Commentationes Mathematicae Universitatis Carolinae

Zuzana Došlá; Ondřej Došlý

A remark on uniqueness criteria for initial value problem

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 3, 447--453

Persistent URL: http://dml.cz/dmlcz/106766

A remark on uniqueness criteria for initial value problem

Z. Došlá, O. Došlý

Dedicated to the memory of Svatopluk Fučík

Abstract. Relations between the recently published uniqueness criteria for the Cauchy's problem for ordinary differential equations are discussed and their equivalence to the Perron's theorem is proved.

Keywords: Ordinary differential equation, uniqueness criterion

Classification: 34A10

1. Introduction. This paper is motivated by the results of the articles [5] and [2] where the uniqueness criteria for the initial value problem

$$x' = f(t,x) \qquad x(t_0) = x_0 \tag{1}$$

are considered. Theorem 1 of [2] is the generalization of [10]. In [5] it is shown that this result can be obtained from [4].

The aim of this paper is to insert the results of the above mentioned papers to the framework of Theorem 1 of [8].

We shall show how the criteria of [10], [4] and the recent one of [2] follow from this general theorem. It will be proved that these results are in some sense equivalent to the Perron's theorem [7].

The case of functions with their values in a Banach space as well as the connection of uniqueness theorem on the convergence of sequences of succesive approximations will be given elsewhere.

2. Historical background. A. Cauchy was the first one dealing with the problem of the uniqueness for (1). It took place in the second decade of the nineteenth century. He was followed by Lipschitz, Osgood, Nagumo, Perron, Kamke and others whose theorems are contained as usual in monographies on ordinary differential equations. These results guarantee the uniqueness of the solution of (1) under the assumption of the form

$$|f(t,x) - f(t,y)| \le g(t, |x-y|) \quad . \tag{2}$$

Here $g(t,u)$ is a nonnegative real-valued function for which $u(t) = 0$ is the unique solution of the initial value problem

$$u' = g(t,u) \qquad u(t_o) = 0$$

and which satisfies various additional assumptions.

Since fourtieth of this century the most general of these results – Kamke's theorem – has been generalized in various directions. The first generalization consists in replacing of the left-hand side of (2) by the derivative of a Ljapunov function along the trajectories of (1) (see, e.g. references in [8]). It was motivated by the fact that the Kamke-type criteria fail to prove the uniqueness in the case like $x' = \sqrt{x} + 1$, $x(0) = 0$; in this case the left-hand side of (2) is the same as for the (nonunique) problem $x' = \sqrt{x}$, $x(0) = 0$.

Recently some efforts were made to uniqueness criteria for (1) under some weaker assumptions on f, often without even assuming $f(t,x)$ to be continuous on the line $t = t_0$. This direction of generalization is followed in our article.

3. List of known results. In this section we enlist some known uniqueness criteria in order to show that they are special cases of the general theorem from [8]. Note that this theorem is a slight generalization of [3].

Denote $R^+ = \langle 0,\infty)$, $R^- = (-\infty, 0\rangle$. Let $C[D_1; D_2]$ be the class of all continuous functions $f : D_1 \to D_2$ and let $f(t) = o(g(t))$ as $t \to t_{0+}$ mean $\lim_{t\to t_{0+}} f(t)/g(t) = 0$. Put

$$D = \{(t,x) = t_0 < t \le T, \quad |x - x_0| \le b\}$$
$$\hat{D} = \{(t,x,y) : t_0 < t \le T + \varepsilon, |x - x_0| \le b + \varepsilon |y - x_0| \le b + \varepsilon, \varepsilon > 0\}$$

Theorem A. *(Theorem 1 and Remark 1, [8]). Let $f(t,x) \in C[D; R]$ and let the following condition be satisfied:*

