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TOPOLOGICALLY COMPLETE SPACES
(Summary of author's results)

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If a complete metric space (P, φ) is homeomorphic with a subspace R of a metrizable space Q, then R is a G_δ -subset of Q. Conversely, if P is a G_δ -subset of a complete metric space, then there exists a metric φ for the space P such that (P, φ) is a complete metric space. In view of these facts we define:

DEFINITION. A Hausdorff topological space is said to be a G_δ -space if P is a G_δ -subset of every one of its Hausdorff extensions, i.e. if P is a dense subspace of a Hausdorff space R, then P is a G_δ -subset of R.

This definition is too general. Indeed, an open subspace of a G_δ -space may fail to be a G_δ -space. This follows at once from the following (for proof see [F 2], 3.2)

LEMMA. Let M_1 and M_2 be two disjoint dense subsets of a H-closed space $(K, \mathfrak{A})$ such that $M_1 \cup M_2 = K$. There exists a topology $\mathfrak{L}$ for the set K such that

(1) $(K, \mathfrak{L})$ is a H-closed space.
(2) $\mathfrak{A}$ and $\mathfrak{L}$ are coincident on both M_1 and M_2.
(3) M_1 is open in $(K, \mathfrak{L})$, i.e. $M_1 \in \mathfrak{L}$.

Let us recall that a Hausdorff space P is said to be H-closed if whenever R is a Hausdorff space containing P, then P is a closed subset of R. Thus every H-closed space is a G_δ-space. A completely regular space is H-closed if and only if it is compact (for further properties see [K], for generalisations see [F 1]). If M_1 is the set of all rational numbers of the unit interval K of real numbers (with the usual topology), then by the preceding lemma there exists a H-closed space containing R as an open subspace. It is well-known, that R is not a G_δ -subset of K . Thus R is not a G_δ -space.

The following special sort of G_δ -spaces was introduced by E. Čech in [Č] . By E. Čech, a space P is said to be topologically complete if P is completely regular anda G_δ -subspace of the Čech-Stone compactification $\beta(P)$ of P .E. Čech proved that whenever a topologically complete space P is a dense subspace of a completely regular space R , then P is a G_δ -subset of R and every G_δ -subspace of a topologically complete spaces is a topologically complete space. The Čech´s method of proofs is based (essentially) on the Čech-Stone mapping theorem, and consequently, this method is not applicable to spaces which are not completely regular.

A class of G_δ-spaces (containing topologically complete spaces) can be characterized internally, i.e. with - out reference to larger spaces. The purpose of this paper is to summarize author´s results concerning this class of G_δ -spaces.

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1
Notation and terminology

The terminology and notation of J. Kelley, General Topology, will be used throughout. For convenience we shall use a few not quite usual symbols and terms which are listed below.

1.1. A system is a synonym for andindexed family. For systems we shall use a notation such as $\{P_a;\ a\in A\}$ or merely $\{P_a\}$. The cartesian product of a system will be denoted by $\times\{P_a\}$. If m is a cardinal, then m-system is a system whose index set is of potency m. A centred family of sets is a family $\mathcal{O}\!\mathcal{L}$ having the finite intersection property, i.e., the intersection of every finite subfamily of $\mathcal{O}\!\mathcal{L}$ is non-void.

1.2. All topological spaces will be supposed to be Hausdorff. The closure of a subset M of a space P will be denoted by $\overline{M}^P$ or merely $\overline{M}$. If $\mathcal{O}\!\mathcal{L}$ is a family of subsets of a space P, then the family of closures of all sets from $\mathcal{O}\!\mathcal{L}$ will be denoted by $\mathcal{O}\!\mathcal{L}$.

1.3. Extensions of spaces. A space P is an extension of a space R, if R is a dense subspace of P. A proper extension of R is an extension P of R with $R\neq P$. The Stone-Čech compactification of a completely regular space P will always be denoted by $\beta(P)$.

1.4. Mappings. A mapping of a space P to a space Q is said to be closed (open), if the image of every closed (open) subset of P is a closed (open, respectively) subset of Q. A continuous mapping of a space P to a space Q is said to be non-extensible, if there exists no proper extension R of P such that for some continuous mapping F from R to Q the mapping f is a restriction of F.

2
Internal characterizations

Complete metric spaces can be defined as follows: a metric space (P,φ) is said to be complete, if the intersection of every Cauchy centred family $\mathfrak{M}$ of closed subsets of P (i.e., $\mathfrak{M}$ contains arbitrarily small sets, that is, for every $\varepsilon>0$ there exists a M in $\mathfrak{M}$ with diameter less than ε) is non-void. Using uniformities instead of metrics to define small sets we obtain complete uniform spaces. A space P is said to be topologically complete if the topology of P is the uniform topology of a complete uniform space. Complete uniform spaces do not possess the

G_δ -property of complete metric spaces. We shall define arbitrarily small sets (i.e. Cauchy families) in such a way that, for the obtained "completeness", the G_δ-property is preserved. We shall use complete diameters, complete sequences and relations of completeness.

