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THE ROBIN PROBLEM IN POTENTIAL THEORY

(Preliminary communication)

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Let G be an arbitrary open set in E_m, the Euclidean space of dimension $m > 2$ and suppose that the boundary B of G is non-void and compact. We denote by $\mathcal{L}$ the Banach space of all finite signed Borel measures with support in B; total variation is taken as a norm in $\mathcal{L}$. In what follows, λ will be a fixed non-negative element of $\mathcal{L}$. With each $\mu \in \mathcal{L}$ we associate its potential

$$\mathcal{U}\mu(x) = \int_B p(x-y)\, d\mu(y)$$

corresponding to the Newtonian kernel $p(x) = |x|^{2-m}/(m-2)$ as well as the class $\mathcal{D}_\mu$ of those infinitely differentiaable functions φ with compact support in E_m for which the integral

$$I_\mu(\varphi) = \int_{B\times B} \varphi(x)\, p(x-y)\, d\lambda(x)\, d\mu(y)$$

converges. Let $\mathcal{T}\mu$ denote the functional over $\mathcal{D}_\mu$ defined by

$$\langle \varphi, \mathcal{T}\mu \rangle = I_\mu(\varphi) + \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} \mathcal{U}\mu(x)\, dx .$$

If B is a smooth surface with the exterior normal n and

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λ is absolutely continuous with respect to the area measure H on $\mathcal{B}$ then, under appropriate assumptions on $\mathcal{U}\mu$, $\langle \varphi, \mathcal{T}\mu \rangle$ transforms into

$$\int_B \varphi \left(\frac{\partial \mathcal{U}\mu}{\partial n} + \frac{d\lambda}{dH} \mathcal{U}\mu \right) dH \quad ,$$

which shows that $\mathcal{T}\mu$ is a natural characterization of $\dfrac{\partial \mathcal{U}\mu}{\partial n} + \dfrac{d\lambda}{dH} \mathcal{U}\mu$.

For $\lambda = 0$, $\mathcal{T}\mu$ reduces to the generalized normal derivative $N\mathcal{U}\mu$ of $\mathcal{U}\mu$ as investigated in [1]. For the case when G is a complementary domain of a simple closed surface in E_3 submitted to some further restrictions, the third boundary value problem (sometimes called the Robin problem) with a weak characterization of boundary values was treated in [3]. Making no a priori restrictions on $\mathcal{B}$ we establish a necessary and sufficient geometric condition guaranteeing, for each $\mu \in \mathcal{B}$, the representability of $\mathcal{T}\mu$ by means of a unique element of $\mathcal{L}$. As in [1], we call x a hit of an open segment or a half-line $S \subset E_m$ on G provided $x \in S$ and each open ball containing x meets both $S \cap G$ and $S - G$ in a set of positive linear measure. Given $y \in E_m$, $0 < r \leq +\infty$ and $\theta \in \Gamma = \{ x \in E_m ; |x| = 1 \}$ consider the number $n_r(\theta, y)$ (possibly zero or infinite) of all the hits of $\{ y + \rho\theta ; 0 < \rho < r \}$ on G . For fixed y and r , $n_r(\theta, y)$ appears to be a Baire function of the variable $\theta \in \Gamma$ and we may define

$$v_r(y) = \int_\Gamma n_r(\theta, y) \, dH_{m-1}(\theta) \quad ,$$

where H_{m-1} is the Hausdorff $(m-1)$-measure.

Results of [1] permit one to obtain the following

Theorem I. The following conditions (1) and (2) are equivalent to each other:

(1) $\sup_{y \in B} [v_\infty(y) + \mathcal{U}\lambda(y)] < \infty$.

(2) For each $\mu \in B$, there is a unique $\nu \in \mathcal{L}$ such that $\langle \varphi, \nu \rangle = \langle \varphi, \mathcal{T}\mu \rangle$ for all $\varphi \in \mathcal{D}_\mu$.

Let us now assume (1). In view of Theorem I, $\mathcal{T}\mu$ can be identified with a unique element of $\mathcal{L}$. The operator $\mathcal{T} : \mu \longmapsto \mathcal{T}\mu$ is bounded on $\mathcal{L}$.

It is natural to investigate the applicability of the Riesz-Schauder theory to the third boundary value problem in the following formulation: Given $\nu \in B$, determine a $\mu \in \mathcal{L}$ with $\mathcal{T}\mu = \nu$. For this purpose it is useful to consider the decomposition

$$\mathcal{T} = \alpha A \mathcal{J} + \mathcal{T}_\alpha$$

where α is a real number, $A = H_{m-1}(\dot{\Gamma})$ and $\mathcal{J}$ is the identity operator on $\mathcal{L}$, and investigate the quantity

$$\omega \mathcal{T}_\alpha = \inf_{Q} \| \mathcal{T}_\alpha - Q \|$$

where Q runs over all operators acting on $\mathcal{L}$ of the form

$$Q\ldots = \sum_{j=1}^{n} \langle f_j, \ldots \rangle m_j \quad ,$$

where n is an integer, $m_j \in \mathcal{L}$ and f_j are bounded Baire functions on B . In a similar way as in [2] it is possible to determine the optimal value γ of the parameter α and evaluate the quantity

(3) $$a = \frac{\omega\mathcal{T}_\gamma}{A|\gamma|} = \inf_{\alpha \neq 0} \frac{\omega \mathcal{T}_\alpha}{A|\alpha|}$$

in geometric terms connected with G and λ.

