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ON DIFFERENCE LINEAR PERIODIC SYSTEMS I – HOMOGENEOUS CASE

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INTRODUCTION

The method used is based on the observation of the periodic repetition of the matrix of coefficients on the N-periodic finite difference system

$$x(n+1) = A(n)\, x(n)\,, \quad n \in \mathbb{N}\,, \tag{1}$$

where $A(n + N) = A(n)$ for all $n \in \mathbb{N}$.

If $x(n)$ is a solution of the system (1), we consider its N "interlaced" subsequences which are defined in the following way:

The subsequence z^0 is formed by the terms of $x(n)$ which occupy the places $0, N, 2N, \ldots$. The subsequence z^1 is formed by the terms of $x(n)$ which occupy the places $1, N + 1, 2N + 1, \ldots$. In this way we proceed until we reach z^{N-1} which is formed by the terms of $x(n)$ which occupy the places $N - 1, 2N - 1, 3N - 1, \ldots$.

It is important to observe that if we divide each $n \in \mathbb{N}$ by N and if k and s are, respectively, the quotient and the rest of this division (and so, they depend on n), the subsequence z^s of $x(n)$ is defined, for fixed s, by

$$z^s(k) := x(kN + s)\,, \quad k \in \mathbb{N}\,.$$

It will be demonstrated in this way that any solution of the system (1) can be given explicitly in terms of its N interlaced subsequences and its behavior, including periodicity, boundedness and convergence, is determined by the first of them, that is, by $z^0(k)$.

All that will be done independently of whether $A(n)$ is regular or not, for all $n \in \mathbb{N}$.

1. HOMOGENEOUS SYSTEM

Consider the linear homogeneous N-periodic system of equations in finite differences

$$x(n+1) = A(n)\, x(n)\,, \quad n \in \mathbb{N}\,, \tag{1}$$

where $A(n)$ is an N-periodic matrix sequence of order q with values in $\mathbb{C}$:

$$A : \mathbb{N} \to M_q(\mathbb{C}), \quad A(n + N) = A(n) \quad \text{for all} \quad n \in \mathbb{N},$$

and $x(n)$ is a vector sequence with complex values.

A fundamental matrix of this system is

$$\begin{cases} X(n) = A(n-1)\, A(n-2) \ldots A(1)\, A(0), & n \in \mathbb{N}^*, \\ X(0) := E. \end{cases}$$

Let k and s be, respectively, the quotient and the rest of the division of n by N, for each $n \in \mathbb{N}$. Then $X(n)$ can be written in the form

$$X(n) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k,$$

where $\prod_{j=s-1}^{0} A(j) := A(s-1)\, A(s-2) \ldots A(0)$ $(s = 1, 2, \ldots, N-1)$, and we agree that $\prod_{j=-1}^{0} A(j) := E$, identity matrix.

Under these conditions, the solution $x(n)$ of (1), which for $n = 0$ assumes the value $x(0) = x^0$, can be written in the form

$$x(n) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k\, x^0. \tag{2}$$

Theorem. *Given the initial value problem*

$$\begin{cases} x(n+1) = A(n)\, x(n), & n \in \mathbb{N}, \\ x(0) = x^0, \end{cases} \tag{1}$$

where $A(n)$ is N-periodic, a sequence $x(n)$ is a solution of (1) if and only if any subsequence of $x(n)$ in the form $x(kN + s)$ $(k = 0, 1, 2, \ldots)$ is a solution for $s =$ $= 0, 1, \ldots, N-1$, respectively, of the initial value problem with constant coefficients

$$(\mathrm{P}_s) \qquad \begin{aligned} z^s(k+1) &= [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]\, z^s(k), \quad k \in \mathbb{N}, \\ z^s(0) &= [\prod_{j=s-1}^{0} A(j)]\, x^0, \end{aligned}$$

where $z^s(k)$ for each s fixed is by definition the subsequence $(x(kN + s))_{k=0,1,2,..}$ of $x(n)$.

Proof. Let $x(n)$ be the solution sequence of (1). By definition $z^s(k) := x(kN + s)$ $(s = 0, 1, 2, \ldots, N-1)$.

Hence

$$z^s(k) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k x^0$$

for each $s = 0, 1, \ldots, N-1$.

