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Archivum Mathematicum, Vol. 24 (1988), No. 3, 147--155

Persistent URL: `http://dml.cz/dmlcz/107321`

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 24, No. 3 (1988), 147–156

$\mathscr{W}$-COMPLETENESS AND FIXPOINT PROPERTIES

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(Received May 23, 1986)

Abstract. We present several large classes $\mathscr{F}$ of functions such that a partially ordered set P is $\mathscr{W}$-complete, i.e. each well-ordered subset of P has a supremum, if and only if each selfmap of P belonging to $\mathscr{F}$ has a fixpoint. The classes $\mathscr{F}$ will include all isotone maps and in some cases also all extensive maps. Furthermore, we discuss the problem of the existence of minimal or maximal fixpoints, respectively.

Key words. Poset, $\mathscr{W}$-complete, extensive, isotone, R-monotone, fixpoint.

MS Classification. 06 A 10, 54 H 25.

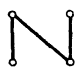
Throughout this paper, P denotes a partially ordered set (*poset*) and $\leqq$ its partial order relation. We call P *$\mathscr{W}_0$-complete* if every nonempty well-ordered subset of P has a supremum. If the empty set has a supremum, too (i.e., P has a least element) then we say P is *$\mathscr{W}$-complete.* Under the assumption of the Axiom of Choice (AC), $\mathscr{W}_0$-completeness is equivalent to the existence of suprema for all nonempty chains and even for all directed subsets. A recent elementary proof of this well-known fact, avoiding any transfinite tools except the Maximal Principle, can be found in [4] (cf. [10]). In the present context, we shall make use of AC only when we are proving the *necessity of $\mathscr{W}$-* (resp. $\mathscr{W}_0$-)completeness for certain fixpoint properties, and in one application concerning *maximal* fixpoints.

The study of fixpoints for certain classes of maps in partially ordered sets is a vital theme in order theory. Since the discovery of the fixpoint theorem for isotone maps in complete lattices, due to Knaster and Tarski (cf. [11]), many authors have found more and more relationships between the existence of fixpoints on one hand and certain completeness properties of the underlying posets on the other hand (see, e.g. [1, 3, 7, 8, 9, 10, 12, 13]). While most of the results in this direction deal with *isotone* (i.e. monotone increasing) maps, there is one basic fixpoint theorem for *extensive* maps (i.e. maps $f: P \to P$ satisfying $x \leqq f(x)$ for all $x \in P$). Its origins go back to Zermelo's celebrated second proof of the Well-Ordering Theorem [14], but an explicit formulation for the general poset setting apparently does not occur in the literature before Bourbaki's 1949 note "Sur le théorème de Zorn" [2] where it is shown (without AC!) that *every extensive selfmap of a $\mathscr{W}$-complete poset has a fixpoint*. Observing that every isotone function $f: P \to P$ maps the set of all $x \in P$ with $x \leqq f(x)$ into itself, one concludes immediately that also *every isotone selfmap of a $\mathscr{W}$-complete poset has a fixpoint* (for a different proof of this fact, see [1]). In the following stronger result due to Markowsky [10] no application of AC is needed either.

Proposition 1. *A poset P is $\mathscr{W}$-complete if and only if every isotone selfmap of P has a least fixpoint.*

Of course, the fixpoint property alone, i.e. the existence of fixpoints for all isotone selfmaps of P, is not sufficient for $\mathscr{W}$-completeness of P. The simplest poset which has the fixpoint property but is neither $\mathscr{W}$-complete nor dually $\mathscr{W}$-complete (missing a least and a greatest element) is the letter N.

DIAGRAM 1



The characterization of all posets with the fixpoint property for isotone maps is still an unsolved problem – even in the finite case. The goal of the present note is slightly different, namely: to give characterizations of $\mathscr{W}$-completeness by the fixpoint property for certain classes of selfmaps including more than only isotone ones. Fixpoint theorems for non-isotone functions play an important rôle not only in analysis and topology, but they are also of interest in computer science, e.g. in logic programming and in deductive data bases, as was pointed out recently by I. Guessarian [6].

