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DERIVATIVE OF THE NORM OF A LINEAR MAPPING AND ITS
APPLICATION TO DIFFERENTIAL EQUATIONS

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Summary. In this paper the notion of the derivative of the norm of a linear mapping in a normed vector space is introduced. The fundamental properties of the derivative of the norm are established. Using these properties, linear differential equations in a Banach space are studied and lower and upper estimates of the norms of their solutions are derived.

Keywords: Normed space, Derivative of the norm of the linear mapping, Solution of the differential equation.

AMS classification: 34A30

Let $(P_1, \|\cdot\|_1)$, $(P_2, \|\cdot\|_2)$ be normed vector spaces over the field of complex numbers. We denote the set of all bounded linear mappings of the space P_1 to the space P_2 by the symbol $L(P_1, P_2)$. We introduce the structure of the linear space on the set $L(P_1, P_2)$ in the standard way. We define the norm of a bounded linear mapping by the relation

$$\|A\| = \sup\{\|Ay\|_2 : \|y\|_1 \leq 1\} \quad \text{for } A \in L(P_1, P_2).$$

Further we define the function $f_1 : L(P_1, P_2) \times L(P_1, P_2) \times \mathbf{R}^+ \rightarrow \mathbf{R}$, where $\mathbf{R}^+ = (0, +\infty)$, $\mathbf{R} = (-\infty, +\infty)$, by the relation

$$(1) \quad f_1(X, A, t) = \frac{\|X + tA\| - \|X\|}{t}.$$

Now for any $\vartheta \in (0, 1)$ and for any $(X, A, t) \in L(P_1, P_2) \times L(P_1, P_2) \times \mathbf{R}^+$ we have

$$(2) \quad \begin{aligned} \vartheta t f_1(X, A, \vartheta t) &= \|X + \vartheta tA\| - \|X\| = \|\vartheta(X + tA) + (1 - \vartheta)X\| - \|X\| \\ &\leq \vartheta(\|X + tA\| - \|X\|) = \vartheta t f_1(X, A, t), \quad \text{i.e.} \end{aligned}$$

$$f_1(X, A, \vartheta t) \leq f_1(X, A, t).$$

For any $t \in \mathbf{R}^+$ we have

$$(3) \quad t f_1(X, A, t) = \|X + tA\| - \|X\| \geq \|X\| - t\|A\| - \|X\| = -t\|A\|,$$

and so $f_1(X, A, t) \geq -\|A\|$.

It follows from the relations (2), (3) that $f_1(X, A, t)$ is a function of the variable t which has a finite limit $\lim_{t \rightarrow 0+} f_1(X, A, t)$ for $t \rightarrow 0+$, for any $X, A \in L(P_1, P_2)$. We denote this limit by the symbol $f_X(A)$.

Definition. The mapping $f_X : L(P_1, P_2) \rightarrow \mathbf{R}$, where $X \in L(P_1, P_2)$, defined by the relation

$$(4) \quad f_X(A) = \lim_{t \rightarrow 0+} \frac{\|X + tA\| - \|X\|}{t},$$

is called the *derivative of the norm of the mapping X*.

Theorem 1. If f_X is the derivative of the norm of the mapping $X \in L(P_1, P_2)$, then

1° $f_X(X) = \|X\|$, $f_X(-X) = -\|X\|$ for any $X \in L(P_1, P_2)$, $f_X(O) = 0$, where O is the zero mapping of the space $L(P_1, P_2)$;

2° $-\|A\| \leq -f_X(-A) \leq f_X(A) \leq \|A\|$ for any $X, A \in L(P_1, P_2)$;

3° $f_X(\alpha A) = \alpha f_X(A)$ for any $\alpha \in \mathbf{R}^+$ and for any $X, A \in L(P_1, P_2)$;

4° $f_X(A + \alpha X) = f_X(A) + \alpha\|X\|$ for any $\alpha \in \mathbf{R}$ and for any $X, A \in L(P_1, P_2)$;

5° $\max\{f_X(A) - f_X(-B), -f_X(-A) + f_X(B)\} \leq f_X(A + B) \leq f_X(A) + f_X(B)$ for any $X, A, B \in L(P_1, P_2)$;

6° f_X is a convex functional on the vector space $L(P_1, P_2)$, i.e. $f_X(\alpha A + (1-\alpha)B) \leq \alpha f_X(A) + (1-\alpha)f_X(B)$ for any $\alpha \in \langle 0, 1 \rangle$ and for any $A, B \in L(P_1, P_2)$.

Proof. 1° The property follows directly from the definition of the derivative of the norm f_X .

