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DEFINABILITY DEGREES FOR CLASSES IN THE ALTERNATIVE SET THEORY

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Abstract. We propose a notion of relative definability using positive formulas and study the induced ordering. We show that the degree of every cut is minimal in this ordering. If $I<J$ and J is semiregular, then the degrees of I, J are different. Also, the ordering contains incomparable elements, Ω-chains and upper bounds for codable classes of degrees.

Key words. Alternative set theory, normal formula, positive formula, cut of natural numbers.

Classification. 02K10, 02B99

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§ 1. **Definability degrees.** Let $\varphi(Z)$ be a normal formula of the language FL_V, where Z is a class variable of φ. (For the definition of the terms just used as well as of any other from the context of the Alternative Set Theory, we refer to [V].) $\varphi(Z)$ is positive in Z, or simply positive, if it belongs to the smallest class of formulas which contains the set-formulas, the formulas "$x \in Z$" and is closed under the positive operations $\vee, \wedge, \exists, \forall$.

Positive formulas were introduced in [MO] to be used in inductive definitions. Given a formula $\varphi(x,Z)$ we put, for every class X

$$\Gamma_\varphi(X)=\{x;\varphi(x,X)\}$$

The main reason of employing positive formulas is that their operator Γ_φ is increasing, i.e.

$$X\subseteq Y \longrightarrow \Gamma_\varphi(X) \subseteq \Gamma_\varphi(Y).$$

From now on every normal formula used will be positive unless otherwise stated.

Definition 1.1. Let X, Y be classes of the extended universe. We say that X is definable in Y iff $X=\Gamma_\varphi(Y)$ for some φ. X, Y are equidefinable if X is definable in Y and Y is definable in X.

Proposition 1.2. The relation "X is definable in Y" is a preorder, that is, it is reflexive and transitive.

Proof. If $\varphi(x,T)\equiv x\in Z$, then $\Gamma_\varphi(X)=X$. On the other hand, if $X=\Gamma_\varphi(Y)$ and $Y=\Gamma_\psi(W)$, and putting

$$\sigma(x,Z)\equiv\varphi(x,\Gamma_\psi(Z)),$$

then σ is positive in Z (cf. [T], Lemma 1.2) and $X=\Gamma_\sigma(W)$. □

It follows that equidefinability is an equivalence relation. The class

$$[X]=\{Y;\ Y \text{ and } X \text{ are equidefinable}\}$$

is called the <u>definability degree</u> of X, or, simply the <u>degree</u> of X.

As usual, we write $[X]\leq[Y]$ to denote the fact that X is definable in Y. Clearly, $\leq$ is a well-defined partial ordering of the degrees. $[X]<[Y]$ means $[X]\leq[Y]$ and $[X]\neq[Y]$.

Examples. 1) If Sd_V is the class of all set-definable classes, then $[X]=Sd_V$ iff $X\in Sd_V$. We denote by $[V]$ the degree of the set-definable classes. Clearly, $[V]\leq[X]$ for every $[X]$, that is, a set-definable class is definable in any class.

2) The classes X,P(X) (the class of subsets of X) are equidefinable since $\cup P(X)=X$ and the operators P, $\cup$ are induced by positive formulas.

3) $[FN]\leq[\Omega]$, since $FN=\{x;x\in\Omega\wedge x\subseteq\Omega\}$ and the formula $x\in\Omega\wedge x\subseteq\Omega$ is positive in Ω (Ω is the class of ordinals).

Proposition 1.3. If F is a 1-1 set-definable function and F"X=Y, then X, Y are equidefinable.

Proof. If F"X=Y, then just note that

$Y=\{y;(\exists x\in X)(f(x)=y)\}$, $X=\{x;(\exists y\in Y)(F(x)=y)\}$,

and the defining formulas of X, Y are equidefinable. □

Corollary 1.4. Any two countable classes are equidefinable.

Proof. If X, Y are countable, then f"X=Y for some 1-1 function f and the conclusion follows from Prop. 1.3. □

One can see, however, that not all classes of [FN] are countable.

A class X is said to be Σ^0 if it can take the form $X=\bigcup\{R''\{n\};n\in FN\}=R''FN$ where R is set-definable. Σ^0-classes were introduced in [M] and are the simplest Σ-classes from the point of view of definability. In [M] it is shown that $\Sigma^0 \neq \Sigma$.

Proposition 1.5. If $X \in \Sigma^0\text{-}Sd_V$, then $[X]=[FN]$.

