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SOME REMARKS ON THE PHYSICAL STRUCTURE OF ENERGY-MOMENTUM TENSORS IN GENERAL RELATIVITY*

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INTRODUCTION

Of the many significant developments in General Relativity in recent years, one of the most important is the classification of the Weyl tensor given by Petrov [1]. His work together with that of Pirani [2], Bel [3], Penrose [4] and others has enabled many branches of Einstein's theory to be rephrased in a more systematic and transparent style. Now whereas the Weyl tensor describes what might loosely be called the purely gravitational properties of space-time, the physical content of space-time is represented by the matter or energy-momentum tensor. This tensor plays the rôle of the source term in Einstein's equations and following initial work by Churchill [5] and Plebanski [6] there has been much recent interest in its classification. The technical details of the classification of second order symmetric tensors in space-time have been described in detail elsewhere and only a brief summary of the relevant parts need be given here. This will be done in section 2, the rest of this section being given over to a summary of the notation used. The remainder of the paper will be devoted to some physical applications of the energy momentum tensor classification. It is appropriate here to keep the discussion brief and to the point with further details being given elsewhere.

A conventional notation will be used throughout, with M representing a space-time, that is a four dimensional real manifold carrying a global Lorentz metric of signature $+2$. Although other structures and restrictions are necessary to make M a realistic model of space-time, these need not concern us here. If $p \in M$, $T_p(M)$ will denote the tangent space to M at p. Latin indices will take the values 0, 1, 2, 3, a comma will denote partial differentiation and a semi-colon covariant differentiation. The Riemann, Weyl, Ricci, metric and energy-momentum tensors will be denoted in component form by R_{abcd}, C_{abcd}, R_{ab}, g_{ab} and T_{ab} respectively and the

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Ricci scalar by R. A second order skew symmetric tensor (2-form) will be called a bivector and round and square brackets will denote respectively the usual symmetrisation and skew-symmetrisation of indices. Finally the symbol $*$ in the approprate place will denote the duality operator.

At $p \in M$, it is convenient to introduce a real null tetrad of members of $T_p(M)$ with components l^a, m^a, x^a, y^a, where the only non-vanishing inner products between the tetrad members are $l^a m_a = x^a x_a = y^a y_a = 1$. From this tetrad one can construct a complex null tetrad $l^a, m^a, t^a, \bar{t}^a$, where $\sqrt{2} t^a = x^a + iy^a$ and where a bar denotes complex conjugation. The only non vanishing inner products here are $l^a m_a = t^a \bar{t}_a = 1$. From the latter tetrad one can construct the complex bivectors

$$(1.1) \qquad V_{ab} = 2l_{[a}\bar{t}_{b]}, \qquad M_{ab} = 2l_{[a}m_{b]} + 2\bar{t}_{[a}t_{b]}, \qquad U_{ab} = 2m_{[a}t_{b]},$$

and their conjugates where t_a is assumed oriented so that the bivectors (1.1) satisfy the self dual conditions $\overset{*}{V}_{ab} = -iV_{ab}$ etc. and their conjugates the anti-self dual conditions $\overset{*}{\bar{V}}_{ab} = i\bar{V}_{ab}$. The only non vanishing inner products between the bivectors (1.1) and their conjugates are $U_{ab}V^{ab} = \bar{U}_{ab}\bar{V}^{ab} = 2$, $M_{ab}M^{ab} = \bar{M}_{ab}\bar{M}^{ab} = -4$.

The following equations will be required in what is to follow.

$$\text{(a)} \qquad R_{abcd} = C_{abcd} + E_{abcd} + \frac{1}{6} R g_{a[c}g_{d]b},$$

$$\text{(b)} \qquad E_{abcd} = \tilde{R}_{a[c}g_{d]b} + \tilde{R}_{b[d}g_{c]a},$$

$$(1.2) \quad \text{(c)} \qquad R_{ab} = R^c_{acb}, \qquad \tilde{R}_{ab} = R_{ab} - \frac{1}{4} R g_{ab}, \qquad R = R_{ab}g^{ab},$$

$$\text{(d)} \qquad {}^*E_{abcd} = -E^*_{abcd}, \qquad E^*_{abcd} = -E^*_{cdab},$$

$$\text{(e)} \qquad E^c_{acb} = \tilde{R}_{ab}, \qquad E^{*c}_{acb} = 0,$$

$$\text{(f)} \qquad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}.$$

Here, $\tilde{R}_{ab}$ is the trace-free Ricci tensor and the equations (f) are the Einstein field equations.

