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ON CORINGS AND COMODULES

HANS–E. PORST

To my friend and colleague Jiří Rosický on his 60th birthday

ABSTRACT. It is shown that the categories of R -coalgebras for a commutative unital ring R and the category of A -corings for some R -algebra A as well as their respective categories of comodules are locally presentable.

INTRODUCTION

The categories under consideration are defined as the categories of comonoids and comonoid-coactions in certain monoidally closed categories as follows:

- a) Given a commutative unital ring R
 - the category \mathbf{Coalg}_R of R -coalgebras is the category of comonoids in $(\mathbf{Mod}_R, - \otimes_R -, R)$,
 - the categories \mathbf{Comod}_A and ${}_A\mathbf{Comod}$ of right resp. left A -comodules for an R -coalgebra A are the corresponding categories of right resp. left A -coactions on R -modules.
- b) Given an R -algebra A ,
 - the category \mathbf{Coring}_A of A -corings is the category of comonoids in $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$, where ${}_A\mathbf{Mod}_A$ denotes the category of A , A -bimodules,
 - the categories $\mathbf{Comod}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathbf{Comod}$ of right resp. left \mathcal{C} -comodules for an A -coring \mathcal{C} again are the respective categories of \mathcal{C} -coactions on left (right) A -modules.

Only scattered results are known about the structure of these categories: co-completeness of these categories is a rather trivial fact (see Fact 2 below), cocommutative coalgebras form a cartesian closed category ([3]), \mathbf{Comod}_A is locally presentable and comonadic over \mathbf{Mod}_R ([11]). A first systematic approach to completeness — limited, however, to the case where the rings involved are regular — was presented in [8] using the dualized construction of colimits in varieties.

In this note we will offer a unified approach to these and many new results by considering the categories in question as subcategories of certain categories of

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functor-coalgebras $\mathbf{Coalg}F$; using methods from the theory of accessible categories (see [2], [8]) we will show first that these categories are complete and then, in a second step, that this also holds for their interesting subcategories \mathbf{Coalg}_R , \mathbf{Coring}_A , and \mathbf{Comod}_A . In fact we will prove even more: all categories mentioned so far are locally presentable categories.

Local presentability of \mathbf{Coalg}_R generalizes Sweedler’s so-called *Fundamental Theorem of Coalgebras* (see [10], [5]), which states that every coalgebra (over some field k) is a directed colimit of coalgebras whose underlying vector space is finite dimensional, hence of finitely presentable coalgebras, since the following is easy to prove:

Proposition 1. *A k -coalgebra is finitely presentable iff its underlying k -vector space is of finite dimension.*

Note, however, that neither this proposition nor Sweedler’s prove generalize to arbitrary rings.

1. THE CATEGORIES $\mathbf{Coalg}T_I$ AND $\mathbf{Coalg}M_A$

Let (\mathbf{C}, \otimes, I) be any of the monoidally closed categories $(\mathbf{Mod}_R, - \otimes_R -, R)$ or $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$ mentioned in the introduction. We consider the following functors:

$$T_n: \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & \otimes^n C \end{array}$$

$$T_I: \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & T_2 C \times I = (C \otimes C) \times I \end{array}$$

$${}_A M: \begin{array}{ccc} \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ C & \longmapsto & A \otimes C \end{array} \qquad M_A: \begin{array}{ccc} \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ C & \longmapsto & C \otimes A \end{array}$$

where A is a monoid in (\mathbf{C}, \otimes, I) and $\mathbf{C}' = \mathbf{C}$ in the commutative case and, for A non-commutative, $\mathbf{C}' = A\text{-Mod}$ and $\mathbf{Mod}\text{-}A$, the categories of left and right A -modules respectively, with $- \otimes -$ the obvious bifunctor $\mathbf{C} \times \mathbf{C}' \longrightarrow \mathbf{C}'$.