Proof. Let X=R"FN. Then

$$x \in X \leftrightarrow (\exists n \in FN)(\langle x,n\rangle \in R)$$

and the r.h.s. formula is positive in FN. Thus $[X] \leq [FN]$.

For the converse consider the set definable function F defined as follows:

$$F(x)=\min\{\alpha ; x \in R"\alpha\}.$$

Then, clearly, $F"X \subseteq FN$ and, since $X \notin Sd_V$, F"X is cofinal in FN. Therefore $\bigcup F"X=FN$ and the operator $\bigcup F$ is positive. This shows that $[FN] \leq [X]$. □

The preceding result can be extended to hold for classes defined as Σ^0 but with FN replaced by an arbitrary cut.

Let us say that X is a Σ I-class if $X= \bigcup\{R"\{\alpha\} ; \alpha \in I\}=R"I$ for some set definable class R. Obviously for I=FN we just get Σ^0-classes.

The following generalizes Prop. 1.5.

Proposition 1.6. If $X \in \Sigma I\text{-}Sd_V$, then $[X]=[I]$.

Proof. Similar to that of 1.5. □

Fully revealed classes Σ-semisets and Π-semisets are totalities of classes essentially disjoint (that is, their common elements are just the set-definable classes). We shall see that for any two of them, the only common predecessor is $[V]$ again.

First a lemma:

Lemma 1.7. If $(u_n)_{n \in FN}$ is an increasing (decreasing) sequence of sets, then for every φ, $\Gamma_\varphi(\bigcup_n u_n)= \bigcup_n \Gamma_\varphi(u_n)$ ($\Gamma_\varphi(\bigcap_n u_n)= \bigcap_n \Gamma_\varphi(u_n)$).

Proof. The Σ-case is just Lemma 2.3 of [T]. The proof of the Π-case is similar. (Both use heavily the prolongation axiom.) □

Proposition 1.8. Let Y be definable in X. Then:

i) If X is fully revealed, then Y is fully revealed.

ii) If X is Σ-semiset, then Y is a Σ-class.

iii) If X is Π-semiset, then Y is a Π-class.

Proof. i) is immediate from the definition of fully revealed classes, while ii), iii) follow from 1.7. □

§2. Minimal degrees. We say that the degree $[X]$ is <u>minimal</u> if $[X] \neq [V]$ and for every φ, either $[\Gamma_\varphi(X)] = [V]$ or $[\Gamma_\varphi(X)] = [X]$.

We shall see in this section that for every cut I, $[I]$ is a minimal degree. And if $I < J$ and J is semi-regular, then $[I] \neq [J]$ (hence incomparable).

Lemma 2.1. Let $\varphi(x,Z)$ be a positive formula. Then there is a set-formula ψ, strings of quantifiers $\bar{Q}_i$ and strings of variables $\bar{x}_i$, $\bar{y}_i$ such that

$$\varphi(x,Z) \longleftrightarrow (\bar{Q}_1\bar{x}_1)(\bar{\exists}\,\bar{y}_1 \in Z)\ldots(Q_k x_k)(\bar{\exists}\,\bar{y}_k \in Z)\psi,$$

where $(\bar{\exists}\,\bar{y} \in Z)$ is an abbreviation of $(\exists\, y_1 \in Z)\ldots(\exists\, y_n \in Z)$, for some n, and $\exists\, y \in Z$ is the usual bounded quantifier.

Proof. By induction on the length of positive formulas. If φ is a set-formula, the assertion is vacuous. If $\varphi \equiv x \in Z$, then $\varphi \longleftrightarrow (\exists\, y \in Z)(x=y)$. The in duction steps for the positive operations are immediate. □

Lemma 2.2. Let I be a cut and suppose the formula $(\bar{\exists}\bar{\alpha} \in I)\varphi$ is given, where φ is a set-formula. Then, there is a set-formula ψ such that $(\bar{\exists}\bar{\alpha} \in I)\varphi \longleftrightarrow (\exists \alpha \in I)\psi$.

