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Bohdan Zelinka

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INFINITE TREE ALGEBRAS

BOHDAN ZELINKA, Liberec

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In [2] L. Nebeský defined a tree algebra.

A tree algebra $A = (M, P)$ is an algebra with the support M and with one ternary operation P which satisfies the following conditions for arbitrary elements u, v, w, x of M:

I. $P(u, u, v) = u$;
II. $P(u, v, w) = P(v, u, w) = P(u, w, v)$;
III. $P(P(u, v, w), v, x) = P(u, v, P(w, v, x))$;
IV. $P(u, v, x) \neq P(v, w, x) \neq P(u, w, x) \Rightarrow P(u, v, x) = P(u, w, x)$.

In the original definition by L. Nebeský it was required that M be finite. Here we omit this condition and deal with infinite tree algebras as well. We consider also infinite trees.

Note that II implies that P is symmetric, i.e. the value of $P(u, v, w)$ does not depend on the ordering of the elements u, v, w.

In [2] it was proved that there is a one-to-one correspondence between finite tree algebras and finite trees, such that every finite tree algebra (M, P) is associated with a tree T whose vertex set is M and for any three vertices u, v, w the vertex $P(u, v, w)$ is the (single) common vertex of the path in T connecting u and v, the path in T connecting u and w and the path in T connecting v and w. For infinite tree algebras this is not true in general, as we shall show.

The fundamental concept of this paper is a bounded segment.

Let (M, P) be a tree algebra, let $u \in M$, $v \in M$. The bounded segment of (M, P) determined by u and v is the set $S(u, v) = \{x \in M \mid P(u, v, x) = x\}$. Instead of "bounded segment" we shall write only "segment", if it does not lead to misunderstanding.

Obviously $\{u, v\} \subseteq S(u, v)$ for any two elements u, v of M.

Proposition 1. *Let (M, P) be a tree algebra, let u, v, w be elements of M such that $w \in S(u, v)$. Then $S(u, w) \cap S(v, w) = \{w\}$, $S(u, w) \cup S(v, w) = S(u, v)$.*

Proof. As $w \in S(u, v)$, we have $P(u, v, w) = w$. As $w \in S(u, w)$, $w \in S(v, w)$, we have $\{w\} \subseteq S(u, w) \cap S(v, w)$. Now let $z \in S(u, w) \cap S(v, w)$. We have $P(u, w, z) = P(v, w, z) = z$. By III now $z = P(u, w, z) = P(u, w, P(v, w, z)) = P(P(u, w, v), w, z) = P(w, w, z) = w$, hence $S(u, w) \cap S(v, w) = \{w\}$. Let $x \in S(u, w)$. Then $P(u, w, x) = x$. By III we have $P(u, v, x) = P(v, u, P(w, u, x)) = P(P(v, u, w), u, x) = P(w, u, x) = x$ and $x \in S(u, v)$. As x is an arbitrary element of $S(u, w)$, we have $S(u, w) \subseteq S(u, v)$. Analogously we can prove $S(v, w) \subseteq S(u, v)$ and therefore $S(u, w) \cup S(v, w) \subseteq S(u, v)$. Suppose that there exists an element y such that $y \in S(u, v)$, $y \notin S(u, w)$, $y \notin S(v, w)$. Then $P(u, w, y) \neq y = P(u, v, y) \neq P(v, w, y)$ and from IV we obtain $P(u, w, y) = P(v, w, y)$. We have $P(u, w, y) \in S(u, w) \cap S(v, w)$, hence $P(u, w, y) = P(v, w, y) = w$. But then $w \in S(u, y) \cap S(v, y)$ and, as $y \in S(u, v)$, we have $S(u, y) \cap S(v, y) = \{y\}$ and $w = y$, which is a contradiction. Hence $S(u, w) \cup S(v, w) = S(u, v)$.