Then \mathbf{Coalg}_R and \mathbf{Coring}_A , respectively, are the full subcategories of $\mathbf{Coalg}T_I$ (w.r.t. the appropriately chosen (\mathbf{C}, \otimes, I) — see above) spanned by those T_I -coalgebras $\mathbf{C} = (C, C \xrightarrow{(m,e)} (C \otimes C) \times I)$ which make the following diagrams commute

$$(1) \quad \begin{array}{ccc} C & \xrightarrow{m} & C \otimes C \\ \downarrow m & & \downarrow m \otimes \text{id}_C \\ C \otimes C & \xrightarrow{\text{id}_C \otimes m} & C \otimes C \otimes C \end{array}$$

$$(2) \quad \begin{array}{ccc} C & & \\ m \downarrow & \searrow r_C & \\ C \otimes C & \xrightarrow{\text{id}_C \otimes e} & C \otimes I \end{array} \quad \text{and} \quad (3) \quad \begin{array}{ccc} C & \xrightarrow{m} & C \otimes C \\ \downarrow l_C & & \downarrow e \otimes \text{id}_C \\ I & & C \end{array}$$

Similarly, the various categories of comodules are subcategories of $\mathbf{Coalg}M_A$ and $\mathbf{Coalg}_A M$, respectively, defined by the obvious diagrammatic axioms.

We will need the following results, which are easy to prove (see [1], [8]).

Fact 2.

1. For each functor $F: \mathbf{C} \rightarrow \mathbf{C}$ the category $\mathbf{Coalg}F$ is cocomplete provided the category \mathbf{C} is so.
2. The categories of comonoids and comonoid-coactions are closed in their respective functor-categories under colimits.

Clearly M_A is accessible since it is a left adjoint. Also, if F is an accessible endofunctor on a category \mathbf{A} with biproducts, then $F \times A = F + A$ is accessible for each object A in \mathbf{A} . Thus, T_I is accessible by the following fact:

Lemma 3. Let $(\mathbf{C}, - \otimes -, I)$ be a monoidally closed category and $F: \mathbf{C} \rightarrow \mathbf{C}$ a finitary functor. Then $\hat{F}: \mathbf{C} \rightarrow \mathbf{C}$ with $\hat{F}(C) = C \otimes FC$ is finitary. In particular, the functor $T_n: \mathbf{C} \rightarrow \mathbf{C}$ with $T_n C = \otimes^n C$ preserves directed limits.

Proof. If $D: \mathbf{I} = (I, \leq) \rightarrow \mathbf{C}$ is a directed diagram in \mathbf{C} with colimit $D_i \xrightarrow{d_i} C$ the colimit of the diagram $\hat{D}: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{C}$ with $\hat{D}(i, j) = D_i \otimes FD_j$ can be computed as $D_i \otimes FD_j \xrightarrow{d_i \otimes Fd_j} C \otimes C$ since F and each $X \otimes -$ and $- \otimes Y$ preserve (directed) colimits. Finally, the diagram $\hat{F} \circ D$ is a cofinal subdiagram of \hat{D} . \square

Remark 4. Since the monoidal categories under consideration are varieties also T_I preserves directed colimits (see also [8]). As a consequence of these observations we obtain that the underlying functors $|-|$ of $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ into \mathbf{C} and \mathbf{C}' , respectively, have right adjoints and thus are comonadic (see [1]); their domains are also accessible by the following observation.

Recall that for functors $F, G: \mathbf{K} \rightarrow \mathbf{L}$ the inserter of F and G is the full subcategory $\mathbf{Ins}(F, G)$ of the comma category $F \downarrow G$ spanned by all arrows $FK \rightarrow GK$ ([2, 2.71]). Since $\mathbf{Coalg}F = \mathbf{Ins}(\text{id}_{\mathbf{C}}, F)$ it follows from [2, 2.72] and the remark above that the categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ are accessible. Since any co-complete accessible category is locally presentable, we obtain

Proposition 5. The categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ are locally presentable.

Remark 6. There is no reason to assume that limits in these categories are respected by their obvious underlying functors $|-|$ into \mathbf{Mod}_R . Consult [8] or [11] for how to possibly describe these limits.

2. THE CATEGORIES \mathbf{Coalg}_R AND \mathbf{Comod}_A

The defining axioms for R -coalgebras, i.e., the commutativity of the diagrams (1), (2), and (3) above can be interpreted as follows:

Denote by φ and ψ the natural transformations

$$\begin{aligned} \varphi & : \quad | - | \longrightarrow T_3 \circ | - | \\ \varphi_{\mathbb{C}} & = \quad C \xrightarrow{m} C \otimes C \xrightarrow{m \otimes \text{id}_C} C \otimes C \otimes C \end{aligned}$$

and

$$\begin{aligned} \psi & : \quad | - | \longrightarrow T_3 \circ | - | \\ \psi_{\mathbb{C}} & = \quad C \xrightarrow{m} C \otimes C \xrightarrow{\text{id}_C \otimes m} C \otimes C \otimes C \end{aligned}$$

(Naturality of φ and ψ is a consequence of functoriality of $- \otimes -$ and the definition of coalgebra homomorphism.)