2. THE CLASSIFICATION

The Petrov classification of the Weyl tensor (Riemann tensor in vacuo) splits up gravitational fields into the (algebraically) general type I and the more specialised types II, III, N, D and 0. (For a comprehensive and recent account see [7]). The classification of a symmetric second order tensor at a point $p \in M$ is based on the Segré type of this tensor when it is considered in the usual way as a linear map $T_p(M) \to T_p(M)$. The Lorentz signature of the metric and the symmetry of the tensor forbids certain Segré types and at $p \in M$, one may always choose a null tetrad such that the symmetric tensor in question (here taken to be the Ricci tensor for convenience although the same comments apply to any symmetric tensor) takes one of the following forms

$$
\begin{aligned}
&\text{(a)} \quad R_{ab} = 2\varrho_1 l_{(a}m_{b)} + \varrho_2(l_a l_b + m_a m_b) + \varrho_3 x_a x_b + \varrho_4 y_a y_b,\\
&\text{(b)} \quad R_{ab} = 2\varrho_{1(a}m_{b)} \pm l_a l_b + \varrho_2 x_a x_b + \varrho_3 y_a y_b,\\
(2.1)\ &\text{(c)} \quad R_{ab} = 2\varrho_1 l_{(a}m_{b)} + 2l_{(a}x_{b)} + \varrho_1 x_a x_b + \varrho_2 y_a y_b,\\
&\text{(d)} \quad R_{ab} = 2\varrho_1 l_{(a}m_{b)} + l_b\varrho_2(l_a - m_a m_b) + \varrho_3 x_a x_b + \varrho_4 y_a y_b \ (\varrho_2 \neq 0),
\end{aligned}
$$

where the ϱ's are real numbers. The Segré types here are respectively $\{1, 1, 1, 1\}$, $\{2, 1, 1\}$, $\{3, 1\}$ and $\{z, \bar{z}, 1, 1\}$, the latter being the only one where complex eigenvalues occur. A detailed review of the derivation of these types can be found in [8]. An alternative way of looking at this is to note that the algebraic properties of R_{ab} and $\tilde{R}_{ab}$ are essentially the same and that $\tilde{R}_{ab}$ and the tensor E_{abcd} are related in a one to one correspondence by (1.2)(b). Hence one could, alternatively, classify the tensor E_{abcd} or equivalently the tensor $\overset{+}{E}_{abcd} = E_{abcd} + iE^*_{abcd}$. (A tensor similar to E_{abcd} can be constructed from any tracefree symmetric second order tensor). The tensor E_{abcd} has the algebraic symmetries of the Riemann tensor and $\overset{+}{E}_{abcd}$ satisfies

$$
(2.2) \qquad \overset{+}{E}_{abcd} = \overline{\overset{+}{E}_{cdab}}, \qquad \overset{+*}{E}_{abcd} = -\overset{+}{E}_{abcd} = -\overset{*+}{E}_{abcd}, \qquad \overset{+}{E}{}^c_{acb} = \tilde{R}_{ab}.
$$

When the classification is cast into this form, one may borrow many techniques used in the Petrov classification. In fact it follows from (2.2) that the tensor $\overset{+}{E}_{abcd}$ may be decompósed in terms of the complex bivectors (1.1) and their conjugates and corresponding to the canonical forms (2.1) one has respectively the following forms for $\overset{+}{E}_{abcd}$ in the appropriate complex null tetrad [8, 9]

$$
\begin{aligned}
&\text{(a)} \quad \overset{+}{E}_{abcd} = \mu_1(\bar{U}_{ab}U_{cd} + \bar{V}_{ab}V_{cd}) + \mu_2\bar{M}_{ab}M_{cd} + \mu_3(\bar{U}_{ab}V_{cd} + \bar{V}_{ab}U_{cd}),\\
(2{,}3)\ &\text{(b)} \quad \overset{+}{E}_{abcd} = \mu_1\bar{V}_{ab}V_{cd} + \mu_2\bar{M}_{ab}M_{cd} + \mu_3(\bar{U}_{ab}V_{cd} + \bar{V}_{ab}U_{cd})(\mu_1 \neq 0),\\
&\text{(c)} \quad \overset{+}{E}_{abcd} = \mu_1(\bar{M}_{ab}M_{cd} - 2\bar{U}_{ab}V_{cd} - 2\bar{V}_{ab}U_{cd}) + \mu_2(\bar{V}_{ab}M_{cd} + \bar{M}_{ab}V_{cd})(\mu_2 \neq 0),\\
&\text{(d)} \quad \overset{+}{E}_{abcd} = \mu_1(\bar{U}_{ab}U_{cd} - \bar{V}_{ab}V_{cd}) + \mu_2\bar{M}_{ab}M_{cd} + \mu_3(\bar{U}_{ab}V_{cd} + \bar{V}_{ab}U_{cd})(\mu_1 \neq 0),
\end{aligned}
$$