DEFINITION. 2.1. A diameter on a space P is a non-negative function d (the values of d are real numbers and ∞) defined on the family of all subsets of P such that

(d 1) If $M \subset N \subset P$, then $d(M) \leqq d(N)$

(d 2) $d(M) = \inf\{d(U) \; ; \; U \text{ open}, U \supset M\}$

(d 3) $d(M) = 0$ for every one-point set M.

Let d be a diameter on a space P. A d-Cauchy family is a centred family $\mathfrak{A}$ of subsets of P such that

$$\inf\{d(A); \; A \in \mathfrak{A}\} = 0$$

A diameter on a space P will be called complete if the intersection of every d-Cauchy family consisting of closed sets is non-void.

Note. Let φ be a pseudometric on a space P. The function $d(M) = \sup\{\varphi(x,y); \; x \in M, \; y \in M\}, \; d(\emptyset) = 0$ is a diameter on P. This diameter is said to be generated by φ.

Let d be a diameter on a space P. An extension D of d is a diameter on an extension R of P such that d is the restriction of D. A diameter d on P is said to be non-extensible, if there exists no extension of d onto any proper extension of P.

THEOREM 2.1. Every complete diameter is non-extensible. If d is a non-extensible diameter on a completely regular space, then d is complete.

For proof see [F 6] .

THEOREM 2.2. If there exists a complete diameter on a space P, then P is a G_δ-space. If P is G_δ-subset of a regular space Q and if there exists a complete diameter on Q, then there exists a complete diameter on P. Evidently, $d \equiv 0$ is a complete diameter on every compact space. Thus a completely regular space is a G_δ-space if and only if there exists a complete diameter on P.

DEFINITION. 2.2. A space P will be called topologically complete in the sense of E. Čech (or merely complete) if there exists a complete diameter on P and P is a regular space.

Let d be a diameter on a space P and let $\mathfrak{M}$ be a maximal centred family of subsets of P. If for every $\varepsilon > 0$ the union of a finite number of sets M with $d(M) < \varepsilon$ belongs to $\mathfrak{M}$, then $\mathfrak{M}$ is a d-Cauchy family. Thus we have the following

THEOREM 2.3. Let d be a complete diameter on a space P. For every $M \subset P$ let $D(M)$ be the greatest lower bound of the set of all $\varepsilon > 0$ for which there exists a finite number of sets $M_1, \dots M_k$ such that $\bigcup_{i=1}^{k} M_i \supset M$ and $d(M_i) \leqq \varepsilon$ $(i = 1, \dots, k)$. Then D is a complete diameter on the space P.

If d is a complete diameter on a regular space P and if $d(M) = 0$, then the closure of M is a compact subspace of P. If D is the diameter from the theorem 2, 3, then for every compact subspace K of P we have $d(K) = 0$.

THEOREM. 2.4. A diameter d on a space P is complete if and only if the following two conditions are fulfilled

(1) If M is closed in P and $d(M) = 0$, then M is a compact space.

(2) If $\{F_n\}$ is a centred sequence of closed sets and $\lim_{n \to \infty} d(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

If (P, φ) is a metric space and d is the diameter generated by φ, then the condition (1) is satisfied. If P is a countably compact space, then the condition(2) is fulfilled for every diameter d on P. Thus no one of the conditions (1) and (2) is sufficient for d to be a complete diameter (see [F 2], 3.10).

DEFINITION 2.3. Let $\alpha = \{\mathfrak{A}_n\}$ be a sequence of open coverings of a space P. An α-Cauchy family is a centred family $\mathfrak{A}$ of subsets of P that for every $n = 1, 2, \dots$ some $A_n \in \mathfrak{A}_n$ contains a $A \in \mathfrak{A}$. A sequence α is said to be complete if the intersection of every α-Cauchy family consisting from closed sets is non-void.

Let d be a complete diameter on a space P. Let $\mathfrak{A}_n$ $(n = 1, 2, \dots)$ be the family of all open subsets U of P with $d(U) < \frac{1}{n}$. It is easy to see that $\alpha = \{\mathfrak{A}_n\}$ is a complete sequence of open coverings. Moreover

(c 1) If an open set A is contained in some set from $\mathfrak{A}_n$, then A belongs to $\mathfrak{A}_n$

(c 2) $\mathfrak{A}_{n+1} \subset \mathfrak{A}_n$ $(n = 1, 2, \dots)$

Conversely, if α is a complete sequence of open coverings satisfying (c 1) and (c 2), let $d(M)$ be the greatest lower bound of the set of all $\frac{1}{n}$ for which there exists a $A \in \mathfrak{A}_n$ containing M. If there exists no such n, put $d(M) = 1$. It is easy to see that d is a complete diameter on P. Finally, if α is a complete sequence of open coverings, it is easy to construct a complete sequence satisfying both conditions (c 1) and (c 2). Thus we have proved the following

THEOREM 2.5. The following conditions on a space P are equivalent

(1) There exists a complete diameter on P.