Decomposing

$$\prod_{j=N-1}^{0} A(j) = [\prod_{j=N-1}^{s} A(j)]\,[\prod_{j=s-1}^{0} A(j)],$$

$z^s(k+1)$ can be written as

$$z^s(k+1) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]\,[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k x^0,$$

and consequently,

$$z^s(k+1) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]\,z^s(k), \quad k \in \mathbb{N} \quad (s = 0, 1, \ldots, N-1).$$

Moreover,

$$z^s(0) = x(s) = [\prod_{j=s-1}^{0} A(j)]\,x^0.$$

This demonstrates that the condition is necessary.

On the other hand, any solution sequence of (P_s) for each $s = 0, 1, \ldots, N-1$, has the following form:

$$z^s(k) = [[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]]^k z^s(0), \quad k \in \mathbb{N}.$$

Then

(2) $$z^s(k) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^{k-1}\,[\prod_{j=N-1}^{s} A(j)]\,[\prod_{j=s-1}^{0} A(j)]\,x^0 =$$

$$[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k x^0 \quad \text{for all} \quad k \in \mathbb{N} \quad (s = 0, 1, \ldots, N-1).$$

Let now $x(n)$ be a sequence determined by its N subsequences

$$x(kN+s) := z^s(k), \quad k \in \mathbb{N} \quad (s = 0, 1, \ldots, N-1),$$

where $z^s(k)$ for each s is the solution of the problem (P_s).

We will see that the sequence $x(n)$ thus constructed is a solution of the problem (1). In fact, every term of $x(n)$ is a term of a certain subsequence $z^s(k)$.

Hence

$$x(n) = x(kN+s) \quad \text{and} \quad x(n+1) = x(kN+s+1).$$

There are two situations possible well distinguished from each other:

i) $s = N-1$ and so $s+1 = N$,

ii) $0 \leqq s < N-1$ and so $0 < s+1 \leqq N-1$.

In the case of i), $x(n+1) = x((k+1)N)$; however, by (2), $z^0(k+1) = = [\prod_{j=N-1}^{0} A(j)]^{k+1} x^0$. On the other hand,

$$A(n)\,x(n) = A(kN + N - 1)\,x(kN + N - 1) = A(N-1)\,z^{N-1}(k),$$

provided $A(n)$ is N-periodic.

Thus, using (2) we get

$$A(n)\,x(n) = A(N-1)\,[\prod_{j=N-2}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k\,x^0$$

and so, in the case of i), $x(n+1) = A(n)\,x(n)$.

In the case of ii), we reach the same conclusion by a similar argument.

In order to complete the proof of the theorem it remains to be demonstrated that $x(0) = x^0$, which is deduced from

$$x(0) = z^0(0) = [\prod_{j=-1}^{0} A(j)]\,x^0 = x^0,$$

where we have used the convention $\prod_{j=-1}^{0} A(j) := E$.

2. COMMENTS TO THEOREM 1

i) Theorem 1 shows the objective expressed in the introduction: it is always possible to obtain the solutions of an N-periodic system in finite differences in an explicit way through its N "interlaced" subsequences.

It is also important to observe that it is not necessary to solve the N systems with constant coefficients; it suffices to solve a single one: P_0. In fact, having obtained the sequence $z^0(k)$ as a solution of the problem (P_0), we find all the other subsequences that when interlaced form $x(n)$ – the solution of the periodic system – as the expressions

$$\text{(1)} \qquad z^s(k) = [\prod_{j=s-1}^{0} A(j)]\,z^0(k), \quad k \in \mathbb{N} \quad (s = 1, 2, \ldots, N-1),$$

which are consequences of (2) of Theorem 1.

ii) Similarly, it can be seen that the results obtained in Theorem 1 are independent of whether the matrix $A(n)$ is regular or not.