Given an arbitrary selfmap f of P, a subset X of P is said to be *f-invariant* if $x \in X$ implies $f(x) \in X$, and X is called *sup-closed* if for every nonempty subset of X possessing a supremum in P, this supremum belongs to X. For each $a \in P$, we denote by $W_f(a)$ the intersection of all f-invariant and sup-closed subsets of P containing a. It is evident that $W_f(a)$ is again f-invariant and sup-closed. Now consider the set

$$E_f = \{x \in P\text{: For all } y \in P,\ x \leqq y \text{ implies } x \leqq f(y)\},$$

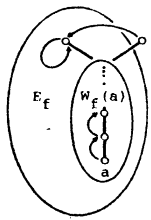
consisting of all elements which generate an f-invariant principal filter. Of course, the restriction of f to E_f is always extensive, and one has $E_f = P$ iff f is extensive on the whole poset P. The crucial observation on E_f is:

Proposition 2. *Let f be any selfmap of P such that E_f is f-invariant. Then for each $a \in E_f$, $W_f(a)$ is a well-ordered subset of E_f with least element a. Furthermore, each* *non-maximal element x of $W_f(a)$ is covered in $W_f(a)$ by its image $f(x)$. If $W_f(a)$ has a supremum then this is the greatest element of $W_f(a)$ and a fixpoint of f.*

Proof. Clearly E_f is sup-closed and, by hypothesis, also f-invariant, whence $W_f(a) \subseteq E_f$ whenever $a \in E_f$. In particular, the restriction of f to $W_f(a)$ is extensive, and for the remaining statements, one may proceed as in Bourbaki's proof (see [2]; cf. also [1] and [5]). □

The reduction from P to E_f is a bit delicate because it may happen that $W_f(a)$ fails to be sup-closed in E_f (considered as a poset with respect to the induced order), as Diagram 2 demonstrates.

DIAGRAM 2



But a careful analysis of the proof in [2] shows that the property $x \leqq f(x)$ is needed only for elements x of $W_f(a)$.

Corollary 1. *Every selfmap f of a $\mathscr{W}_0$-complete poset P such that E_f is nonempty and f-invariant has a fixpoint.*

We are now going to establish some large classes of selfmaps f for which E_f is automatically f-invariant. Certainly the class of all isotone maps has this property; so Corollary 1 includes Theorem 2 in [1]. Notice that $E_f = \{x \in P : x \leqq f(x)\}$ for isotone f. More generally, J. Klimeš [7] has shown recently that every relatively isotone selfmap of a $\mathscr{W}$-complete poset has a fixpoint, where $f : P \to P$ is called *relatively isotone* if

$$x \leqq y,\ x \leqq f(y) \text{ and } f(x) \leqq y \text{ together imply } f(x) \leqq f(y).$$

This fixpoint theorem is also an immediate consequence of Corollary 1, since E_f is obviously f-invariant for every relatively isotone map $f : P \to P$, and the least element of P (which exists by $\mathscr{W}$-completeness) belongs to E_f. This result suggests to generalize the property of isotonicity as follows.

Let R denote any function assigning to each selfmap f of any poset P a relation $R(f)$ on P, and call the map f *R-monotone* if $xR(f)\,y$ implies $f(x) \leqq f(y)$. For any two functions R and R' of this type, $R \succ R'$ means that for all posets P, all selfmaps $f : P \to P$ and all $x, y \in P$, $xR'(f)\,y$ implies $xR(f)\,y$. In this case every R-monotone map is also R'-monotone. Here are some examples we shall encounter in the sequel:

$$xR_0(f)\,y \Leftrightarrow x \leqq y$$
$$xR_1(f)\,y \Leftrightarrow x \leqq y \text{ and } x \leqq f(x)$$
$$xR_2(f)\,y \Leftrightarrow x \leqq f(x) \leqq y$$
$$xR_3(f)\,y \Leftrightarrow x \leqq f(x) \leqq y \text{ and } x \leqq f(y)$$
$$xR_4(f)\,y \Leftrightarrow x \in E_f \text{ and } f(x) \leqq y$$
$$xR_i(f)\,y \Leftrightarrow x \leqq y, f(x) \leqq y \text{ and } x \leqq f(y).$$

