Daniel A. Moran

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 14 (1973), No. 4, 647--650

Persistent URL: http://dml.cz/dmlcz/105515

Commentationes Mathematicae Universitatis Carolinae

14,4 (1973)

# MINIMAL CELL COVERINGS OF SOME SPHERE BUNDLES

Daniel A. MORAN, East Lansing

Abstract: It is shown that certain sphere bundles over spheres admit coverings by three open cells.

Key words: Fibre space, Ljusternik-Schnirelman category, total space.

AMS, Primary: 55F05, 57A15 Ref. Ž. 3.976.1,3.985

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Let $M$ be the total space of a locally trivial fibre space $\pi: M \longrightarrow S^p$ with base space $S^p$ and fibre $S^q$, and let $m = p + q$ be the dimension of $M$. This note deals with the determination of the smallest number of open $m$-cells necessary to cover $M$. (This number has been called the "strong Ljusternik-Schnirelmann category"; the ordinary Ljusternik-Schnirelmann category of $M$ has been computed [2].

It is a simple matter to construct a collection of three open cells which cover a product of two spheres. Such a covering will in all cases be minimal, because a compact manifold can be covered by two open cells if and only if it is a sphere. Further, there is no difficulty in finding a covering of four open cells for an arbitrary sphere bundle over a sphere. We contribute the

**Theorem. If $M$ admits a cross-section, then $M$ can be covered by three open $n$-cells.**

*Proof*. Let $F$ be the fibre over the point $x$ of $S^p$, and $\sigma : S^p \longrightarrow M$ a cross-section. $M - F$ is homeomorphic with the product $E^p \times S^q$, and the further removal of $\sigma S^p$ yields an open $n$-cell $C_1$.

Because $M$ is locally trivial, there is an open $p$-cell neighborhood $U$ of $x$ in the base space such that $\pi^{-1}\overline{U}$ is homeomorphic with $\overline{U} \times S^q$ in such a way as to preserve fibres. Since the fibres are here homogeneous, $\sigma\overline{U}$ can be considered a slice in this product, and there is a slice parallel to it corresponding to some local cross-section $\sigma' : \overline{U} \longrightarrow \pi^{-1}\overline{U}$. Then $\sigma\overline{U} \cap \sigma'\overline{U} = \emptyset$. Let $C_2$ be the open $n$-cell $\pi^{-1}U - \sigma'\overline{U}$.

Let $y \in U - \{x\}$. There is a self-homeomorphism $f$ of $\overline{U}$ fixed on $\operatorname{bdry} \overline{U}$ which carries $x$ into $y$. Define the self-homeomorphism $g$ of $\pi^{-1}\overline{U}$ by employing the product structure on this space and setting $g(u, v) = (fu, v)$. Lastly, extend $g$ by the identity to a self-homeomorphism $h$ of all of $M$.

Now again consider $M - F$, a copy of $E^p \times S^q$ by way of some fibre-preserving homeomorphism. Once again the fibres are homogeneous, so the image $X = \sigma(S^p - \{x\})$ is a slice relative to some product structure on $M - F$. Let $Y$ be any slice relative to this structure, but chosen parallel to $X$ and such that $h\sigma'x \notin Y$. Removing $Y$ from $M - F$ yields an open $n$-cell $D$. Set $C_3 = h^{-1}D$.

Then $M = C_1 \cup C_2 \cup C_3$ . This completes the proof.

Remark 1. The above theorem is not stated in the fullest possible generality justified by the proof. Minor tampering with the argument yields the same conclusion under the following weakened hypothesis: there exists a map $\sigma : S^p - \{x\} \longrightarrow M$ with $\pi\sigma$ = identity, such that $F \cap cl_M$ (image of $\sigma$ ) $\neq F$ . (Here as above, F denotes the fibre over $x$ .)

Remark 2. E. Luft [1] has determined an upper bound for the strong Ljusternik-Schnirelmann category of any $k$-connected $n$-manifold. If $M$ is an $S^q$-bundle over $S^p$, the exact homotopy sequence of the bundle can be exploited to infer that $M$ is $k$-connected, where $k + 1 = \min\{p, q\}$, if $p, q > 1$ . By Luft's results, it follows that $M$ can be covered by three $n$-cells if $\frac{1}{2}(p + 1) \leq q \leq 2p - 1$ . (This pair of inequalities is symmetric in $p$ and $q$ .)

The question for arbitrary sphere bundles over spheres remains unanswered.

## References

[1] E. LUFT: Covering of manifolds with open cells, Illinois J.Math.13(1969),321-326.

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Michigan State University
East Lansing
Michigan 48823
U.S.A.

(Oblatum 30.7.1973)