2° For any $t \in \mathbf{R}^+$ we have

$$-\|A\| = \frac{\|X\| - t\|A\| - \|X\|}{t} \leq \frac{\|X + tA\| - \|X\|}{t} \leq \frac{\|X\| + t\|A\| - \|X\|}{t} = \|A\|,$$

thus

$$(5) \quad -\|A\| \leq \lim_{t \rightarrow 0+} \frac{\|X + tA\| - \|X\|}{t} = f_X(A) \leq \|A\|.$$

For any $t \in \mathbf{R}^+$ we have

$$-\|A\| = \frac{\|X\| - t\|A\| - \|X\|}{t} = \frac{\|X\| - t\| -A\| - \|X\|}{t} \leq -\frac{\|X + t(-A)\| - \|X\|}{t},$$

thus

$$(6) \quad -\|A\| \leq -\lim_{t \rightarrow 0^+} \frac{\|X + t(-A)\| - \|X\|}{t} = -f_X(-A).$$

Further,

$$\begin{aligned} 0 &= \|X + t(A - A)\| - \|X\| \leq \frac{\|X + 2tA\| - \|X\| + \|X - 2tA\| - \|X\|}{2} \\ &= \frac{\|X + 2tA\| - \|X\|}{2t} t + \frac{\|X + 2t(-A)\| - \|X\|}{2t} t, \quad \text{thus} \\ 0 &\leq \lim_{t \rightarrow 0^+} \frac{\|X + 2tA\| - \|X\|}{2t} + \lim_{t \rightarrow 0^+} \frac{\|X + 2t(-A)\| - \|X\|}{2t}, \quad \text{i.e.} \end{aligned}$$

$$(7) \quad 0 \leq f_X(A) + f_X(-A).$$

Now the relations (5), (6) and (7) imply

$$-\|A\| \leq -f_X(-A) \leq f_X(A) \leq \|A\|.$$

3° The proof of this property is evident.

4° According to the definition of the derivative of the norm f_X we have

$$\begin{aligned} f_X(A + \alpha X) &= \lim_{t \rightarrow 0^+} \frac{\|X + t(A + \alpha X)\| - \|X\|}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{(1 + \alpha t)\|X + tA/(1 + \alpha t)\| - \|X\|}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{\|X + tA/(1 + \alpha t)\| - \|X\|}{t/(1 + \alpha t)} + \alpha\|X\| \\ &= f_X(A) + \alpha\|X\|. \end{aligned}$$

5° For any $t \in \mathbb{R}^+$ we have

$$\begin{aligned} \frac{\|X + t(A + B)\| - \|X\|}{t} &= \frac{\|X + 2tA + X + 2tB\| - 2\|X\|}{2t} \leq \\ &\leq \frac{\|X + 2tA\| - \|X\|}{2t} + \frac{\|X + 2tB\| - \|X\|}{2t}, \quad \text{thus} \end{aligned}$$

$$\begin{aligned} (8) \quad f_X(A + B) &= \lim_{t \rightarrow 0^+} \frac{\|X + t(A + B)\| - \|X\|}{t} \leq \\ &\leq \lim_{t \rightarrow 0^+} \frac{\|X + 2tA\| - \|X\|}{2t} + \lim_{t \rightarrow 0^+} \frac{\|X + 2tB\| - \|X\|}{2t} = \\ &= f_X(A) + f_X(B). \end{aligned}$$

The first inequality in 5° follows from the inequality

$$f_X(A) \leq f_X(A + B) + f_X(-B)$$

and from the inequality arising by the interchanging A and B .

6° The proof follows from the properties 3°, 5°. □

Theorem 2. Let $I \in L(P_1, P_1)$ be the identical mapping of the space P_1 to the space P_1 and let λ be an eigenvalue of the linear mapping $A \in L(P_1, P_1)$. Then

- (i) $-f_I(-A) \leq \operatorname{Re} \lambda \leq f_I(A)$,
- (ii) $-f_I(-A)\|x\|_1 \leq \|Ax\|_1, -f_I(A)\|x\|_1 \leq \|Ax\|_1$
for any $x \in P_1$.

Proof. Ad (i). Let $v \in P_1$ be a vector, where $Av = \lambda v$ and $\|v\|_1 = 1$. For any $t \in \mathbf{R}^+$ we have $\|I + t(-A)\| \geq \|Iv - tAv\|_1 = \|v - t\lambda v\|_1$, and so

$$-\frac{\|I + t(-A)\| - 1}{t} \leq -\frac{\|v - t\lambda v\|_1 - 1}{t} = -\frac{|1 - t\lambda| - 1}{t}.$$

Consequently,

$$-f_I(-A) = -\lim_{t \rightarrow 0^+} \frac{\|I + t(-A)\| - 1}{t} \leq -\lim_{t \rightarrow 0^+} \frac{|1 - t\lambda| - 1}{t} = \operatorname{Re} \lambda.$$