Proof. It suffices to observe that for every set formula $g(\bar{x})$ and every cut I,

$$(\exists\bar{\alpha} \in I)\varphi(\bar{\alpha}) \longleftrightarrow (\exists \alpha \in I)\ \ (\exists\bar{\alpha} < \alpha)\varphi(\bar{\alpha}).\ \ \square$$

Lemma 2.3. For any formula of the form $(\forall x)(\exists \alpha \in I)\varphi$, where φ is a set-formula, there is a set-formula ψ such that

$$(\forall x)(\exists \alpha \in I)\varphi \longleftrightarrow (\exists \alpha \in I)(\forall x)\psi.$$

Proof. Define the (Skolem) function $G:V \longrightarrow N$ as follows: $G(x)$ = the least α such that $\varphi(x,\alpha)$. If the given formula is true, then $G''V \subseteq I$. Clearly, $G''V$ is bounded in I, whence

$$(\forall x)(\exists \alpha \in I)\varphi \longleftrightarrow (\exists \beta \in I)(\forall x)(\exists \alpha < \beta)\varphi.$$

Putting $\psi \equiv (\exists \alpha < \beta)\varphi$, we are done. □

Theorem 2.4. For every cut I and any formula φ, $\Gamma_\varphi(I)$ is a ΣI-class.

Proof. We have to show that given $\varphi(x,I)$, we can find a set-formula σ such that $\varphi(x\ ,I) \longleftrightarrow (\exists \alpha \in I)\sigma$. The algorithm is as follows: Write

$\varphi(x,I)$ in the form described in Lemma 2.1. Then, in the subformula $(\bar{\exists}\,\bar{\alpha}_k \in I)\psi$ contract the string of existential quantifiers to a single existential quantifier $\exists\, \alpha \in I$ by the help of Lemma 2.2. Then, using Lemma 2.3, carry, step by step, the quantifier $\exists\, \alpha \in I$ in front of the string $Q_k x_k$. This way, $\exists\, \alpha \in I$ joins the string $\bar{\exists}\,\bar{\alpha}_{k-1} \in I$. Contract again and so on. It is clear that the finally resulting equivalent formula is as required. $\square$

Theorem 2.5. For any I, [I] is minimal.

Proof. By Theorem 2.4 $\Gamma_\varphi(I)$ is a ΣI-class for any φ. And by Prop. 1.6, either $\Gamma_\varphi(I) \in Sd_V$ or $[\Gamma_\varphi(I)] = [I]$. $\square$

Remark. P. Vopěnka pointed out that, as regards semisets, the converse of Th. 2.5 is also true, that is, every minimal degree is the degree of some cut. In fact, given the semiset X, the cut $I=\{\alpha ;(\exists\, x \subseteq X)(|x|= \alpha)\}$ (a kind of "inner measure" of X) is positively definable in X, hence $[I] \leq [X]$.

To show that $[I] \neq [J]$ in the case that $I < J$ and J is semi-regular, we need some terminology.

Let I be a cut, X is an I-class if there is a 1-1 function f such that $I \subseteq dom(f)$ and X=f"I.

A class X is I-revealed if for every I-class $Y \subset X$ there is a set u such that $Y \subseteq u \subseteq X$.

Recall that a cut I is semi-regular if for every $\alpha \in I$ and every f,f"α is not cofinal in I.

Lemma 2.6. a) Let $I < J$ and J be semiregular. Then, every J-class is I-revealed.

b) Let X be a (proper) ΣI-class. Then, for some $K \leq I$, X is not K-revealed.

Proof. a) Let X=f"J, Y=g"I, with f, g 1-1, such that $Y \subseteq X$. Then,

$$(\forall \alpha \in I)(\exists \beta \in J)(g(\alpha)=f(\beta)),$$

and if we define h by

$$h(\alpha)=\min\{\beta ;g(\alpha)=f(\beta)\},$$

then h is 1-1 and $h"I \subseteq J$. Since J is semi-regular, there is some $\gamma \in J$ such that $h"I \subseteq \gamma$. Then it is easy to see that $Y \subseteq f"\gamma \subseteq X$.

b) Let $X= \bigcup\{ R"\{\alpha\} ;\alpha \in I\}$ be a ΣI-class. Define recursively:

$$f(o),\ f(\alpha +1)=\min\{\beta ;R"\{\beta\} - R"\{f(\alpha)\} \neq \emptyset\}.$$

Since X is proper (non set-definable), if $K=F^{-1}(I)$, then f"K is cofinal in I and $K \not\leq I$. Put

$$g(\alpha) = \text{least element of } R''\{f(\alpha+1)\} - R''\{f(\alpha)\}.$$

We easily see, then, that there is no u such that $g''K \subseteq u \subseteq X$, which shows that X is not K-revealed. □

Theorem 2.7. If $I < J$ and J is semiregular, then $[I] \neq [J]$.