(2) There exists a complete sequence of open coverings of P.

The theory of complete spaces using complete sequences of coverings had been built in [F 2].

DEFINITION 2.4. A relation of completeness of a space P is a binary relation r defined for open subsets of P such that

(r 1) $r(A, B)$ implies $A \supset B$

(r 2) If $r(A, B)$, C and D are open, $C \supset A$ and $B \supset D$, then $r(C, D)$

(r 3) If A is an open set, then the family

$$\{B;\ r(A,B)\}$$

is a base for open subsets of A.

(r 4) If $\mathfrak{M}$ is a centred family of sets such that for every positive integer n there exists $A_1,\dots,A_{n+1}$ with $r(A_i, A_{i+1})$ for all $i=1,\dots,n$ and A_{n+1} contains a $M\in\mathfrak{M}$, then the intersection of $\overline{\mathfrak{M}}$ is non-void.

Let r be a relation of completeness of a space P. For every $M\subset P$ let $d(M)$ be the greatest lower bound of the set of all $\frac{1}{n}$ for which there exists $A_1,\dots,A_{n+1}$ such that $r(A_i,A_{i+1})$, $i=1,\dots,n$, and $A_{n+1}\supset M$. If there exists no such n put $d(M)=1$. It is easy to see that d is a complete diameter on the space P.

Conversely, let d be a complete diameter on a space P. For arbitrary open sets U and V let us define $r(U,V)$ if and only if $U\supset V$ and

$$2d(V)\leqq \min(1, d(U))$$

It is easy to see that r is a relation of completeness of the space P. Thus we have proved the following

THEOREM 2.6. The following conditions on a space are equivalent:

(1) There exists a complete diameter on P.

(2) There exists a relation of completeness on P.

In the connection with relations of completeness see [Ch]. The preceding theorem is the solution of a problem of G. Choquet.

The most useful special sorts of complete spaces are complete completely regular spaces which have been introduced by E. Čech. For these spaces from the preceding results it follows at once

THEOREM 2.7. The following properties of a completely regular space P are equivalent:

(1) P is a G_δ-subset of $\beta(P)$

(2) P is a G_δ-subset of a compactification of P.

(3) P is a G_δ-subset of every completely regular extension of P

(4) P is a G_δ-space

(5) There exists a complete diameter on P

(6) There exists a complete sequence of open coverings of P

(7) There exists a relation of completeness on the space P.

Note. The equivalence of conditions (1) - (3) had been proved by E. Čech in [Č].

3

Subspaces and products

The theorems of this section are immediate consequences of internal characterizations stated in the section 2.

THEOREM 3.1. The following conditions on a subspace M of a complete space P are equivalent:

(1) M is a G_δ-space

(2) M is a complete space

(3) M is a G_δ-subset of the closure of M in P.

From the preceding theorem it follows at once

THEOREM 3.2. A subspace P of a completely normal complete space R is complete if and only if P is a G_δ-subset of R.

Using complete diameter it is easy to prove the following

THEOREM 3.3. The topological product of every countably family of complete spaces is a complete space. If an uncountable system $\{P_a\}$ of spaces contains no H-closed space, then the topological product of $\{P_a\}$ is not a complete space.

4

Invariance under mappings and paracompact spaces

THEOREM 4.1. Let f be a continuous and closed mapping of a regular space P onto a regular space Q such that the inverses of points are compact. Then P is a complete space if and only if Q is complete.

For proof see [F 3] and [F 7].

THEOREM 4.2. Let f be an open and continuous mapping of a completely regular space P onto a completely regular space Q. If P is complete, then Q is a complete space.

For proof see [F 3]. This proof is not based on internal characterizations

THEOREM 4.3. The following conditions on a space P are equivalent:

(1) P is complete and paracompact

(2) There exists a closed and continuous mapping of P onto a complete metric space such that the inverses of points are compact.

(3) There exists a complete diameter on P generated by a pseudometric

For proof see [F 7].

THEOREM 4.4. A completely regular space P is paracompact if and only if the following condition is fulfilled:

If Q is a locally compact completely regular extension of P, then there exists a paracompact complete space R with $P \subset R \subset Q$.

5
Localization and further properties

DEFINITION 5.1. A space P is said to be locally complete if every point of P is contained in some open complete subspace of P, i.e., if every point of P has a neighborhood which is a complete space.