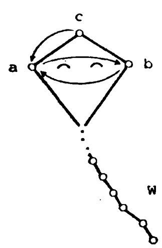
Of course, "R_0-monotone" means "isotone" and "R_i-monotone" means "relatively isotone". An extensive map is always R_2-monotone but not necessarily R_1-monotone. Furthermore, we observe that f is R_4-monotone iff the set E_f is f-invariant. The relationships

$$R_0 \succ R_1 \succ R_2 \succ R_3 \succ R_4 \qquad \text{and} \qquad R_0 \succ R_i \succ R_3$$

are clear by definition.

The next notion is a bit technical but helpful for the study of fixpoint properties. By a *kite* of a poset P we mean a subset $W \cup \{a, b, c\}$ such that W is well-ordered (by the induced order), a and b are distinct minimal upper bounds of W, and c is an upper bound of $\{a, b\}$. Notice that a kite fails to be $\mathscr{W}$-complete although each of its isotone selfmaps has a fixpoint (the *dual* of a kite is $\mathscr{W}$-complete). Given any map $f: P \to P$, we say an element $a \in P$ is a *disturbation point* for f if there is a kite $W \cup \{a, b, c\}$ such that $f(a) = b$ and $f(c) = f(b) = a$.

DIAGRAM 3



Now we define the function R_ε by

$xR_\varepsilon(f)\,y \Leftrightarrow a \neq x \leqq y$ if there is exactly one disturbation point a for f

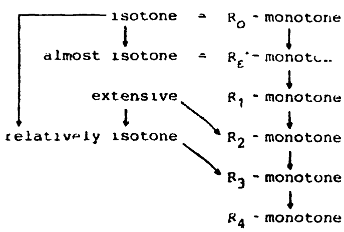
$xR_\varepsilon(f)\,y \Leftrightarrow x \leqq y$ otherwise.

We observe at once that

$$R_0 \succ R_\varepsilon \succ R_1.$$

In fact, if $x \leqq f(x)$ then x cannot be a disturbation point. An R_ε-monotone map might be called *almost isotone*, because such a map f is either isotone (and has therefore no disturbation point), or f has exactly one disturbation point a and for all x distinct from a, $x \leqq y$ implies $f(x) \leqq f(y)$. However, an almost isotone map need not be relatively isotone, as the simple (and typical) example in Diagram 3 shows. Summarizing the previous remarks, we obtain a hierarchy of R-monotonicity properties, as depicted in Diagram 4.

DIAGRAM 4



Perhaps almost isotone maps are only a curiosity, but – in contrast to the class of isotone maps – the fixpoint property for the class of almost isotone maps characterizes $\mathscr{W}$-completeness! In order to prove this fact, we need a straightforward but helpful

Lemma 1. *Let W be any well-ordered subset of P, and let $W^\dagger$ denote the set of all upper bounds for W. Then every selfmap g of $W^\dagger$ extends to a selfmap f of P with the following properties:*

(i) *$x \leqq y$ implies $f(x) \leqq f(y)$ or $\{x, y\} \subseteq W^\dagger$.*
In particular, if g is (almost) isotone then so is f.

(ii) *$x \in P \setminus W^\dagger$ implies $f(x) \nleqq x$.*
In particular, if g is fixpoint free then so is f.

In fact, the map f with $f(x) = g(x)$ for $x \in W^\dagger$ and $f(x) = \min \{w \in W : w \nleqq x\}$ for $x \in P \setminus W^\dagger$ has the required properties. Notice that the "if" part in Proposition 1 is an immediate consequence of Lemma 1: If W fails to have a least upper bound then the identity on $W^\dagger$ has an isotone extension without least fixpoint.