Proof. Suppose $[I]=[J]$. Then $J = \Gamma_\varphi(I)$ for some φ. By Th. 2.4, J is a Σ I-class, hence (by 2.6 b)) not K-revealed for some $K \not\leq I$. But J is a J-class, hence (by 2.6 a)) K-revealed for every $K < J$. A contradiction. □

Remark. Concerning Theorem 2.5, K. Čuda made the following comment:

The theorem is no longer true if we replace the cut by an arbitrary class. That is, we can find classes X, Y such that $Y = \Gamma_\varphi(X)$ but Y cannot be put in the form $Y = \bigcup\{R''\{x\}; x \in X\} = R''X$ for some $R \in Sd_V$ (cf. [Č]). Indeed, take the classes

$$Y = FN \times (\alpha - FN) \text{ and } X = (FN \times \{0\}) \cup ((\alpha - FN) \times \{1\})$$

for some $\alpha > FN$.

Then, clearly, $[Y] \not\leq [X]$. Suppose $Y = R''X$.

Then, there are sets r_1, r_2 such that $Y = r_1''FN \cup r_2''(\alpha - FN)$. Define

$$f_1(\beta) = \min\{\gamma; (\exists\delta)(\langle\delta,\gamma\rangle \in r_1''\beta)\}$$

$$f_2(\beta) = \max\{\gamma; (\exists\delta)(\langle\gamma,\delta\rangle \in r_2''(\alpha-\beta)\}.$$

Then,

$$(\forall n \in FN)(f_1(n) \in \alpha - FN)$$

and

$$(\forall \gamma \in \alpha - FN)(f_2(\gamma) \in FN).$$

Hence, there are $\beta \in \alpha - FN$, $k \in FN$ such that

$$(\forall n \in FN)(f_1(n) > \beta)$$

and

$$(\forall \gamma \in \alpha - FN)(f_2(\gamma) < k).$$

Therefore $r_1''FN \subseteq FN \times (\alpha - \beta)$, $r_2''(\alpha - FN) \subseteq k \times (\alpha - FN)$. It follows that

$$FN \times (\alpha - FN) \subseteq FN \times (\alpha - \beta) \cup k \times (\alpha - FN)$$

which is false.

§ 3. Incomparable degrees and chains of degrees

Theorem 3.1. For any $[X] \neq [V]$, there is a Y such that $[X]$, $[Y]$ are incomparable.

Proof. Suppose first that X is real (cf. [Č-V] for the notions of real and imaginary class). It suffices to choose a Y fully revealed and $Y \notin Sd_V$. Then for every φ, $\Gamma_\varphi(X)$ is real. If $\Gamma_\varphi(X)$ is not revealed, then $\Gamma_\varphi(X) \neq Y$. If $\Gamma_\varphi(X)$ is revealed, then it is a Π-class, thus again $\neq Y$. On the other hand, $\Gamma_\varphi(Y)$ is fully revealed, therefore $\neq X$.

Now, let X be imaginary. There are codably many real classes definable in X, while all real classes are uncodable. Choose a real Y not definable in Y. Then [X], [Y] are incomparable. □

Theorem 3.2. For any X there is a Y such that $[X] < [Y]$.

Proof. Given X take Z so that X, Z be incomparable. Put $Y = (X \times \{0\}) \cup \cup (Z \times \{1\})$. Then obviously $[X] \leq [Y]$. Suppose $[Y] \leq [X]$, that is $\Gamma_\varphi(X) = = (X \times \{0\}) \cup (Z \times \{1\})$. Then

$$x \in Z \leftrightarrow \langle x,1 \rangle \in \Gamma_\varphi(X)$$

and the r.h.s. formula is positive in X. Thus $[Z] \leq [X]$, a contradiction. □

Corollary 3.3. Any codable class of degrees has an upper bound.

Proof. Let $\{X''\{c\}; c \in C\}$ be a codable class with code $\langle X,C \rangle$. Then, obviously, $[X''\{c\}] \leq [X]$ for any $c \in C$. □

Corollary 3.4. i) For any X there is an Ω-chain of degrees above X.

ii) The class of degrees above [X] is uncodable.

iii) The class of degrees below [X] is codable.

Proof. i) It follows from 3.2 (for the successor stages) and from 3.3 (for the limit stages).

ii) If $\mathfrak{M} = \{[Y]; [X] \leq [Y]\}$ were codable, there would be, by 3.3 an upper bound [W] of $\mathfrak{M}$ and, by 3.2, a $[U] > [W]$. Then $[U] \in \mathfrak{M}$, while $[U] > [Y]$ for every $[Y] \in \mathfrak{M}$, a contradiction.

iii) Immediate from the fact that the class of positive formulas is codable.

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