In [F 7] it is shown that locally complete completely regular space may fail to be complete. However, every locally complete space contains an open dense complete subspace.

THEOREM 5.1. Every paracompact locally complete space is complete.

Now we proceed to state further properties of complete spaces (for proof see [F 7]).

THEOREM 5.2. Let P be a locally complete space. Let us suppose that a countable subset N of P has an accumulation point. Then there exists a compact subspace K of P such that the set $K \cap N$ is infinite. Moreover, if an open subset U of P contains an accumulation point of the set N, then K may be chosen with $K \subset U$.

As a consequence of the preceding theorem we have at once the following

THEOREM 5.3. Let P be a countably compact locally complete space. Then for every countably compact space Q the topological product $P \times Q$ is a countably compact space.

For families of open sets there is result analoguous to theorem 5.2. Thus we obtain the following

THEOREM 5.4. Let P be a pseudocompact completely regular locally complete space. Then for every completely regular pseudocompact space P the topological product $P \times Q$ is a pseudocompact space.

By a theorem of P. Urysohn and P.S. Alexandrov, in compact spaces the characters and pseudocharacters of points coincide (for definition see [K 2] .

THEOREM 5.5. If P is a locally complete space, then the characters and pseudocharacters of points coincide.

From the preceding theorem using well-known theorems about images of metrizable spaces under continuous closed mappings it follows at once:

THEOREM 5.6. The image P of a metrizable complete space under a continuous closed mapping is a metrizable space if and only if P is a complete space.

It is well-known that every locally compact space is a k-space. A completely regular complete space may fail to be a k-space. Indeed, it is easy to prove the following

THEOREM 5.7. Let $\beta(N)$ be the Čech-Stone compactification of the countably infinite discrete space N. A space P, $N \subset P \subset \beta(P)$, is a k-space if and only if

P is a locally compact space.

The property of being locally complete is invariant under continuous closed mappings of regular spaces, i.e., a theorem analoguous to the theorem 4.2. holds.

6
Generalization ($G(m)$-spaces)

DEFINITION 6.1. A space P is said to be a $G(m)$-space, m being a cardinal number, if for every extension R of P there exists a m-system $\{U_a\}$ of open subsets of R with $\bigcap\{U_a\} = P$.

Thus G_δ-space and $G(\aleph_0)$-space are synonymous. $G(m)$-spaces were introduced and studied in [F 2]. Using complete m-systems of open coverings (defined below) to characterize completely regular $G(m)$-spaces we obtain theorems analoguous to those concerning complete spaces. Here we shall state the definitions and a theorem, only.

Let $\alpha = \{\mathfrak{A}_a ; a \in A\}$ be a m-system of open coverings of a space P. A α-Cauchy family is a centred family $\mathfrak{M}$ of subsets of P such that for every $a \in A$ some $U \in \mathfrak{A}_a$ contains a $M \in \mathfrak{M}$. A m-system is said to be complete if the intersection of every α-Cauchy family consisting from closed sets is non-void.

THEOREM 6.1. The following conditions on a completely regular space P are equivalent:

(1) P is a $G(m)$-space.

(2) There exists a complete m-system of open coverings of P.

(3) P is the intersection of a m-system of open subsets in some compactification of P.

From the definition of $G(m)$-spaces it follows at once that " $G(0)$-space" is a synonym for " H-closed space" and from the preceding theorem it follows that in the case of completely regular spaces "locally compact space" and " $G(1)$-space" are synonymous. Thus some theorems about compact or locally compact spaces can be deduced from analoguous theorems concerning $G(m)$-spaces.

7

Specializations

($N(m)$-spaces and $\mathfrak{Q}$-spaces)

In this section all spaces are supposed to be cpmpletely regular.

For every real-valued continuous function on a space P let $\mathcal{W}(f)$ be the covering of P consisting from the sets of the form

$$\{x ; |f(x)| < n\}$$

where n is running over all positive integers. A family $\mathcal{F}$ of continuous functions is said to be complete, if the system $\{\mathcal{W}(f) ; f \in \mathcal{F}\}$ is complete in the sense of the section 6.

Complete families of functions are studied in [F 4].

THEOREM. A space P is a $\mathfrak{Q}$-space (in the terminology of E. Hewitt) if, and only if, there exists a complete family of continuous functions on P, i.e., if, and only if, the family of all continuous functions is complete.

A space P is said to be a $N(m)$-space, if there exists a complete m-system of continuous function on P. A space P is a $N(m)$-space if, and only if, there exists a m-system $\{N_a\}$ of N-sets in $\beta(P)$ with $\bigcap\{N_a\} = P$.

If we consider countable open coverings instead of coverings of the form $\mathcal{W}(f)$, we obtain an interesting class of spaces which will be investigated in a prepared paper.

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