Further,

$$\begin{aligned} \frac{\|I + tA\| - 1}{t} &\geq \frac{\|v + t\lambda v\|_1 - 1}{t} = \frac{|1 + t\lambda| - 1}{t}, \quad \text{thus} \\ f_I(A) &= \lim_{t \rightarrow 0^+} \frac{\|I + tA\| - 1}{t} \geq \lim_{t \rightarrow 0^+} \frac{|1 + t\lambda| - 1}{t} = \operatorname{Re} \lambda. \end{aligned}$$

Ad (ii). For any $t \in \mathbf{R}^+$ and for any vector $x \in P_1$ we have

$$\begin{aligned} \|Ax\|_1 &= \frac{\|x - (x - tAx)\|_1}{t} \geq \\ &\geq \frac{\|x\|_1 - \|I - tA\| \|x\|_1}{t} = -\frac{\|I + t(-A)\| - 1}{t} \|x\|_1, \quad \text{thus} \\ \|Ax\|_1 &\geq -\lim_{t \rightarrow 0^+} \frac{\|I + t(-A)\| - 1}{t} \|x\|_1 = -f_I(-A)\|x\|_1. \end{aligned}$$

Replacing the mapping A by the mapping $-A$ in this relation we get the second relation.

The theorem is proved. □

In what follows we consider a differential equation

$$(11) \quad \frac{dx}{ds} = A(s)x$$

in a Banach space $(P_1, \|\cdot\|_1)$, where $A \in L(P_1, P_1)$ is a continuous mapping on an open, unbounded from above interval $J \subset \mathbf{R}$, and $A(t)$ is a continuous mapping of J into $L(P_1, P_1)$. It has been shown in the theory of differential equations (see [2], p. 353, Theorem 10.8.4) that there exists just one continuous mapping $x(\cdot, s_0, x_0): \{s \in$

$J: s \geq s_0\} \rightarrow P_1$ for any $(s_0, x_0) \in J \times P_1$ such that $dx(s)/ds = A(s)x(s)$ for any $s \geq s_0$ and $x(s_0) = x_0$. This mapping is called the solution of the differential equation (11). Besides, for any s , where $s \geq s_0 \in J$, there exists a mapping $F(s) \in L(P_1, P_1)$ —the so-called fundamental mapping of the equation (11)—and there exists its inverse mapping $F^{-1}(s) \in L(P_1, P_1)$ such that $x(s, s_0, x_0) = F(s) \circ F^{-1}(s_0)x_0$.

If $x(\cdot)$ is a solution of the differential equation (11) then there exists $y(s, t) \in P_1$ for any $(s, t) \in J \times \mathbf{R}^+$ with $\lim \|y(s, t)\|_1 = 0$ for $t \rightarrow 0+$ and such that

$$x(s+t) = x(s) + \frac{dx(s)}{ds}t + ty(s, t) = x(s) + tA(s)x(s) + ty(s, t).$$

This implies

$$(12) \quad \lim_{t \rightarrow 0+} \frac{\|x(s+t)\|_1 - \|x(s)\|_1}{t} = \lim_{t \rightarrow 0+} \frac{\|x(s) + tA(s)x(s)\|_1 - \|x(s)\|_1}{t}.$$

Theorem 3. If $(P_1, \|\cdot\|_1)$ is a Banach space and $x(\cdot, s_0, x_0)$ is a solution of the differential equation (11), $F \in L(P_1, P_1)$ is its fundamental mapping, f_I is the derivative of the norm of the identical mapping $I \in L(P_1, P_1)$, then for any $s \geq s_0 \in J$ the estimates

$$(13) \quad \|x_0\|_1 \exp\left[-\int_{s_0}^s f_I(-A(\sigma)) d\sigma\right] \leq \|x(s)\|_1 \leq \|x_0\|_1 \exp\left[\int_{s_0}^s f_I(A(\sigma)) d\sigma\right],$$

$$(14) \quad \exp\left[-\int_{s_0}^s f_I(-A(\sigma)) d\sigma\right] \leq \|F(s) \circ F^{-1}(s_0)\| \leq \exp\left[\int_{s_0}^s f_I(A(\sigma)) d\sigma\right],$$

hold whenever the integrals involved are defined.