(i) *there exist a positive function $B(t) \in C[(t_0,T\rangle; R^+]$ and a function $g(t,u) \in C[(t_0,T\rangle \times R^+; R]$ such that for every $t_1 \in (t_0,T)$ the function $u(t) \equiv 0$ is the only differentiable function satisfying*

$$u'(t) = g(t,u(t)) \quad \text{for} \quad t \in (t_0, t_1) \tag{3}$$
$$u(t) = o(B(t)) \quad \text{as} \quad t \to t_{0+}; \tag{4}$$

(ii) *there exists a function $V(t,x,y) \in C[\hat{D}; R^+]$ such that $V(t,x,y)$ is locally Lipschitzian in x,y for $(t,x,y) \in \hat{D}$ and for any two solutions $x(t)$, $y(t)$ of (1)*

$$V(t,x(t),y(t)) \equiv 0 \Longleftrightarrow x(t) \equiv y(t) \qquad \text{on} \quad (t_0,T\rangle$$
$$V(t,x(t),y(t)) = o(B(t)) \text{ as } t \to t_{0+} \quad \text{for} \quad x(t) \not\equiv y(t);$$

(iii) *for $(t,x),(t,y) \in D$, $x \ne y$, $t < T$ the inequality*

$$D_{+f}V(t,x,y) \le g(t,V(t,x,y)) \quad \textit{holds, where}$$

$$D_{+f}V(t,x,y) = \liminf_{h\to 0_+}[V(t+h, x + hf(t,x), y + hf(t,y)) - V(t,x,y)]/h.$$

Then the initial value problem (1) has at most one solution.

Theorem B. *(Witte [10]). Let $f(t,x) \in C[D;R]$ and let the following assumptions be satisfied:*

$$(5) \qquad |f(t,x) - f(t,y)| \le h(t)|x-y| \quad on \quad D$$

where $h(t) \in C[(t_0,T);R^+]$;

$$|f(t,x)| \le \varrho(t)h(t)\exp\{\int_T^t h(s)ds\} \quad on \quad D$$

where $\varrho(t) \in C[\langle t_0,T\rangle;R]$, $\varrho(t_0)=0$.
Then (1) has at most one solution.

Theorem C. *(Lemmert [4]). Let $f(t,x) \in C[D;R]$, (5) holds and let*

$$(6) \qquad \text{the function} \quad H(t,x) := f(t,x)\exp\{\int_t^T h(s)ds\}/h(t) \quad \text{is uniformly continuous on } D.$$

Then (1) has at most one solution.

For completeness' sake we introduce the following unpublished generalization of Witte's theorem presented by the first author in the Czechoslovak student competition in 1978.

Theorem D. *Let $f(t,x) \in C[D;R]$, (5) hold and let*

$$(7) \qquad |f(t,x)| = o(H(t)) \quad for \quad t \to t_{0+} \text{ uniformly with respect to } x$$

where $H(t)$ is a non-negative function defined for $t \in (t_0,T\rangle$ and satisfying

$$(8) \qquad \int_{t_0}^T H(s)ds < \infty, \qquad \liminf_{t\to t_0+}[\int_{t_0}^t H(s)ds \exp\int_t^T h(s)ds] < \infty.$$

Then (1) has at most one solution.

Remark 1. Theorem D can be obtained as a corollary of Theorem A (see proof of Theorem 2). However, its original proof was based on the usual method for uniqueness.

Putting $H(t) = h(t)\exp\int_T^t h(s)ds$ one can easily verify that Witte's theorem is a corollary of Theorem D.

Theorem E. *(Banas'-Rivero [2]). Let $f(t,x) \in C[(t_0,T\rangle \times R; R]$ satisfying*

$$
\begin{aligned}
&|f(t,x) - f(t,y)| \le h(t)|x-y| \\
&|f(t,x) - f(t,y)| = o(h(t)\exp(A(t)) \quad \text{as} \quad t \to t_{0+}
\end{aligned} \tag{9}
$$

uniformly with respect to $x, y \in \langle x_0 - \delta, x_0 + \delta\rangle$, $\delta > 0$ arbitrary, where $h(t) \in C[(t_0,T\rangle; R^+]$ and $A(t) : (t_0,T\rangle \to R$ such that $A'(t) = h(t)$ for almost all $t \in (t_0,T\rangle$ and there exists a limit $\lim_{t\to t_{0+}} A(t)$ (finite or not).