Proposition 2. *Let (M, P) be a tree algebra, let u, v, w be elements of M. Then $S(u, v) \cap S(u, w) \cap S(v, w) = \{P(u, v, w)\}$.*

Proof. Let u, v, w be elements of M, let $z = P(u, v, w)$. We have $P(u, v, z) = P(u, v, P(u, v, w)) = P(P(u, v, u), v, w) = P(u, v, w) = z$ and hence $z \in S(u, v)$. Analogously we can prove $z \in S(u, w)$, $z \in S(v, w)$, therefore $z \in S(u, v) \cap S(u, w) \cap S(v, w)$. By Proposition 1 we have $S(u, v) = S(u, z) \cup S(v, z)$, $S(u, w) = S(u, z) \cup S(w, z)$, $S(v, w) = S(v, z) \cup S(w, z)$, $S(u, z) \cap S(v, z) \cap S(w, z) = \{z\}$. Let $x \in S(u, v) \cap S(u, w) \cap S(v, w)$. As $x \in S(u, v)$, we have either $x \in S(u, z)$ or $x \in S(v, z)$. Suppose $x \in S(u, z)$. As $x \in S(v, w)$, we have $x \in S(v, z) \cup S(w, z)$. Hence $x \in (S(v, z) \cup S(w, z)) \cap S(u, z) = (S(v, z) \cap S(u, z)) \cup (S(w, z) \cap S(u, z)) = \{z\} \cup \{z\} = \{z\}$ and $x = z$. This implies the assertion. If $x \in S(v, z)$, the proof is analogous.

Theorem 1. *Let M be a set, let S be a mapping of $M \times M$ into the set $\mathcal{P}(M)$ of all non-empty subsets of M fulfilling the following conditions for any elements u, v, w, x of M:*

(i) $S(u, v) = S(v, u)$;
(ii) $S(u, v) \cap S(u, w) \cap S(v, w) \neq \emptyset$;
(iii) *if* $z \in S(u, v)$, *then* $S(u, z) \cap S(v, z) = \{z\}$, $S(u, z) \cup S(v, z) = S(u, v)$;
(iv) *if* $S(u, v) = S(w, x)$, *then* $\{u, v\} = \{w, x\}$.

Then there exists a tree algebra (M, P) with the support M such that each $S(u, v)$ is the segment of (M, P) determined by u and v.

Proof. Let u, v, w be elements of M and consider the set $R = S(u, v) \cap S(u, w) \cap S(v, w)$. According to (ii) it is non-empty. Let $x \in R$. Then from (iii) we have $S(u, v) = S(u, x) \cup S(v, x)$, $S(u, w) = S(u, x) \cup S(w, x)$, $S(v, w) = S(v, x) \cup S(w, x)$, $S(u, x) \cap S(v, x) \cap S(w, x) = \{x\}$. Now $R = S(u, v) \cap S(u, w) \cap S(v, w) = (S(u, x) \cup S(v, x)) \cap (S(u, x) \cup S(w, x)) \cap (S(v, x) \cup S(w, x)) = S(u, x) \cap (S(v, x) \cup$

$\cup S(w, x)) = (S(u, x) \cap S(v, x)) \cup (S(u, x) \cap S(w, x)) = \{x\} \cup \{x\} = \{x\}$. Now consider $S(u, u)$ for an arbitrary $u \in M$. If $z \in S(u, u)$, then by (iii) we have $S(u, u) = S(u, z) \cup \cup S(u, z) = S(u, z)$; by (iv) we have $z = u$ and thus $S(u, u) = \{u\}$. For any three elements u, v, w of M put $P(u, v, w) = y$, where y is such an element of M that $\{y\} = S(u, v) \cap S(u, w) \cap S(v, w)$. Then $\{P(u, u, v)\} = S(u, u) \cap S(u, v) \cap S(u, v) = = \{u\}$ and $P(u, u, v) = u$; the condition I holds for (M, P). The condition II follows immediately from (i). Let u, v, w, x be four elements of M and consider $a = = P(P(u, v, w), v, x)$, $b = P(u, v, P(w, v, x))$. Put $y = P(u, v, w)$, $z = P(w, v, x)$. Then $y \in S(v, w)$, $z \in S(v, w)$. This implies either $S(v, w) = S(v, y) \cup S(y, z) \cup \cup S(z, w)$, $S(v, y) \cap S(y, z) = \{y\}$, $S(y, z) \cap S(z, w) = \{z\}$, or $S(v, w) = S(v, z) \cup \cup S(z, y) \cup S(y, w)$, $S(v, z) \cap S(z, y) = \{z\}$, $S(z, y) \cap S(y, w) = \{y\}$. In the first case $S(v, x) = S(v, y) \cup S(y, z) \cup S(z, x)$, hence $a = P(y, v, x) = y$. Further $S(u, v) = = S(u, y) \cup S(y, v)$, $S(u, z) = S(u, y) \cup S(y, z)$, $S(v, z) = S(v, y) \cup S(y, z)$, hence $b = P(u, v, z) = y$ and $a = b$. Analogously in the second case $a = b = z$; the condition III holds. Now suppose $P(u, v, x) \neq P(v, w, x) \neq P(u, w, x)$. Put $c = = P(u, v, x)$, $z = P(v, w, x)$, $d = P(u, w, x)$. Both c and z are in $S(v, x)$; hence either $c \in S(v, z)$ or $z \in S(v, c)$. If $c \in S(v, z)$, then $S(u, w) = S(u, c) \cup S(c, z) \cup S(z, w)$, $S(u, x) = S(u, c) \cup S(c, z) \cup S(z, w)$, $S(u, x) = S(u, c) \cup S(c, x)$, $S(w, x) = S(w, z) \cup \cup S(z, c) \cup S(c, x)$ and $d = c$, which was to be proved; the condition IV holds and (M, P) is a tree algebra.