Lemma 7. $\mathbb{C} = (C, \langle m, e \rangle)$ satisfies (1) iff $\varphi_{\mathbb{C}} = \psi_{\mathbb{C}}$.

Similarly,

$$\begin{aligned} \varrho & : \quad | - | \longrightarrow | - | \otimes R \\ \varrho_{\mathbb{C}} & = \quad C \xrightarrow{m} C \otimes C \xrightarrow{\text{id}_C \otimes e} C \otimes R \end{aligned}$$

and

$$\begin{aligned} \lambda & : \quad | - | \longrightarrow R \otimes | - | \\ \lambda_{\mathbb{C}} & = \quad C \xrightarrow{m} C \otimes C \xrightarrow{e \otimes \text{id}_C} R \otimes C \end{aligned}$$

are natural transformations and the following obviously hold

Lemma 8.

- 1. $\mathbb{C} = (C, \langle m, e \rangle)$ satisfies (2) iff $\varrho_{\mathbb{C}} = r_{| \mathbb{C} |}$.
- 2. $\mathbb{C} = (C, \langle m, e \rangle)$ satisfies (3) iff $\lambda_{\mathbb{C}} = l_{| \mathbb{C} |}$.

Recall now that (see [2, 2.76]), for accessible functors $F^t, G^t: \mathbf{K} \longrightarrow \mathbf{L}_t$ and families of natural transformations $\mu^t, \nu^t: F^t \longrightarrow G^t$ ($t \in T$) the equifier of (μ^t) and (ν^t) is the full subcategory $\mathbf{Eq}(\mu^t, \nu^t)$ of \mathbf{K} spanned by all K in \mathbf{K} with $\mu_K^t = \nu_K^t$ for all $t \in T$, and that this subcategory is accessible.

Theorem 9. *The categories \mathbf{Coalg}_R and \mathbf{Coring}_A are locally presentable categories.*

Proof. By Lemmas 7 and 8 the category of comonoids in \mathbb{C} is the equifier of the three pairs $(\varphi, \psi), (\lambda, l_{| - |}), (\varrho, r_{| - |})$ of natural transformations. Since all categories and functors under consideration are accessible, it is accessible as well. Moreover the categories under consideration are closed under colimits in their respective \mathbf{Coalg}_{T_I} by Fact 2 and hence cocomplete. Now the same argument used in the proof of Proposition 5 gives the result. □

In a completely analogous way one obtains

Theorem 10. *The categories $\mathbf{Comod}_A, {}_A\mathbf{Comod}, \mathbf{Comod}_C$ and ${}_C\mathbf{Comod}$ are locally presentable categories and therefore have all limits.*

We now get, as simple corollaries,

Proposition 11.

1. \mathbf{Coalg}_I is coreflective in $\mathbf{Coalg}T_I$.
2. \mathbf{Comod}_A is coreflective in $\mathbf{Coalg}M_A$.

Proof. The proof is the same in both cases: the embedding of the respective subcategory preserves colimits and both subcategories, being locally presentable, are co-wellpowered and have a generator. Now apply the (dual of the) Special Adjoint Functor Theorem. □

Theorem 12.

1. \mathbf{Coalg}_R is comonadic over \mathbf{Mod}_R ,
2. \mathbf{Coring}_A is comonadic over ${}_A\mathbf{Mod}_A$,
3. \mathbf{Comod}_A is comonadic over \mathbf{Mod}_R , and
4. \mathbf{Comod}_C is comonadic over $\mathbf{Mod}\text{-}A$.

Proof. The respective underlying functors have right adjoints by Remark 4 and Proposition 11. They also create split equalizers by Remark 6 and because all of these categories are closed in their respective categories of functor-coalgebras w.r.t. subobjects carried by split monos (see [8] or Fact 17 below). □

Remark 13. The existence of cofree comodules certainly can be obtained directly. The argument given in [6] generalizes to our somewhat more general situation. See also [11]. Note also that the existence of a cofree coalgebra is known (see [3] for the cocommutative case with an argument similar or ours, and [10] with an explicit construction via the tensor algebra for the case of a field, which however generalizes to rings).

Generalizing a result in [1] we might reformulate the statement of the last theorem as follows

Theorem 14. *All categories \mathbf{Coalg}_R , \mathbf{Coring}_A , \mathbf{Comod}_A , and \mathbf{Comod}_C respectively, are covarieties.*

Remark 15. Obviously, the the above results can be extended to the categories of cocommutative coalgebras.