Lemma 2. *A poset P does not contain any kite iff every (fixpoint free) almost isotone selfmap of P is already isotone.*

Proof. By definition, in a poset without kites any almost isotone map must be isotone. Conversely, if $W \cup \{a, b, c\}$ is a kite then we may define a map $g : W^\dagger \to W^\dagger$ by setting $g(a) = b$ and $g(x) = a$ for $x \in W^\dagger \{a\}$. On account of Lemma 1, g extends to a fixpoint free map $f : P \to P$ such that a is a perturbation point of f and for $a = x < y$ we have either $f(x) \leqq f(y)$ or $\{x, y\} \subseteq W^\dagger$, whence $f(x) = a = f(y)$. Thus f is almost isotone but not isotone. □

Now we are ready for the main result of this note:

Proposition 3. *Let $\mathscr{F}$ be any set of R_4-monotone selfmaps of a poset P such that every almost isotone selfmap of P belongs to $\mathscr{F}$. Then P is $\mathscr{W}$-complete iff each $f \in \mathscr{F}$ has a fixpoint.*

Proof. By Proposition 2 every R_4-monotone selfmap f of a $\mathscr{W}$-complete poset P has a fixpoint, because E_f is f-invariant and contains the least element of P.

Conversely, assume P is a poset containing a well-ordered subset W without supremum, and let M denote the set of all minimal upper bounds of W.

Case 1. There exists some $x \in W^\dagger$ with $m \nleq x$ for all $m \in M$. Then AC gives a dually well-ordered subset V of $W^\dagger$ which has no lower bound in $W^\dagger$, and the dual of Lemma 1 (applied to V instead of W) provides a fixpoint free isotone selfmap of $W^\dagger$ which then, by Lemma 1, extends to a fixpoint free isotone selfmap of the whole poset P (cf. [8, 9, 12, 13]).

Case 2. For each $x \in W^\dagger$ there is exactly one $m_x \in M$ with $m_x \leqq x$. Then, as M has at least two elements (otherwise W would possess a supremum), we may take any fixpoint free selfmap h of M and define a fixpoint free isotone selfmap g of $W^\dagger$ by setting $g(x) = h(m_x)$; again, Lemma 1 applies.

Case 3. There exist different $a, b \in M$ with a common upper bound c. Then $W \cup \{a, b, c\}$ is a kite, and by Lemma 2, there exists an almost isotone but fixpoint free selfmap of P.

Hence, in all three cases, we find an (almost isotone) $f \in \mathscr{F}$ which has no fixpoint. □

Now a look at Diagram 4 immediately yields:

Corollary 2. *Let $k \in \{1, 2, 3, 4\}$. Then a poset P is $\mathscr{W}$-complete iff every R_k-monotone selfmap of P has a fixpoint.*

Observe that the existence of fixpoints for all relatively isotone selfmaps of P is necessary but *not* sufficient for $\mathscr{W}$-completeness (take any poset P whose dual is $\mathscr{W}$-complete while P itself is not; e.g. a kite).

If a well-ordered set W without supremum has a least element w_0 then the above proof provides a fixpoint free almost isotone, hence R_4-monotone selfmap f of P with $w_0 \leqq f(x)$ for all $x \in P$; in particular, w_0 belongs to E_f. Thus, assuming AC, we arrive at the following improvement of Corollary 1:

Corollary 3. *A poset P is $\mathscr{W}_0$-complete iff every selfmap f of P with $E_f \neq \emptyset$ and $f[E_f] \subseteq E_f$ has a fixpoint.*

Of course, the fixpoint criterion of Proposition 3 becomes particularly convenient if every almost isotone map is already isotone; in other words, if the poset in question does not contain any kite. Important examples of such posets are all *forests*, i.e. posets in which no two incomparable elements have a common upper bound, and all *meet-semilattices*; more generally, all posets such that all well-ordered subsets have a down-directed or empty set of upper bounds (cf. [8, 9, 12]).

Corollary 4. *A poset P is $\mathscr{W}$-complete iff it contains no kite and every isotone selfmap of P has a fixpoint. In particular, a forest, respectively, a meet-semilattice P is $\mathscr{W}$-complete iff every isotone selfmap of P has a fixpoint.*

Let us have a quick look at *chains*, i.e. totally ordered sets. Of course, a chain is $\mathscr{W}$-complete iff it is a complete lattice. A straightforward comparison of the involved relations shows that for a selfmap f of a chain P, the following conditions are equivalent:

(a) f is relatively isotone.