Theorem 2. *The conditions* (i), (ii), (iii), (iv) *from Theorem* 1 *are independent.*

Proof. Let M be the set of all points of a circle. Put $S(u, u) = \{u\}$ for each $u \in M$ and for any two distinct elements u, v of M let $S(u, v)$ be the set of all points of the arc of the circle which joins u and v and has the property that passing along this arc from u into v we move on the circle in the positive direction. Then (ii), (iii) and (iv) are fulfilled while (i) is not.

Let M be the set of vertices of a pentagon. For each $u \in M$ put $S(u, u) = \{u\}$. If u and v are distinct vertices of the same edge of the pentagon, then $S(u, v) = = \{u, v\}$; if they do not belong to the same edge, then $S(u, v) = \{u, v, w\}$, where w is the vertex of the pentagon joined by edges with both u and v. Then (i), (iii) and (iv) are fulfilled while (ii) is not.

Let M be the set of all real numbers. For each $u \in M$ let $S(u, u) = \{u, 0\}$ and for any two distinct elements u, v let $S(u, v)$ be the union of the set $\{0\}$ and the closed interval bounded by u and v. Then (i), (ii) and (iv) are fulfilled while (iii) is not.

Finally, let M be an arbitrary set with at least two elements, let $a \in M$. For any two elements u, v of M put $S(u, v) = \{a\}$. Then (i), (ii) and (iii) are fulfilled while (iv) is not.

Theorem 3. *Let* (M, P) *be a tree algebra, let* $\emptyset \neq G \subseteq M$, *let* (M_0, P_0) *be the subalgebra of* (M, P) *generated by* G. *Then* $M_0 = \{P(u, v, w) \mid \{u, v, w\} \subseteq G\}$.