Then (1) has at most one solution.

Remark 2. Note that the criteria given above could be formulated also for vector-valued functions. Some further uniqueness criteria have been obtained as corollaries of Theorem A in [8].

4. Application of Theorem A. The main goal of this section is to prove Theorems B-E as corollaries of Theorem A. In all proofs which follow, we use the same choice of $V(t,x,y) = |x-y|$ and $g(t,u) = h(t)u$. In case of the first function we have according to (5)

$$
\begin{aligned}
D_{+f}V(t,x,y) &= \frac{1}{|x-y|}(x-y)\cdot(f(t,x) - f(t,y)) \le h(t)|x-y| = \\
&= g(t, |x-y|)
\end{aligned}
$$

for $(t,x),(t,y) \in D$, $t \ne t_0$, $x \ne y$. In case of the function $g(t,u)$ observe that the differential equation $u'(t) = h(t)u$ has the general solution $u(t) = u(t_1)\exp[-\int_t^{t_1} h(s)ds]$ for arbitrary fixed $t_1 \in (t_0,T\rangle$. Thus, to obtain a uniqueness criterion as a corollary of Theorem A it remains to find a function $B(t)$ such that

(i) $u(t) \equiv 0$ is the only differentiable solution of $u' = h(t)u$ satisfying (4) and
(ii) $|x(t) - y(t)| = o(B(t))$ for every two solutions $x(t)$, $y(t)$ of (1).

Theorem 1. *Theorem C is the corollary of Theorem A.*

PROOF : Put $B(t) = \exp\int_T^t h(s)ds$. Then

$$
\lim_{t\to t_{0+}} \frac{u(t)}{B(t)} = u(t_1) \lim_{t\to t_{0+}} \frac{\exp(-\int_t^{t_1} h)}{\exp(\int_T^t h)} = u(t_1)\exp\int_{t_1}^T h(s)ds
$$

and $u(t) = o(B(t))$ if and only if $u(t_1) = 0$. So $u(t) \equiv 0$ because $t_1 \in (t_0,T\rangle$ is arbitrary.

Further, it follows from (6) that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for $(t_1,x),(t_2,x) \in D$, $|t_1 - t_2| + |x-y| < \delta(\varepsilon)$ it holds

$$
|f(t_1,x)/[h(t_1)\exp\int_T^{t_1} h] - f(t_2,y)/[h(t_2)\exp\int_T^{t_2} h]| < \varepsilon. \tag{10}
$$

Let $x(t)$, $y(t)$ be arbitrary two solutions of (1). With respect to the initial condition there exists $\tau \in (t_0, T)$ such that $|x(t) - y(t)| < \delta$ for $t_0 < t < \tau$. Thus, from (10) it follows

(11) $$\lim_{t \to t_0+} \frac{|f(t, x(t) - f(t, y(t))|}{h(t) \exp \int_T^t h} = 0.$$

If $\lim_{t \to t_0+} \int_t^T h = K$ is finite then $\lim_{t \to t_0+} B(t) = e^{-K}$ and $|x(t) - y(t)| = o(B(t))$ as $t \to t_0+$.

If $\lim_{t \to t_0+} \int_t^T h = \infty$ then $\lim_{t \to t_0+} B(t) = 0$ then, using l'Hospital rule, we get from (11) $|x(t) - y(t)| = o(B(t))$ as $t \to t_0+$. ∎

Theorem 2. *Theorem D is the corollary of Theorem A.*

PROOF : Put $B(t) = \int_{t_0}^t H(s)ds$. According to (8) there exists a sequence $\{t_n\}$, $\lim t_n = t_0$, $t_n \in (t_0, t_1)$ such that

$$\lim_{n \to \infty} \int_{t_0}^{t_n} H(s) \exp \int_{t_n}^{t_1} h(s)ds = K > 0.$$

If a solution $u(t)$ of $u' = h(t)u$ satisfies $u(t) = o(B(t))$ as $t \to t_0+$ then also $\lim_{n \to \infty} u(t_n)/B(t_n) = \lim_{n \to \infty} u(t_1) \exp(-\int_{t_n}^{t_1} h)/\int_{t_n}^{t_1} H = u(t_1)/K = 0$. As $K > 0$ we get $u(t_1) = 0$ and thus $u(t) \equiv 0$ on (t_0, t_1).