where the μ's may always be chosen real. From these equations it is noted that the Ricci tensor can be associated with up to three complex self dual bivectors (c. f. Maxwell's theory). These equations are intimately connected with the invariant 2-space structure of R_{ab} [5, 8, 9].

The tensor E_{abcd} ($\neq 0$) allows a natural analogue of the Bel criteria [3] for the Weyl tensor. This can be summarised as follows [8, 9]:

(i) The Ricci tensor has Segré type $\{(2, 1, 1)\}$ if and only if there exists $l^a \in T_p(M)$, $l^a \neq 0$, such that $l^a E_{abcd} = 0$. The vector l^a is necessarily null, unique up to a real factor and coincides with the (unique) null Ricci eigendirection.

(ii) The Ricci tensor has Segré type $\{(3, 1)\}$ if and only if there exists a non-zero null bivector F_{ab} at p and a vector $l^a \in T_p(M)$ such that $l^a E_{abcd} = l_b F_{cd}$. Again l^a is necessarily null, unique up to a real factor and coincides with the (unique) null Ricci eigendirection and the (unique) repeated principal null direction of F_{ab}.

(iii) A non-zero null vector $l^a \in T_p(M)$ is a Ricci eigenvector if and only if there exists $\alpha \in \mathbf{R}$ such that $l^a l^c E_{abcd} = \alpha l_b l_d$.

(iv) If l^a is a non-zero null vector in $T_p(M)$ then $R_{ab} l^a l^b = 0$ if and only if $l^b l^c l_{[e} E_{a]bc[d} l_{f]} = 0$.

The criterion (iv) is the natural analogue of the Debever-Penrose condition on the Weyl tensor.

One can now see from (1.2) (a) the algebraic structure of the Riemann tensor completely in terms of the Petrov type of C_{abcd} and the canonical type for E_{abcd}.

3. ENERGY CONDITIONS

The well known energy-momentum tensors used in general relativity can be classified according to the scheme outlined in section 2. Thus, for example, null Maxwell fields have an energy-momentum tensor of type $\{(2, 1, 1)\}$ with zero eigenvalue, non-null Maxwell fields have type $\{(1, 1)(1, 1)\}$ with equal and opposite eigenvaues and perfect fluids have type $\{1, (1, 1, 1)\}$. A major restriction on the possible type of non-zero energy-momentum tensor is provided by the "energy conditions" [10]. These are (i) for each timelike vector u^a, $T_{ab} u^a u^b \geqq 0$, (ii) for each timelike vector u^a, $T^a_b u^b$ is non-spacelike. It turns out that no energy momentum tensor of Segré type $\{3, 1\}$ or $\{z, \bar{z}, 1, 1\}$ can satisfy either of the energy conditions, that if T_{ab} is of type $\{2, 1, 1\}$, then (i) and (ii) hold if and only if $\varrho_1 \leqq 0$ and $\varrho_1 \leqq \varrho_2, \varrho_3 \leqq -\varrho_1$ in (2.1) (b) and the optional sign in this equation is positive and that if T_{ab} is of type $\{1, 1, 1, 1\}$ then (i) and (ii) hold if and only if $\varrho_1 \leqq 0$, $\varrho_2 \geqq 0$ and $\varrho_1 - \varrho_2 \leqq \varrho_3, \varrho_4 \leqq \varrho_2 - \varrho_1$ in (2.1) (a) [6, 8, 10].