Proof. If $u \in G$, then $u = P(u, u, u)$ and therefore $G \subseteq \{P(u, v, w) \mid \{u, v, w\} \subseteq G\}$. Let $u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3$ be elements of G, let $x_1 = P(u_1, v_1, w_1)$, $x_2 = P(u_2, v_2, w_2)$, $x_3 = P(u_3, v_3, w_3)$. Let $y = P(x_1, x_2, x_3)$. We have $S(u_1, v_1) = S(u_1, x_1) \cup S(v_1, x_1)$, $S(u_1, w_1) = S(u_1, x_1) \cup S(w_1, x_1)$, $S(v_1, w_1) = S(v_1, x_1) \cup S(w_1, x_1)$, $S(u_1, x_1) \cap S(v_1, x_1) \cap S(w_1, x_1) = \{x_1\}$. Suppose that $S(x_1, y)$ has a common element different from x_1 with each of the sets $S(u_1, x_1)$, $S(v_1, x_1)$, $S(w_1, x_1)$; let these elements be respectively z_1, z_2, z_3. Without loss of generality suppose $S(x_1, z_1) \subseteq S(x_1, z_2) \subseteq S(x_1, z_3)$. Then $S(x_1, z_1) = S(x_1, z_1) \cap S(x_1, z_2) \cap S(x_1, z_3) \subseteq S(u_1, x_1) \cap S(v_1, x_1) \cap S(w_1, x_1) = \{x_1\}$, which contradicts the assumption $z_1 \neq x_1$. We have proved that at least one of the set $S(u_1, x_1)$, $S(v_1, x_1)$, $S(w_1, x_1)$ has the intersection with $S(x_1, y)$ equal to $\{x_1\}$; without loss of generality let this set be $S(u_1, x_1)$. Therefore $S(u_1, y) = S(u_1, x_1) \cup S(x_1, y)$. Analogously we obtain $S(u_2, y) = S(u_2, x_2) \cup S(x_2, y)$, $S(u_3, y) = S(u_3, x_3) \cup S(x_3, y)$. Further $S(u_1, u_2) = S(u_1, y) \cup S(u_2, y)$, $S(u_1, u_3) = S(u_1, y) \cup S(u_3, y)$, $S(u_2, u_3) = S(u_2, y) \cup S(u_3, y)$ and this implies $y = P(u_1, u_2, u_3)$. As u_1, u_2, u_3 are elements of G, we have $y \in \{P(u, v, w) \mid \{u, v, w\} \subseteq G\}$. We have proved that this set contains G and is closed under P. It is evidently the least (with respect to set inclusion) set with this property, hence it equals M_0.

Corollary 1. *Let (M, P) be a tree algebra, let (M_0, P_0) be its subalgebra generated by a finite subset G of M. Then (M_0, P_0) is a finite tree algebra.*

Analogously to the degree of a vertex in a graph we may define the degree of an element of a tree algebra.

Let (M, P) be a tree algebra, let $u \in M$. We introduce a binary relation E_u on $M - \{u\}$ so that xE_uy if and only if $u \notin S(x, y)$. This relation is evidently reflexive and symmetric. Let x, y, z be three elements of $M - \{u\}$ and let xE_uy, yE_uz. Let $w = P(x, y, z)$. Then $S(x, z) = S(x, w) \cup S(w, z)$. If $u \in S(x, z)$, then either $u \in S(x, w)$ or $u \in S(w, z)$. But $S(x, y) = S(x, w) \cup S(w, y)$, $S(y, z) = S(y, w) \cup S(w, z)$ and hence either $u \in S(x, y)$ or $u \in S(y, z)$, which is a contradiction. This proves xE_uz and E_u is an equivalence relation. The cardinality of the set of equivalence classes of E_u will be called the degree of u.

Proposition 3. *Let (M, P) be a tree algebra, let u, v, w, x be elements of M such that $x = P(u, v, w)$, $x \notin \{u, v, w\}$. Then u, v, w belong to pairwise different equivalence classes of E_x.*

Proposition 4. *Let x be an element of a tree algebra (M, P) of degree 2. If $x = P(u, v, w)$ for some elements u, v, w of M, then at least one of the elements u, v, w is equal to x.*

Proposition 5. *Let x be an element of a tree algebra (M, P) of degree 1. If $x = P(u, v, w)$ for some elements u, v, w of M, then at least two of the elements u, v, w are equal to x.*

Proposition 6. *Let (M, P) be a tree algebra, let u be its element of the degree* 1 *or* 2. *Then $M - \{u\}$ is the support of a subalgebra of (M, P).*

Proposition 7. *Let (M, P) be a tree algebra, let (M_0, P_0) be its subalgebra generated by a non-empty subset G of M. Let u be an element of M of the degree* 1 *or* 2 *in (M, P). Then $u \in M_0$ if and only if $u \in G$.*

Proofs are straightforward.

Now we shall study unbounded segments of tree algebras.

Let (M, P) be a tree algebra. Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of pairwise different elements of M with the following properties:

(α) $S(x_0, x_i) \subset S(x_0, x_{i+1})$ for each positive integer i;
(β) For each $y \in M$ there exists a positive integer i such that $x_i \notin S(x_0, y)$.

Then the set $\bigcup_{i=1}^{\infty} S(x_0, x_i)$ is called an unbounded segment of (M, P) with the initial element x_0.