Now we have from (7) $\lim_{t \to t_0+} \int_{t_0}^t |f(s, x(s))|ds / \int_{t_0}^t H(s)ds = 0$ and thus for every two solutions $x(t)$, $y(t)$ of (1) it holds

$$\lim_{t \to t_0+} \frac{|x(t) - y(t)|}{B(t)} \leq \lim \frac{|x(t) - x_0|}{B(t)} + \lim \frac{|-y(t) + x_0|}{B(t)} = 0.$$

∎

Theorem 3. *Theorem E is the corollary of Theorem A.*

PROOF : Let $B(t) = \exp A(t)$. Then

$$\lim_{t \to t_0+} \frac{u(t)}{B(t)} = u(t_1) \lim_{t \to t_0+} \exp[\int_{t_1}^t h - A(t)] = u(t_1)e^{\text{const}}$$

since $A' = h$ a.e. on (t_0, t_1). Thus (4) holds iff $u(t_1) = 0$, i.e. $u(t) \equiv 0$.

In the same way as in the proof of Theorem 1 (starting from the relation (11)) we get from (9) $|x(t) - y(t)| = o(B(t))$ as $t \to t_0+$. ∎

5. Uniqueness criteria and Perron theorem. In [6] and [9] it was shown that several uniqueness theorems (among them the theorem of Kamke) - formally more general then the following Perron's theorem - are actually equivalent to this criterion.

We are going to show that Theorems B-E are also (in some sense) equivalent to the following criterion.

Theorem F. *(Perron [7]). Let* $\bar{D}: t_0 \le t \le T, |x - x_0| \le b$. *Let* $f(t,x) \in C[\bar{D}; R]$ *and let*

$$|f(t,x) - f(t,y)| \le h(t, |x-y|) \quad in \quad \bar{D}$$

hold, where $h(t,u) \in C[\langle t_0, T\rangle \times R^+, R^+]$ *and for every* $t_1 \in (t_0, T)$ *the function* $u(t) \equiv 0$ *is the only differentiable function on* $\langle t_0, t_1)$ *such that*

$$u' = h(t,u) \qquad u(t_0) = 0. \tag{12}$$

Then (1) has only one solution.

The function $h(t,u)$ from Theorem F will be called the Perron function of (1).

Theorem 4. *Let* $B(t) \in C^1[(t_0,T\rangle; R^+]$, $\lim_{t\to t_0+} B(t) = B(t_{0+})$ *exist (finite or not),* $f(t,x) \in C[\bar{D}; R]$ *and*

$$|f(t,x) - f(t,y)| \le g(t, |x-y|) \quad on \quad \bar{D},$$

where $g(t,u)$ *satisfies the assumption (i) of Theorem A. Suppose*

$$|f(t,x) - f(t,y)| = o(B'(t)) \qquad t \to t_{0+} \tag{13}$$

uniformly with respect to $x, y \in \langle x_0 - \delta, x_0 + \delta\rangle$, $\delta > 0$ *arbitrary.*
Let

$$h(t,u) = \begin{cases} \sup |f(t,x) - f(t,y)| & for \quad 0 \le u \le \delta \\ |x-y| = u & \\ x, y \in [x_0 - \delta, x_0 + \delta] & \\ h(t,\delta) \qquad for \quad u > \delta. & \end{cases} \tag{14}$$

Then $h(t,u)$ *is the Perron function of (1).*

PROOF : From (13) we get $h(t,u) = o(B'(t))$, $t \to t_{0+}$, uniformly for $0 \le u \le \delta$. Let $u(t)$ be a solution of (12). Then $u'(t) = h(t, u(t)) = o(B'(t))$, $t \to t_{0+}$. If $B(t_{0+}) = 0$, using l'Hospital rule $\lim_{t\to t_0+} u(t)/B(t) = \lim_{t\to t_0+} u'(t)/B'(t) = 0$. If $B(t_{0+}) > 0$, then also obviously $u(t) = o(B(t))$, $t \to t_{0+}$.

