostmi popsaných algoritmů je jejich dobrá přehlednost, obecnost, relativní jednoduchost příslušných numerických výpočtů a velmi snadná programovatelnost pro samočinné číslicové počítače.

Author's address: Prof. Ing. *Daniel Mayer*, CSc., Vysoká škola strojní a elektrotechnická, katedra teoretické elektrotechniky, Nejedlého sady 14, Plzeň.