Evidently $\bigcup_{i=1}^{\infty} S(x_0, x_i) = \bigcup_{j=1}^{\infty} S(x_j, x_{j+1})$, $S(x_j, x_{j+1}) \cap S(x_{j+1}, x_{j+2}) = \{x_{j+1}\}$, $S(x_i, x_{i+1}) \cap S(x_j, x_{j+1}) = \emptyset$ for $j \notin \{i - 1, i, i + 1\}$. An unbounded segment of a tree algebra is an analogon of a one-way infinite path of a tree, while a bounded segment is an analogon of a finite path. We may also define a two-way unbounded segment of a tree algebra as the union of two unbounded segments whose intersection consists of exactly one element which is the initial element of both those segments; this is an analogon of a two-way infinite path in a tree.

Proposition 8. *Let D be an unbounded segment of a tree algebra (M, P) with the initial element x_0, let $u \in D$. Then $D = S(x_0, u) \cup D_0$, $S(x_0, u) \cap D_0 = \{u\}$, where D_0 is an unbounded segment of (M, P) with the initial element u.*

Proof. According to the definition of an unbounded segment there exists an infinite sequence $x_0, x_1, x_2, \ldots$ of elements of D such that $D = \bigcup_{j=0}^{\infty} S(x_j, x_{j+1})$. As $u \in D$, there exists a positive integer k such that $u \in S(x_k, x_{k+1})$. Then $S(x_k, x_{k+1}) = S(x_k, u) \cup S(u, x_{k+1})$, $S(x_k, u) \cap S(u, x_{k+1}) = \{u\}$. We have $D = \bigcup_{j=0}^{\infty} S(x_j, x_{j+1}) = (\bigcup_{j=0}^{k-1} S(x_j, x_{j+1})) \cup S(x_k, x_{k+1}) \cup \bigcup_{j=k+1}^{\infty} S(x_j, x_{j+1}) = (\bigcup_{j=0}^{k-1} S(x_j, x_{j+1})) \cup S(x_k, u) \cup S(u, x_{k+1}) \cup \bigcup_{j=k+1}^{\infty} S(x_j, x_{j+1})$. Evidently $(\bigcup_{i=1}^{\infty} S(x_j, x_{j+1})) \cup S(x_k, u) = S(x_0, u)$. Denote $D_0 = S(u, x_{k+1}) \cup \bigcup_{j=k+1}^{\infty} S(x_j, x_{j+1})$. The set D_0 is evidently an unbounded segment of (M, P) and from the above described properties of unbounded segments it follows that $S(x_0, u) \cap D_0 = \{u\}$.

Proposition 9. *Let D, D' be two unbounded segments of a tree algebra (M, P) with the elements x_0, u, respectively, let $D' \subseteq D$. Then $D = S(x_0, u) \cup D'$, $S(x_0, u) \cap \cap D' = \{u\}$.*

Proof. According to Proposition 8 there exists an unbounded segment D_0 with the initial element u such that $S(x_0, u) \cup D_0 = D$, $S(x_0, u) \cap D_0 = \{u\}$. Evidently then $D_0 = \{v \in D \mid P(x_0, u, v) = u\}$. As D' is an unbounded segment, there exists an infinite sequence $y_0, y_1, y_2, \ldots$ fulfilling (α) and (β) and such that $y_0 = u$ and $D' = \bigcup_{i=1}^{\infty} S(y_0, y_i)$. The condition ($\alpha$) implies that either all y_i are in $S(x_0, u)$, or all y_i are in D_0. In the first case (β) is not fulfilled, hence all y_i are in D_0 and $D' \subseteq D_0$. Each y_i is in $S(u, x_j)$ for some j, but none of these segments contains all y_i, hence also each x_l is in $S(u, y_m)$ for some m and $D' = D_0$. Then the assertion follows from Proposition 8.