Proof. For any $x \in P_1$, $t \in \mathbf{R}^+$ we have

$$2\|x\|_1 = \|(I + tA)x + (I - tA)x\|_1 \leq \|(I + tA)\| \cdot \|x\|_1 + \|I + t(-A)\| \cdot \|x\|_1,$$

and this implies

$$(15) \quad -\frac{\|I + t(-A)\| - 1}{t} \|x\|_1 \leq \frac{\|x + tAx\|_1 - \|x\|_1}{t} \leq \frac{\|I + tA\| - 1}{t} \|x\|_1.$$

According to (12) the relation (15) implies the inequality

$$(16) \quad -f_I(-A(s))\|x(s)\|_1 \leq \lim_{t \rightarrow 0+} \frac{\|x(s+t)\|_1 - \|x(s)\|_1}{t} \leq f_I(A(s))\|x(s)\|_1.$$

If $x_0 = o \in P_1$, the relation (13) is obviously true. Thus, let us suppose $x_0 \neq o$. Then also $\|x(s, s_0, x_0)\|_1 > 0$ for any $s \geq s_0 \in J$. From the inequality (16) the relation

$$-f_I(-A(\sigma)) \leq \lim_{t \rightarrow 0+} \frac{\|x(\sigma+t)\|_1 - \|x(\sigma)\|_1}{t\|x(\sigma)\|_1} \leq f_I(A(\sigma)),$$

follows. By its integration we obtain

$$-\int_{s_0}^s f_I(-A(\sigma)) \, d\sigma \leq \ln \frac{\|x(s)\|_1}{\|x_0\|_1} \leq \int_{s_0}^s f_I(A(\sigma)) \, d\sigma$$

provided $(s_0, s) \subset J$.

This yields the inequality (13) as well as the relation (14). □

Theorem 4. Let $(P_1, \|\cdot\|_1)$ be a Banach space, in which a differential equation

$$(17) \quad \frac{dx}{ds} = (A + B(s))x$$

is given, where $A \in L(P_1, P_1)$ is a constant mapping and $B(s) \in L(P_1, P_1)$ is a continuous mapping on an open, unbounded from above interval $J \subset \mathbf{R}$. Let $x(\cdot, s_0, x_0)$ be a solution of the equation (17) and let f_I be the derivative of the norm of the identical mapping $I \in L(P_1, P_1)$. Then the following implications hold:

- (i) $f_I(A) = 0$, $\int_{s_0}^{+\infty} f_I(B(s)) \, ds < +\infty \Rightarrow$ the solution $x(\cdot)$ is bounded;
- (ii) $f_I(A) < 0$, $\int_{s_0}^{+\infty} f_I(B(s)) \, ds < +\infty \Rightarrow \lim_{s \rightarrow +\infty} \|x(s)\|_1 = 0$;
- (iii) $-f_I(-A) > 0$, $-\int_{s_0}^{+\infty} f_I(-B(s)) \, ds > -\infty$, $x_0 \neq o \Rightarrow \lim_{s \rightarrow +\infty} \|x(s)\|_1 = +\infty$;
- (iv) $-f_I(-A) = 0$, $-\int_{s_0}^{+\infty} f_I(-B(s)) \, ds = +\infty$, $x_0 \neq o \Rightarrow \lim_{s \rightarrow +\infty} \|x(s)\|_1 = +\infty$.

Proof. The inequality (13) and the property 5° from Theorem 1 imply

$$\begin{aligned} \|x_0\|_1 \exp \left[-\int_{s_0}^s f_I(-A) \, d\sigma - \int_{s_0}^s f_I(-B(\sigma)) \, d\sigma \right] \\ \leq \|x_0\|_1 \exp \left[-\int_{s_0}^s f_I(-A - B(\sigma)) \, d\sigma \right] \leq \|x(s)\|_1 \\ \leq \|x_0\|_1 \exp \left[\int_{s_0}^s f_I(A + B(\sigma)) \, d\sigma \right] \\ \leq \|x_0\|_1 \exp \left[\int_{s_0}^s f_I(A) \, d\sigma + \int_{s_0}^s f_I(B(\sigma)) \, d\sigma \right], \end{aligned}$$

i.e.

$$\begin{aligned} & \|x_0\|_1 \exp[-f_I(-A)(s-s_0)] \exp\left[-\int_{s_0}^s f_I(-B(\sigma)) d\sigma\right] \\ & \leq \|x(s)\|_1 \leq \|x_0\|_1 \exp[f_I(A)(s-s_0)] \cdot \exp\left[\int_{s_0}^s f_I(B(\sigma)) d\sigma\right], \end{aligned}$$

from which the validity of the implications (i), (ii), (iii), (iv) is evident. \square

References

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- [2] *J. Dieudonné*: Foundations of modern analysis, Academic Press, New York-London, 1960; Russian translation, Mir, Moscow.

Souhrn

DERIVACE NORMY LINEÁRNÍHO ZOBRAZENÍ A JEJÍ APLIKACE V DIFERENCIÁLNÍCH ROVNICÍCH

FRANTIŠEK TUMAJER

V článku je zaveden pojem derivace normy lineárního zobrazení v normovaném vektorovém prostoru. Odvozují se základní vlastnosti derivace normy. Užitím těchto vlastností jsou studovány lineární diferenciální rovnice v Banachově prostoru a jsou odvozeny dolní i horní odhady pro normu jejich řešení.

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