Denote by I_B the set of all isolated points of B and put $E = B - I_B$ or $E = B$ according as I_B is finite or not and write B_1 for the set of all points $y \in E$ that have a neighborhood $\Omega(y)$ such that $\Omega(y) - G$ has Lebesgue measure zero. Let B_2 stand for the set of those $y \in B$ at which the m-dimensional density of G equals $\frac{1}{2}$. Then B_2 is a Borel set with $H_{m-1}(B_2) < \infty$ and one may consider the Lebesgue decomposition $\lambda = \lambda_0 + \hat{\lambda}$ with respect to the restriction $\hat{H}$ of H_{m-1} to B_2; here λ_0 is absolutely continous $(\hat{H})$ and $\hat{\lambda}$ and $\hat{H}$ are mutually singular. For each $r > 0$ and $y \in E_m$, denote by $\Omega_r(y)$ the open ball of radius r and center y and put

$$\hat{v}_r(y) = \frac{\hat{\lambda}[\Omega_r(y)]}{(m-2)r^{m-2}} + \int_0^r \rho^{1-m}\, \hat{\lambda}[\Omega_\rho(y)]\, d\rho .$$

(Note that $\hat{v}_r(y)$ is just the value of the potential induced at y by the restriction of $\hat{\lambda}$ to $\Omega_r(y)$.)

For $j = 1, 2$ set

$$k_j = \limsup_{r \to 0+} \sup_{y \in B_j} [v_r(y) + \hat{v}_r(y)]$$

if $B_j \neq \emptyset$; in the opposite case define $k_j = 0$. With this notation we have the following theorem which we state here for the simplest case when $U\lambda_0$ is continuous.

Theorem II. If a and γ are defined by (3), then $a < 1$ if and only if, simultaneously,

$$k_1 < A \,, \quad k_2 < \frac{1}{2} A \,. \tag{4}$$

If (4) holds then one of the following cases must take place:

(i*) $B_1 = \emptyset$,

(ii) $B_2 = \emptyset$ or $k_1 \geqq \frac{1}{2} A + k_2$,

(iii) $B_1 \neq \emptyset \neq B_2$ and $|k_1 - k_2| < \frac{1}{2} A$.

In the case (i*)

$$a = 2 k_2 / A \,, \qquad \gamma = \frac{1}{2} \;;$$

if (ii) occurs then

$$a = k_1 / A \,, \qquad \gamma = 1 \,,$$

while in the case (iii)

$$a = \frac{k_1 + k_2 + \frac{1}{2} A}{k_1 - k_2 + \frac{3}{2} A} \,, \quad \gamma = \frac{3}{4} + \frac{k_1 - k_2}{2A} \,.$$

Under suitable conditions the corresponding theorem for discontinuous $\mathcal{U}\lambda_0$ is the same, only the definition of the constants k_1, k_2 must be generalized and becomes more complicated. On the other hand, if $\mathcal{U}\lambda$ happens to be continuous on $\mathbf{B}$ (especially if $\lambda = 0$) then $\hat{v}_x(y)$ can be omitted in the definition of k_1, k_2 .

Using some ideas of J. Radon, we are in a position to prove the following

Theorem III. Let α, β be real numbers, $A|\beta| > \omega \mathcal{T}_\alpha$ and denote by $d(y)$ the m-dimensional density of G at y. Suppose that

$$d(y) \neq \beta - \alpha$$

for every $y \in B$. If $\mu \in \mathcal{L}$ satisfies

$$[A \beta \mathcal{I} + \mathcal{T}_\alpha]\mu = 0$$

then the corresponding potential $\mathcal{U}\mu$ is quasi-everywhere bounded (and thus possesses finite Dirichlet integral).

This proposition is a basic tool for the proof of the following theorem that is stated only for the case of continuous $\mathcal{U}\lambda_0$ here.

Theorem IV. Assume G to be a domain (= connected and open set) satisfying (4). Then

$$\mathcal{T}(\mathcal{L}) = \mathcal{L}$$

with the only exception which occurs if G is bounded and $\lambda = 0$. In this case the range of $\mathcal{T}$ consists precisely of those $\nu \in \mathcal{L}$ with $\nu(B) = 0$.

The proof of the announced theorems together with further related results and details and the corresponding bibliography will be given elsewhere.

R e f e r e n c e s

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