The condition (i) above may be replaced by the condition $T_{ab} u^a u^b > 0$ for all timelike vectors u^a without changing the joint conditions (i) and (ii) since if (i) and (ii) hold but $T_{ab} u^a u^b = 0$ for some non-zero timelike vector u^a, then by (ii), $T_{ab} u^b = 0$. So T_{ab} has a timelike eigenvector and is thus of type $\{1, 1, 1, 1\}$ [8]. But the corresponding eigenvalue is zero ($\varrho_1 = \varrho_2$ in (2.1) (a)) and so the above conditions imply that $T_{ab} = 0$. The result then follows.

4. APPLICATIONS

In this final section some brief remarks and discussion will be presented about the applications of the classification. Further details will be given elsewhere.

(i) An energy-momentum tensor of a given type will have a uniquely determined minimal polynomial which will yield a certain contracted identity on T_{ab}. For example, if T_{ab} has Segré type $\{(2, 1, 1)\}$ with zero eigenvalue, this identity is $T_a^b T_b^c = 0$, a relation which in fact characterises this type if $T_{ab} \neq 0$. This relation together with the positivity of energy condition constitutes the algebraic Rainich condition for a null Maxwell field. For a non-null Maxwell field, T_{ab} has Segré type $\{(1, 1)(1, 1)\}$ with equal and opposite eigenvalues and a minimal polynomial relation $T_a^b T_b^c = \frac{1}{4}(T_{bd}T^{bd})\,\delta_a^c$ where $T_{bd}T^{bd} \neq 0$. Here however one cannot deduce the converse result because the Segré type and the minimal polynomial for T_{ab} do not always have a one to one correspondence. Here one needs an extra condition, say $T_a^a = 0$, in order to be able to deduce the Segré type. These relations together with the positivity of energy condition are the algebraic Rainich conditions for a non-null Maxwell field. In general, the minimal polynomial relation allows the setting up of generalised algebraic Rainich conditions for each T_{ab} (however a few ambiguities like the one mentioned above exist and need extra relations).

(ii) Pirani [2] and Szekeres [11] have studied the scattering effects of a gravitational field on a cloud of uncharged non-rotating test particles by considering the equation of geodesic deviation

$$\delta \ddot{x}^a = R^a_{bcd} u^b u^d \delta x^c, \tag{4.1}$$

where $\delta \ddot{x}^a$ represents the connecting vector joining a particular particle (the observer) to a neighbouring particle, u^a is the tangent vector to the observer's geodesic world line and $u^a u_a = -1$, $\delta x^a u_a = 0$. Equation (4.1) can be thought of as representing the individual contributions from the Weyl tensor C_{abcd} and the energy-momentum tensor $\left(\text{represented by } E_{abcd} + \frac{1}{6} R g_{a[c} g_{d]b}\right)$. The canonical forms for E_{abcd} given in (2.3) enable this latter contribution to be evaluated (the Petrov forms of course enable the former to be evaluated). As an example, one might reasonably expect the energy-momentum tensor T_{ab} of a "radiation" field in general relativity to satisfy the following conditions: (a) T_{ab} admits a unique null eigendirection l^a, (b) in all the wave surfaces orthogonal to l^a, the scattering is transverse. These two conditions can be shown to imply that $T_{ab} \alpha l_a l_b$, (type $\{(2, 1, 1)\}$ with zero eigenvalue) the form usually assumed for such fields, and necessarily l^a is geodesic.

(iii) If a space-time is *locally isotropic* then the group of motions involved puts certain restrictions on the allowed energy momentum tensors. The canonical forms of section 2 allow these restrictions to be evaluated easily [7, 12, 13].

(iv) One recalls the elegant result concerning the asymptotic "peeling" behaviour of the Petrov types of the vacuum Riemann tensor given by Sachs [14] and by Newman and Penrose [15]. One might hope that under appropriate conditions, a similar asymptotic behaviour would hold for the Segré types of the energy-momentum tensor bearing in mind the comparisons made between the Petrov and Segré classifications mentioned in section 2. A very simple example of this property within special relativity can now be given by considering the energy-momentum tensor associated with the electromagnetic field of a charged particle. Starting with the solutions given in [16, 17] one finds for the (retarded) Maxwell bivector

$$F_{ab} = r^{-1}F_1 l_{[a}x_{b]} + r^{-2}F_2 l_{[a}m_{b]}, \tag{4.2}$$

where r is the spatial distance between the field point and the retarded point in the Lorentz frame in which the charge is at rest at the retarded time, l^a is the null propagation vector and x^a and m^a are appropriately defined unit spacelike and null vectors respectively with $l^a x_a = m^a x_a = 0$. The quantities F_1 and F_2 depend on l^a and the particle's charge and 4-acceleration. The corresponding energy-momentum tensor then turns out to be