Problem 16. It is not clear that the kernel of a morphism in the categories \mathbf{Coalg}_R or \mathbf{Comod}_A is a subobject in the sense of say [4], i.e., whether its carrier map is injective. It has been shown in [8] that each injective homomorphism is a strong monomorphism, but it is clear that the converse doesn't hold: the categories under consideration, being locally presentable, carry an (epi, strong mono)-factorization structure (see [2]) but don't allow for image-factorizations of morphisms (see [8]). It thus would be interesting to characterize the injective homomorphism in these categories categorically and to describe the strong monomorphisms explicitly. If $\mathbf{Ker} f$ could be shown to be a subcoalgebra of f 's domain, it would be an easy consequence to prove that it is the largest subcoalgebra contained in the \mathbf{Mod}_R -kernel of f . This is the case, if the ring R is regular (see [8]).

3. PURITY

It is easy to see that \mathbf{Coalg}_R , \mathbf{Coring}_A , and \mathbf{Comod}_A are closed in their respective categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$ w.r.t. subobjects whose underlying embedding in \mathbf{Mod}_R splits (see [8]). In fact, the proof of this statement given in [8] shows more:

Given a homomorphism $m: \mathbb{C} \rightarrow \mathbb{D}$ in $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$, respectively, then \mathbb{C} is a comonoid and an A -coaction, respectively, provided that \mathbb{D} is and $m \otimes m \otimes m$ and $m \otimes \text{id}$ are monomorphisms in \mathbb{C} . In the commutative case this clearly holds, provided that m is a pure homomorphism. We thus have

Fact 17. *\mathbf{Coalg}_R and \mathbf{Comod}_A are closed in $\mathbf{Coalg}T_R$ and $\mathbf{Coalg}M_A$, respectively, under subobjects carried by pure R -linear maps.*

The categorical concept of λ -purity (λ a regular cardinal) as presented, e.g., in [2] generalizes the notion of a pure module homomorphism in the sense that an \aleph_0 -pure morphism in \mathbf{Mod}_R is simply a pure homomorphism, provided R is a PID. We do not know whether this fact has appeared in print elsewhere but believe it must be well known: an argument would be a straightforward generalization of the proof given for [9, 61.11], considering finitely generated submodules instead of single generated ones.

Also in the non-commutative case the notion of \aleph_0 -purity can be exploited: Since the functor $C \otimes -$ is left adjoint it preserves (directed) colimits and finitely presentable objects, hence \aleph_0 -pure morphisms by [2, 2.38] which are (regular) monomorphisms. Thus the closure-statement of Fact 17 holds also in the non-commutative case.

Since the underlying functors $\mathbf{Coalg}F \rightarrow \mathbf{C}$ ($F = T_I$ or $F = M_A$) are left adjoints and $\mathbf{Coalg}F$ is a λ -presentable category for some λ they preserve λ -pure morphisms by the same argument as above, so that we can deduce

Proposition 18. *Each of the categories \mathbf{Coalg}_R , \mathbf{Coring}_A , \mathbf{Comod}_A , and \mathbf{Comod}_C is closed in its respective category of functor coalgebras under λ -pure subobjects for a suitable λ .*

Proof. Let $\mathbf{Coalg}F$ be λ -presentable and \mathbb{C} a λ -pure subobject of \mathbb{D} , \mathbb{D} in the subcategory under consideration. Then, in \mathbf{C} , the embedding $C \hookrightarrow D$ is λ -pure, thus \aleph_0 -pure. Now the claim follows from the above observations. \square

Remark 19. The proposition above allows for an alternative proof of Theorem 9 and Theorem 10. Accessibility of our subcategories is in view of [2, 2.36] an immediate consequence of Proposition 18 since they are clearly closed under colimits (see [8]).

Let us finally relate \aleph_0 -purity in ${}_A\mathbf{Mod}_A$ with purity in ${}_A\mathbf{Mod}$ and \mathbf{Mod}_A .

Proposition 20. *If f is an \aleph_0 -pure morphism in ${}_A\mathbf{Mod}_A$, then f is pure in ${}_A\mathbf{Mod}$ and in \mathbf{Mod}_A .*

Proof. By [2, 2.30] f is a directed colimit of split monomorphisms in ${}_A\mathbf{Mod}_A$, hence it is a directed colimit of split monomorphisms in each of the categories \mathbf{Mod}_A and ${}_A\mathbf{Mod}$ as well and therefore pure in both categories again by [2, 2.30]. \square

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