Theorem 4. *Let D_1, D_2, D_3 be three unbounded segments of a tree algebra (M, P). Let $D_1 \cap D_2$ and $D_2 \cap D_3$ be unbounded segments of (M, P). Then also $D_1 \cap D_3$ is an unbounded segment of (M, P).*

Proof. Let a_1, a_2, a_3 be the initial elements of D_1, D_2, D_3, respectively. Let $D_4 = D_1 \cap D_2$, let a_4 be its initial element. Then by Proposition 8 we have $D_1 = S(a_1, a_4) \cup D_4$, $D_2 = S(a_2, a_4) \cup D_4$, $S(a_1, a_4) \cap D_4 = S(a_2, a_4) \cap D_4 = \{a_4\}$. Further let $D_5 = D_2 \cap D_3$, let a_5 be its initial element. Again $D_2 = S(a_2, a_5) \cup D_5$, $D_3 = S(a_3, a_5) \cup D_5$, $S(a_2, a_5) \cap D_5 = S(a_3, a_5) \cap D_5 = \{a_5\}$. As both a_4, a_5 are in D_2, we have either $a_4 \in S(a_2, a_5)$ or $a_5 \in S(a_2, a_4)$. Without loss of generality suppose $a_4 \in S(a_2, a_5)$. Then $D_1 = S(a_1, a_4) \cup S(a_4, a_5) \cup D_5$, $D_3 = S(a_3, a_5) \cup D_5$ and evidently $D_1 \cap D_3 = D_5$.

Therefore we can define a relation R on the set of all unbounded segments of (M, P) such that $D_1 R D_2$ if and only if $D_1 \cap D_2$ is an unbounded segment; this relation is an equivalence. The equivalence classes of R will be called the ends of (M, P). This is an analogon of the end of a graph defined by R. Halin [1].

Theorem 5. *Let $\mathfrak{E}$ be an end of a tree algebra (M, P), let $u \in M$. Then there exists exactly one unbounded segment in (M, P) which belongs to $\mathfrak{E}$ and whose initial element is u.*

Proof. Let $D_0 \in \mathfrak{E}$, let $x_0, x_1, x_2, \ldots$ be the sequence of elements of D_0 fulfilling (α) and (β), such that $D_0 = \bigcup_{i=1}^{\infty} S(x_0, x_i)$. Consider the elements $P(u, x_0, x_i)$. All of them are contained in $S(x_0, u)$, hence by (β) there exists a positive integer j such that $v = P(u, x_0, x_j) \neq x_j$. We have $S(x_0, u) = S(x_0, v) \cup S(v, u)$, $S(x_0, v) \cap S(v, u) = \{v\}$, $D_0 = S(x_0, v) \cup D_1$, $S(x_0, v) \cap D_1 = \{v\}$, where D_1 is the unbounded segment with the initial element v, such that $D_1 \subseteq D_0$. Then we have $D = S(u, v) \cup D_1$ and this is an unbounded segment belonging to $\mathfrak{E}$ with the initial element u. Now suppose that there exists $D' \in \mathfrak{E}$ with the initial element u. Then $D \cap D'$ is an unbounded segment; let its initial element be w. By Proposition 9 we have $D = S(u, w) \cup \cup (D \cap D') = D'$ and thus the required segment is unique.

Theorem 6. *Let $\mathfrak{E}$ be an end of a tree algebra (M, P), let $u \in M$, $v \in M$. Then the intersection of $S(u, v)$ and the unbounded segments D_1, D_2 from $\mathfrak{E}$ with the initial elements u, v, respectively, consists of exactly one element of M.*

Proof. There exists a sequence $x_0, x_1, x_2, \ldots$ fulfilling (α) and (β) and such that $D_1 = \bigcup_{i=1}^{\infty} S(x_0, x_i)$. Analogously as in the proof of Theorem 5 we prove that there exists a positive integer j such that $w = P(u, v, x_j) \neq x_j$. There exists an unbounded segment D with the initial element w such that $D \subseteq D_1 \cap D_2$. We have $D_1 = = S(u, w) \cup D$, $D_2 = S(v, w) \cup D$, $S(u, v) = S(u, w) \cup S(v, w)$ and $S(u, w) \cap \cap S(v, w) = \{w\}$. Then also $S(u, v) \cap D_1 \cap D_2 = \{w\}$.