On the other hand, suppose that $u(t_1) > 0$ for some $t_1 \in (t_0, t_0 + \kappa)$. According to the continuity of $u(t)$ we can assume, without loss of generality, that $u(t_1) < \delta$. Denote by $v(t)$ the left minimal solution of $v' = g(t,v)$, $v(t_1) = u(t_1)$. Since $h(t,u) \le g(t,u)$ for $t \in (t_0, t_0 + \kappa)$, $0 \le u \le \delta$, we have $v(t) \ne o(B(t))$, $t \to t_{0+}$. Hence also $u(t) \ne o(B(t))$, $t \to t_{0+}$ and thus, in view of (4), $u(t_0) > 0$. Consequently, $u(t) \equiv 0$ is the only solution of (12) on $(t_0, t_0 + \kappa)$. From [9, Folgerung A] it follows that the same holds for $t \in (t_0, T)$, i.e., $h(t,u)$ is the Perron function of (1). ∎

Corollary. *If* $f(t,x) \in C[\bar{D}; R]$ *and the assumptions of Theorem X,* $X \in \{B, C, D, E\}$ *are satisfied then* $h(t,u)$ *given by (14) is the Perron function of (1).*

PROOF : Put $g(t,u) = h(t)u$ and $B(t) = \exp \int_T^t h$ in case of Theorems B,C, $B(t) = \int_{t_0}^t H(t)$ in case of Theorem D, $B(t) = \exp A(t)$ for Theorem E. The conclusion follows from Theorem 4. ∎

REFERENCES

[1] Banas' J., Hajnosz A., Wedrychowicz S., *Relations among various criteria of uniqueness for ordinary differential equations*, Comment. Math. Univ. Carolinae **22** (1981), 59-70.

[2] Banas' J., Rivero J., *Remarks concerning I. Witte's theorem and its applications*, Comment. Math. Univ. Carolinae **28** (1987), 23-31.

[3] Bernfeld S.R., Driver R.D., Lakshmikantham V., *Uniqueness for ordinary differential equations*, Math. Systems Theory **9** (1975/6), 359-367.

[4] Lemmert R., *Über einen Satz von Witte*, Math. Z. **145** (1975), p. 289.

[5] Lemmert R., *Zur Eindentigkeitsbedingung von Nagumo*, Comment. Math. Univ. Carolinae **29** (1988), 293-294.

[6] Olech C., *Remarks concerning criteria for uniqueness of solutions of ordinary differential equations*, Bull. Acad. Polon. Sci., Sér. sci math., astr. et phys. **8** (1960), 661-666.

[7] Perron O., *Über Ein- und Mehrdeutigkeit des Integrals eines Systems von Differentialgleichungen*, Math.Ann. **95** (1926), 98-101.

[8] Tesařová Z., Došlý O., *General uniqueness theorems for ordinary differential equations*, Arch. Math. 4, **16** (1980), 217-224.

[9] Walter W., *Bemerkungen zu verschiedenen Eindeutigkeitskriterien für gewöhnliche Differentialgleichungen*, Math. Z. **84** (1964), 222-227.

[10] Witte J., *Uniqueness theorem for ordinary differential equations $y' = f(x,y)$*, Math. Z. **140** (1974), 281-287.

Department of Mathematics, J.E.Purkyně University, Janáčkovo nám. 2a, 662 95 Brno, Czechoslovakia

(*Received* March 22,1989)