(b) f is R_3-monotone.

(c) There is no pair of elements $x, y \in P$ such that $x \leqq f(y) < f(x) \leqq y$.

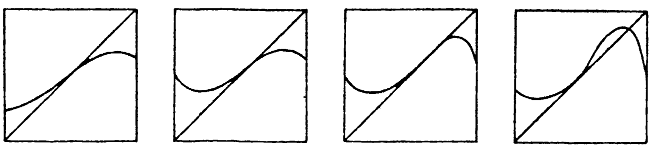
Corollary 5. *A chain P is complete iff every selfmap of P satisfying*

$$(x \leqq f(y) \text{ and } f(x) \leqq y) \Rightarrow f(x) \leqq f(y)$$

has a fixpoint.

Some illustrative examples of R-monotone functions on the real unit interval are given in Diagram 5.

DIAGRAM 5



R_1 - monotone, not R_0 - monotone | R_2 - monotone, not R_1 - monotone | R_3 - monotone, not R_2 - monotone | R_4 - monotone, not R_3 - monotone

We have seen that every R_3-monotone selfmap and, in particular, every relatively isotone selfmap of a $\mathscr{W}$-complete poset has a fixpoint. However, the real function $f : [0, 1] \to [0, 1]$ with $f(0) = f(1) = \frac{1}{2}$ and $f(x) = x$ otherwise shows that neither a minimal nor a maximal fixpoint is guaranteed by this existence criterion. But a bit more can be said about R_1- and R_2-monotone maps, respectively.

Proposition 4. *Let P be a $\mathscr{W}$-complete poset, f any selfmap of P, and F the set of all fixpoints of f.*

(1) *If f is R_1-monotone then each subset of F possessing a supremum in P has also a supremum in F (but the two suprema may be distinct). In particular, F is $\mathscr{W}$-complete and f has a least fixpoint.*

(2) *If f is R_2-monotone then each subset of F possessing a supremum in P has an*

upper bound in F. In particular, each well-ordered subset of F is upper bounded in F.

Proof. We know that for each R_2-monotone map $f: P \to P$ the set E_f is f-invariant and sup-closed, and f restricted to E_f is extensive. Now let Z be a subset of F possessing a supremum x_0 in P. Then $x_0 \leqq y$ implies $z = f(z) \leqq y$ for all $z \in Z$, and if f is R_2-monotone, this gives $z = f(z) \leqq f(y)$ for all $z \in Z$, i.e. $x_0 \leqq f(y)$. Thus x_0 belongs to E_f, and f induces an extensive selfmap of the $\mathscr{W}$-complete subset $G = \{x \in E_f : x_0 \leqq x\}$. Hence f has a fixpoint in G, and this is an upper bound of Z. If, moreover, f is assumed to be R_1-monotone, then the restriction of f to G is even isotone and has therefore a *least* fixpoint (see Proposition 1). But this fixpoint must be the least upper bound of Z in F, since E_f contains all fixpoints of the map f. □

Assuming AC, we derive from (2):

Corollary 6. *Every R_2-monotone selfmap of a $\mathscr{W}$-complete poset has a maximal (but in general no minimal) fixpoint.*

For example, the map $f: [0, 1] \to [0, 1]$ with $f(0) = 1$ and $f(x) = x$ for $x \neq 0$ is extensive, hence R_2-monotone, but has no minimal fixpoint. Easy finite examples show that even in complete lattices R_2-monotone maps need not possess greatest fixpoints.

Part (1) in Proposition 4 generalizes Theorem 9 in [10] (see also [9]). Furthermore, it provides some variants of Proposition 1 and of the Tarski–Davis fixpoint theorem for complete lattices [3, 10].

Corollary 7. *In Proposition* 1, *"isotone" may be replaced with "R_1-monotone" (but not with "R_2-monotone", nor with "relatively isotone").*

Corollary 8. *For any R_1-monotone selfmap f of a complete lattice, the fixpoints of f form a complete lattice, too.*

Corollary 9. *A lattice L is complete iff every R_1- (resp. R_2-, R_3-, R_4-) monotone selfmap of L has a fixpoint.*

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