Theorem 7. *Let $\mathfrak{E}_1, \mathfrak{E}_2$ be two different ends of a tree algebra (M, P). Then there exists exactly one two-way unbounded segment of (M, P) which is the union of one-way unbounded segments from $\mathfrak{E}_1$ and $\mathfrak{E}_2$.*

Proof. Let $u \in M$, let D_1, D_2 be the unbounded segments with the initial element u such that $D_1 \in \mathfrak{E}_1$, $D_2 \in \mathfrak{E}_2$. As D_1, D_2 belong to different ends of (M, P), we have $D_1 \nsubseteq D_2$ and there exists $x \in D_1 - D_2$. Let w be the (single) common element of $S(u, x)$, D_2 and the unbounded segment D_3 from $\mathfrak{E}_1$ with the initial element x. Let D_4, D_5 be unbounded segments with the initial element w such that $D_4 \in \mathfrak{E}_1$, $D_5 \in \mathfrak{E}_2$. Then $D_1 = S(u, w) \cup D_4$, $D_2 = S(u, w) \cup D_5$, $S(u, w) \cap D_4 \cap D_5 = \{w\}$. The required two-way unbounded segment is $D = D_4 \cup D_5$. The uniqueness of this segment can be proved analogously as in the proof of Theorem 5.

Theorem 8. *Let $\mathfrak{E}_1, \mathfrak{E}_2$ be two different ends of a tree algebra (M, P), let $u \in M$. Let D_1 (or D_2) be the one-way unbounded segment with the initial element u belonging to $\mathfrak{E}_1$ (or $\mathfrak{E}_2$, respectively). Let D be the two-way unbounded segment which is the union of one-way unbounded segments from $\mathfrak{E}_1$ and $\mathfrak{E}_2$. Then $D_1 \cap \cap D_2 \cap D$ consists of exactly one element.*

Theorem 9. *Let $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ be three pairwise different ends of a tree algebra (M, P). For $\{i, j\} \subseteq \{1, 2, 3\}$, $i \neq j$, let D_{ij} be the two-way unbounded segment which is the union of one-way unbounded segments from $\mathfrak{E}_i$ and $\mathfrak{E}_j$. Then $D_{12} \cap \cap D_{13} \cap D_{23}$ consists of exactly one element.*

Proofs of these theorems are analogous to those of the preceding theorems.

Therefore, if $\mathscr{E}$ is the set of all ends of a tree algebra (M, P), we may introduce an algebra (M^*, P^*) with a ternary operation P^* in the following way. We put $M^* = M \cup \mathscr{E}$. For any two elements u, v of M^* we define the segment $S^*(u, v)$.

If $u \in M$, $v \in M$, then $S^*(u, v) = S^*(u, v)$. If $u \in M$, $v \in \mathscr{E}$, then $S^*(u, v)$ is the union of $\{v\}$ and the one-way unbounded segment with the initial element u belonging to the end v. If $u \in \mathscr{E}$, $v \in M$, then $S^*(u, v) = S^*(v, u)$. If $u \in \mathscr{E}$, $v \in \mathscr{E}$, $u \neq v$, then $S^*(u, v)$ is the union of $\{u, v\}$ and the two-way unbounded segment which is the union of one-way unbounded segments belonging to the ends u and v. If $u \in \mathscr{E}$, then $S^*(u, u) = \{u\}$. Further, for any three elements u, v, w of M^* the element $P^*(u, v, w)$ is the element for which $\{P^*(u, v, w)\} = S^*(u, v) \cap S^*(u, w) \cap S^*(v, w)$. According to the above stated theorems (M^*, P^*) is a tree algebra and (M, P) is its subalgebra. Each element of $\mathscr{E}$ has the degree 1 in (M^*, P^*).

Now we turn to tree algebras without unbounded segments.

Theorem 10. *Let (M, P) be a tree algebra without unbounded segments and with at least two elements, let $u \in M$. Then each equivalence class of the relation E_u contains at least one element of the degree* 1.

Proof. Let C be an equivalence class of the relation E_u. If $x \in C$, $y \in C$, we write $x \precsim y$ if and only if $S(u, x) \subseteq S(u, y)$; the relation $\precsim$ is evidently a partial ordering on C. Each chain in this ordering is bounded from above; otherwise it would determine an unbounded segment. According to Zorn's Lemma there exists a maximal element z of this ordering. Suppose that the degree of z is greater than 1. Then there exists an equivalence class C_0 of E_z such that $u \notin C_0$. If $v \in C_0$, then $z \in S(u, v)$ and thus $z \precsim v$ and obviously $z \neq v$, which is a contradiction. Therefore E_z has only one equivalence class and the degree of z is 1.

