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### SHORT PROOFS OF SOME INEQUALITIES OF HORST ALZER

#### MALCOLM T. MCGREGOR

ABSTRACT. We provide elementary proofs of some inequalities of Horst Alzer.

Let the real numbers  $a_i$  be such that  $0 < a_i \leq \frac{1}{2}$  for all  $i = 1, \ldots, n$ , and let

$$\begin{split} A_n &= \frac{1}{n} \sum_{i=1}^n a_i, \qquad G_n = \prod_{i=1}^n a_i^{1/n}, \qquad H_n = n/\sum_{i=1}^n 1/a_i \\ A'_n &= \frac{1}{n} \sum_{i=1}^n (1-a_i), \quad G'_n = \prod_{i=1}^n (1-a_i)^{1/n}, \quad H'_n = n/\sum_{i=1}^n 1/(1-a_i) \,. \end{split}$$

In [1] Horst Alzer establishes the inequalities

(1) 
$$1/H'_n - 1/H_n \le 1/G'_n - 1/G_n \le 1/A'_n - 1/A_n,$$

where the sign of equality holds if and only if  $a_1 = \cdots = a_n$ . Furthermore, in [1] he refers to the inequalities

(2) 
$$H_n/H'_n \le G_n/G'_n \le A_n/A'_n$$

discovered by W.-L. Wang and P.-F. Wang, and Ky Fan. (See [3] and [2, p. 5].)

In this note we use the classical arithmetic, geometric, harmonic mean inequalities

$$(3) A_n \ge G_n \ge H_n$$

together with (2) to establish the inequalities (1). We begin by showing that

(4) 
$$\frac{1}{H_n} \left( 1 - \frac{H_n}{H'_n} \right) \ge \frac{1}{G_n} \left( 1 - \frac{G_n}{G'_n} \right)$$

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which is equivalent to the left hand inequality in (1). Clearly, since  $0 < a_i \leq \frac{1}{2}$  for all  $i = 1, \ldots, n$  we have  $H'_n \geq H_n > 0$  and  $G'_n \geq G_n > 0$ , and using (2) we deduce that

(5) 
$$1 - \frac{H_n}{H'_n} \ge 1 - \frac{G_n}{G'_n} \ge 0$$
.

In addition, the right hand inequality in (3) yields

(6) 
$$\frac{1}{H_n} \ge \frac{1}{G_n} > 0 ,$$

and forming the products of corresponding sides in the inequalities (5) and (6) readily produces (4). The right hand inequality in (1) is proved in an analogous way.

**Remark.** It is worth noting that the above argument may be used to show that if p > 0 then

$$(1/H'_n)^p - (1/H_n)^p \le (1/G'_n)^p - (1/G_n)^p \le (1/A'_n)^p - (1/A_n)^p$$

#### References

- Alzer, H., Inequalities for arithmetic, geometric and harmonic means, Bull. London Math. Soc. 22 (1990), 362-366.
- [2] Beckenbach, E. F., Bellman, R., Inequalities, Springer, Berlin, 1961.
- [3] Wang, W.-L., Wang, P.-F., A class of inequalities for the symmetric functions (Chinese), Acta Math. Sinica 27 (1984), 485-497.

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