$$T_{ab} = r^{-2}T_1 l_a l_b + r^{-3}T_{2(}l_a x_{b)} + r^{-4}T_3(2l_{(a}m_{b)} - g_{ab}), \tag{4.3}$$

where T_1, T_2 and T_3 depend on l^a and the particle's charge and 4-acceleration. In (4.2), one recognises the peeling property of the Maxwell bivector, the two terms being null and non-null bivectors respectively. In (4.3), one sees the peeling off of the energy-momentum tensor, the terms being respectively of type $\{(2, 1, 1)\}$ with zero eigenvalue (radiation term), type $\{(3, 1)\}$ with zero eigenvalue and type $\{(1, 1)(1, 1)\}$ with equal and opposite eigenvalues (Coulomb term) (c.f. [18]).

(v) Suppose now that a space time admits a group of motions. The Lie derivative (denoted by $\mathscr{L}_\xi$, where ξ is the appropriate Killing vector) of $\overset{+}{E}_{abcd}$ is zero by Einstein's equations and this puts restrictions on the bivectors in terms of which $\overset{+}{E}_{abcd}$ is expressed in its canonical decomposition and hence restrictions on the physical characteristics of the field (for example the Maxwell bivector in an electromagnetic field or the fluid flow vector in a fluid field). For example, in a source-free Maxwell field (null or non-null) one has $\overset{+}{E}_{abcd} = C\overset{+}{F}_{ab}\overset{+}{F}_{cd}$, where F_{ab} is the Maxwell bivector, $\overset{+}{F}_{ab} = F_{ab} + i\overset{*}{F}_{ab}$ and C is a constant. Taking the Lie derivative of this equation and performing an obvious contraction gives $\mathscr{L}_\xi \overset{+}{F}_{ab} = \alpha \overset{+}{F}_{ab}$ $(\alpha \in \mathbf{C})$. A second similar contraction gives $\alpha = i\beta$ $(\beta \in \mathbf{R})$ and Maxwell's equations $\overset{+}{F}{}^{b}_{a;b} = 0$ then imply that β is a constant for a non-null field and that $\beta_{,a}$ is proportional to the repeated principal null direction of F_{ab} for a null field. This gives a simple derivation of the results first given by Woolley [19] in the non-null case and by Coll [20] in the null case (see also [21, 22]).

(vi) The canonical expression of the tensor E_{abcd} in terms of bivectors together with the similar Petrov decomposition for C_{abcd} enables one in principle to calculate the infinitesimal holonomy group of a space-time in a similar fashion to that for vacuum space-times [23, 24].

(vii) In this final section, an alternative approach to the classification problem is briefly discussed. This approach was first given by Penrose [18] and was considered further by Cormack and Hall [25] where it was shown to be essentially equivalent to one given by Ludwig and Scanlon [26]. Consider the set $P^3(\mathbf{C})$ of all non-zero complex directions at p. The trace-free Ricci tensor and the metric tensor define two quadric surfaces $\tilde{R}_{ab}x^a x^b = 0$ and $g_{ab}x^a x^b = 0$ in $P^3(\mathbf{C})$ where the x^a are homogeneous coordinates. The intersection of these quadrics is a *quartic curve* μ which is representative of the algebraic structure of $\tilde{R}_{ab}$. One then classifies $\tilde{R}_{ab}$ by classifying μ according to (a) whether μ is irreducible or decomposes into irreducible curves of lower order and the exact nature of such a decomposition, (b) the multiple point structure of μ, (c) the number and nature of the real points in which μ intersects the quadric $g_{ab}x^a x^b = 0$. With regard to the apparent complexity of the criteria (a), (b) and (c), the following remarks are reassuring. Firstly, the possibility of a twisted cubic component in (a) is ruled out by the fact that the two quadrics considered are simulateneously real. Secondly, one only needs to consider the real multiple points in (b) since the complex multiple points do not refine the classification. Finally, one only needs to consider the number of one dimensional real parts in which μ intersects $g_{ab}x^a x^b = 0$, since discrete intersections again do not refine the classification. The relationship between the resulting classification scheme and that arising from the Segré type and the Ludwig–Scanlon approaches is given in detail in [25].

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