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ON 0—1 MEASURE FOR PROJECTORS

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Gleason has demonstrated [1] that every σ-additive measure μ on projectors in separable Hilbert space is of the form

$$\mu(P) = \mathrm{tr}\,(WP)$$

where the density matrix W is hermitian operator with $\mathrm{tr}\, W = 1$.

This implies that there is no $0-1$ σ-additive measure on projectors in separable Hilbert space.

In a finite dimensional space the σ-additivity of measure is redundant. There must exist a finite number of projectors for which it is impossible to define a non-trivial $0-1$ measure. This follows from the compactness of the space of all functions having values 0 or 1 on projectors in the space. We shall restrict ourselves to E_3 and exhibit an example of such a set of projectors.

We shall begin with the following auxiliary lemma:

Given two planes ϱ, σ which are nearly orthogonal $(\mathrm{tg} \sphericalangle (\varrho, \sigma) > 2\sqrt{2})$, there exist two orthogonal vectors, one in ϱ, the other in σ, and another couple of such vectors so that the third orthogonal vectors to these couples are mutually orthogonal.

Proof. ϱ is the plane $z = 0$, σ is the plane $z = Kx$. A vector in ϱ is $(\cos\varphi, \sin\varphi, 0)$ in σ it is $v = (a, b, Ka)$. The third orthogonal vector is (r, s, t) with

$$a = k \,.\, \sin\varphi\,, \quad b = -k \,.\, \cos\varphi\,,$$

$$r \,.\, \cos\varphi + s \,.\, \sin\varphi = 0\,, \quad r \,.\, a + s \,.\, b + t \,.\, Ka = 0$$

and hence

$$r = \varkappa \,.\, \sin\varphi\,, \quad s = -\varkappa \,.\, \cos\varphi\,, \quad t = -\varkappa/K \,.\, \sin\varphi\,.$$

The second couple has a vector in ϱ $(\cos\psi, \sin\psi, 0)$ and to this one we find the third orthogonal vector (r', s', t'). The orthogonality of $(r, \ldots)$ and $(r', \ldots)$ gives

$$\sin\varphi \,.\, \sin\psi + \cos\varphi \,.\, \cos\psi + 1/(K^2 \,.\, \sin\varphi \sin\psi) = 0$$

or

$$V = K^2 \sin\varphi \sin\psi \cos(\varphi - \psi) + 1 = 0\,.$$

For $\psi = 0$, $V = 1$ independently of the value for φ. Taking

$$\varphi = \pi/6\,, \quad \psi = -\pi/6\,, \quad \varphi - \psi = \pi/3$$

we have $V = -K^2 \sin^2 \pi/6 \,.\, \cos \pi/3 + 1$ and this is <0 for $K > 2\sqrt{2}$.

Hence there exist φ, ψ with $V = 0$.

Now we can construct a finite set of vectors for which no $0-1$ measure exists. We take three orthogonal vectors. The measure for one must be 1, for the other two 0 (we identify the one-dimensional projector and the vector). We shall choose the first as the axis z, the other two as the axes x, y. In the plane (x, z) we take the vector $z^{(1)} = (\cos 4\pi/10, 0, \sin 4\pi/10)$.

We shall apply the lemma to the planes $\varrho = (x, y)$ and $\sigma = (y, z^{(1)})$. We have $\mu(x) = \mu(y) = 0$ and if $\mu(z^{(1)}) = 0$ we find two orthogonal vectors $(r, \ldots)$, $(r', \ldots)$ with $\mu = 1$.

Hence we must $\mu(z^{(1)}) = 1$.

Now we take $x^{(1)}$ orthogonal to $y, z^{(1)}$ and proceed with the triplet $(x^{(1)}, y, z^{(1)})$ in the same manner. Finally, the vector $z^{(5)}$ is identical with the vector x and we must have $\mu(z^{(5)}) = 1$, $\mu(x) = 0$ which is impossible.

Remark. In order to find vectors $(r, \ldots)$, $(r', \ldots)$ we must solve the equation $V = 0$. This is done by solving quadratic equations. $\cos 4\pi/10$ and $\sin 4\pi/10$ are expressions in rational numbers and quadratic roots. It is evident that the construction is possible if we take coordinates in E_3 to be elements in an appropriate algebraic field and not all real numbers.

Each step of construction involves six vectors and so there are 30 vectors together. As the construction is needed for every possible choice of values of the measure μ for the first triple (x, y, z), we have at most 3. 30 vectors. This number is somewhat smaller then the number 117 in [2].

References

[1] *A. M. Gleason:* Measure on the closed subspaces of a Hilbert space. Journal of Math. and Mech. 6 (1957), 885–893.

[2] *S. Kochen, E. P. Specker:* The problem of hidden variables in quantum mechanics. Journal of Math. and Mech. 17 (1967), 59–67.

Souhrn

O 0–1 MÍŘE PRO PROJEKTORY

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Je nalezena konečná množina jednorozměrných projektorů v E_3, pro které neexistuje žádná 0–1 míra. Konstrukce je založena na tomto tvrzení: existují dvě trojice ortogonálních vektorů, jejichž dva vektory jsou v daných dvou rovinách a jestliže roviny svírají dostatečně velký úhel, pak třetí vektory jsou navzájem kolmé.

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