3. PERIODIC SOLUTIONS

Consider the N-periodic system

$$\text{(1)} \qquad x(n+1) = A(n)\,x(n), \quad n \in \mathbb{N}.$$

The fact that one solution of the system (1) is pN-periodic if and only if $x(Np) = x(0)$ and that the sequence $z^0(k)$, solution of the problem (P_0), is p-periodic if and only if $z^0(p) = z^0(0)$, immediately implies the following

Proposition. *The sequence $x(n)$, solution of (1), is Np-periodic and non-trivial if and only if the sequence $z^0(k)$ defined in Theorem 1 is p-periodic and non-trivial. Or, equivalently, if and only if the matrix $\prod_{j=N-1}^{0} A(j)$ admits an eigenvalue that is the p^{th} root of the unity and $x(0)$ is an eigenvector associated with the eigenvalue 1 of $[\prod_{j=N-1}^{0} A(j)]^p$.*

Consequently, $x(n)$ is N-periodic if and only if its subsequence $z^0(k)$ is constant; or equivalently, if the matrix $\prod_{j=N-1}^{0} A(j)$ admits 1 as an eigenvalue and $x(0)$ is an eigenvector associated with this eigenvalue. The same results can be obtained from (1) of Sec. 2 merely by observing that if $z^0(k)$ is p-periodic (constant), then all $z^s(k)$ ($s = 1, \ldots, N-1$) are p-periodic (constant) and so $x(n)$ is Np-periodic (N-periodic, respectively). Further, if $x(n)$ is Np-periodic (N-periodic), then all the $z^s(k)$ ($s = 0, 1, \ldots, N-1$) are p-periodic (constant) and thus, in particular, $z^0(k)$ is p-periodic (constant, respectively).

4. ASYMPTOTIC BEHAVIOR

Theorem. *Consider the N-periodic system*

$$x(n+1) = A(n)\,x(n)\,, \quad n \in \mathbb{N}\,, \tag{1}$$

and the system with constant coefficients

$$z^0(k+1) = [\prod_{j=N-1}^{0} A(j)]\, z^0(k)\,, \quad k \in \mathbb{N}\,. \tag{2}$$

Let $X(n)$ be a fundamental matrix of (1) and $Z^0(k) := X(kN)$ the corresponding fundamental matrix of (2). Then

(a) *$X(n)$ is bounded if and only if $Z^0(k)$ is bounded.*
(b) *If $X(n)$ converges as $n \to \infty$, then $Z^0(k)$ converges as $k \to \infty$ and their limits coincide.*
(c) *If $Z^0(k) \to Z$, then $X(n)$ converges to Z if and only if*

$$[\prod_{j=s-1}^{0} A(j)]\, Z = Z \quad \textit{for each} \quad s = 0, 1, \ldots, N-1.$$

Proof. A fundamental matrix solution of (1) is

$$X(n) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k\,, \tag{3}$$

where k and s are the quotient and the rest, respectively, of the division of n by N, for each $n \in \mathbb{N}$.

On the other hand, the corresponding fundamental matrix solution of (2) is

$$Z^0(k) = [\prod_{j=N-1}^{0} A(j)]^k . \tag{4}$$

So, $Z^0(k)$ is the subsequence $X(kN)$ of $X(n)$ and, therefore, if $X(n)$ converges or is bounded, the subsequence $Z^0(k)$ converges to the same limit as $X(n)$ or is bounded, respectively.

Moreover,

$$\|X(n)\| \leqq \|\prod_{j=s-1}^{0} A(j)\| \, \|[\prod_{j=N-1}^{0} A(j)]^k\|$$

and so, if the sequence $Z^0(k)$ is bounded, $X(n)$ is bounded as well.

Finally, let $Z^s(k)$ be the subsequence $X(kN + s)$ of $X(n)$, i.e.

$$Z^s(k) := [\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{0} A(j)]^k .$$

Then, if the sequence $Z^0(k)$ converges to Z as $k \to \infty$, $Z^s(k)$ converges to

$$[\prod_{j=s-1}^{0} A(j)] Z \quad \text{for each} \quad s = 1, 2, \ldots, N - 1 \tag{5}$$

and hence $X(n)$ is bounded but, in general, not convergent.

From these facts we can deduce that $X(n)$ is convergent if and only if

$$[\prod_{j=s-1}^{0} A(j)] Z = Z \quad \text{for each} \quad s = 1, 2, \ldots, N - 1 , \tag{6}$$

where $Z := \lim_{k \to \infty} Z^0(k)$, and in this case, obviously $\lim_{n \to \infty} X(n) = Z$.

Moreover, if $Z^0(k)$ converges but does not verify (6), even if $X(n)$ does not converge we know exactly its behavior as n tends to ∞: it approaches "interlacely" the N values Z and those expressed in (5), which are its only cluster values (i.e. subsequential limits).

From (3) and (4) we can see, obviously, that $X(n)$ is not bounded if and only if $Z^0(k)$ is not bounded, which completes the proof.

5. COMMENTS TO THEOREM 4

i) We have to mention that the problems of convergence and boundedness of the sequence of powers of a constant matrix were studied by Pullman ([1], Sec. 11 of Chap. 1), which completes Theorem 4.

ii) On the other hand, if $X(n)$ is convergent to $Z \neq 0$, it follows that $Z^0(k)$ converges to Z and so, (Pullman [1], loc. cit.) $\prod_{j=N-1}^{0} A(j)$ admits 1 as an eigenvalue, which guarantees that the N-periodic system admits a non-trivial N-periodic solution.

iii) Moreover, if for some $n \in \mathbb{N}$, $X(n)$ does not admit 1 as an eigenvalue, then either $X(n)$ does not converge as $n \to \infty$, or in case of convergence its limit is the zero matrix.

In fact, if $X(n)$ converges to $Z \neq 0$ as n tends to ∞, then the system admits a periodic non constant and non trivial solution, since $X(n)$ does not admit 1 as an eigenvalue for all $n \in \mathbb{N}$.

iv) Finally, if $X(n)$ converges to Z as $Z^0(k)$ converges to Z, then $Z^0(k - r)$ converges to Z as well, and since

$$Z^0(k) = [\prod_{j=N-1}^{0} A(j)]^r Z^0(k - r),$$

we obtain

$$[\prod_{j=N-1}^{0} A(j)]^r Z = Z$$

and, passing to the limit as $r \to \infty$, we conclude that

$$Z^2 = Z.$$

Consequently, we can say that if the matrix $X(n)$ converges to a matrix Z, then the latter matrix is idempotent.

FLOQUET'S THEORY

The Floquet-Liapunov theory (see [2] and [3]) is known for linear periodic differential systems. We could develop such a theory for finite difference systems in a similar way to [2]. Doing this would allow us to obtain some results which coincide with those presented in this article. Nevertheless, other results put forth here are not obtained directly from the above theory. In this section we mention some of these coincidences and differences.

Given the N-periodic system

$$x(n + 1) = A(n)\, x(n), \quad n \in \mathbb{N}, \tag{1}$$

we say (as in [2]) that C is a *monodromy matrix* of this system if it satisfies the condition $X(n + N) = X(n)\, C$, for all $n \in \mathbb{N}$, where $X(n)$ is a fundamental matrix of (1).

Thus, under these conditions we say that C is the monodromy matrix of the system (1), associated with the fundamental matrix $X(n)$.

It is easy to verify that all the monodromy matrices of the system (1) are similar.

A little calculation shows us that the monodromy matrix of the system (1) associated with the fundamental matrix $X(n)$, which verifies $X(0) = E$, is

$$C = \prod_{j=N-1}^{0} A(j) .$$

A theorem similar to that of Floquet ([2] p. 139) would lead us to a result analogous to that of Proposition of Section 3 on the existence of periodic solutions of the system (1).

Nevertheless, by using the Floquet-Liapunov theory it is impossible to obtain some of the results established by our method, namely those relative to the availability in an explicit form of the solutions (Section 2), as well as the results concerning the boundedness and convergence of the solutions (Sections 4 and 5).

To conclude we mention some open problems:

1) One problem whose resolution is of unquestionable interest is that which concerns the search for the solutions of linear periodic differential systems. Would it be possible to use a method similar to ours to reach an approximation of the above mentioned solutions?

2) The study of problems dealt with in this paper can be continued in the following direction: how to obtain in a closed form solutions of linear periodic finite difference systems which are non-homogeneous. Could a method similar to ours be used to reach this? Could such a method be used for the study of asymptotic behavior and stability of the solutions?

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Souhrn

O DIFERENČNÍCH LINEÁRNÍCH PERIODICKÝCH SYSTÉMECH I – HOMOGENNÍ PŘÍPAD

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Článek pojednává o redukci lineárního homogenního periodického systému diferenčních rovnic na lineární homogenní systém s konstantními koeficienty. Tato redukce umožňuje studovat existenci a vlastnosti periodických řešení, asymptotické chování, a obdržet všechna řešení v uzavřeném tvaru.

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