Theorem 11. *Let (M, P) be a tree algebra without unbounded segments. Let x be an element of (M, P) of a degree at least* 3. *Then there exist three elements u, v, w of M of the degree* 1 *such that $x = P(u, v, w)$.*

Proof. As the degree of x is at least 3, there exist three pairwise different equivalence classes C_1, C_2, C_3 of E_x. According to Theorem 10 there exist elements $u \in C_1$, $v \in C_2$, $w \in C_3$ of the degree 1. As these elements belong to pairwise different equivalence classes of E_x, we have $x = P(u, v, w)$.

Corollary 2. *Let (M, P) be a tree algebra without unbounded segments, let G be the set of all elements of M of the degree* 1, *let (M_0, P_0) be the subalgebra of (M, P) generated by the set G. Then M_0 is the set of all elements of M of degrees different from* 2.

This immediately follows from Theorem 11 and Proposition 4.

Corollary 3. *Let (M, P) be a tree algebra without unbounded segments, let the number of its elements of the degree* 1 *be finite. Then the number of elements of M of degrees greater than* 2 *is finite.*

Now we shall define discrete tree algebras.

A tree algebra (M, P) is called discrete if and only if the segment $S(u, v)$ in (M, P) for any two elements u, v of M is finite.

A tree algebra (M, P) is called realizable by a tree, if there exists a tree T such that the vertex set of T is M and for any three elements u, v, w of T the element $P(u, v, w)$ is the vertex of T which is common to the path in T from u into v, the path in T from u into w and the path in T from v into w.

Theorem 12. *A tree algebra (M, P) is realizable by a tree if and only if it is discrete.*

Proof. Let (M, P) be discrete. We construct such a graph T with the vertex set M that two vertices u, v are joined by an edge if and only if $u \neq v$ and $S(u, v) = \{u, v\}$. Now let $S(x, y)$ be a segment of (M, P). As $S(x, y)$ is finite, the property (iii) implies that $S(x, y) = \{u_0, \ldots, u_m\}$ so that $u_0 = x$, $u_m = y$ and $S(u_i, u_{i+1}) = \{u_i, u_{i+1}\}$ for $i = 0, \ldots, m - 1$. Therefore each segment $S(x, y)$ is associated with a path in T connecting x and y. As the segment $S(x, y)$ exists for any two elements of M, the graph T is connected. Suppose that T contains a circuit C with elements $v_1, \ldots, v_p$ and edges v_iv_{i+1} for $i = 1, \ldots, p - 1$ and v_pv_1. Then the segment $S(v_1, v_p)$ contains the elements $v_2, \ldots, v_{p-1}$ and the vertices v_1, v_p are not joined by an edge, which is a contradiction. Therefore T is a tree and the assertion on $P(u, v, w)$ follows from Proposition 2. On the other hand, if (M, P) is realizable by a tree, then each segment $S(x, y)$ of (M, P) is associated with a path in this tree; such a path contains only a finite number of vertices, hence (M, P) must be discrete.

We shall show an example of a tree algebra which is not discrete. Let M be the set of all ordered pairs of real numbers. Let $u = [u_1, u_2]$, $v = [v_1, v_2]$, where u_1, u_2, v_1, v_2 are real numbers. The segment $S(u, v)$ will be defined as follows. It is the set of all ordered pairs $[x, y]$ of real numbers for which one of the following conditions holds:

(a) $x = u_1$, $0 \leqq y \leqq u_2$ or $u_2 \leqq y \leqq 0$;
(b) $u_1 \leqq x \leqq u_2$ or $u_2 \leqq x \leqq u_1$, $y = 0$;
(c) $x = v_1$, $0 \leqq y \leqq v_2$ or $v_2 \leqq y \leqq 0$.

The segments so defined fulfil the conditions of Theorem 1, therefore P can be defined in the usual way. Evidently this tree algebra is not discrete.

In the end we shall prove an assertion on direct products of tree algebras. A direct product of two tree algebras (M_1, P_1), (M_2, P_2) is the algebra (M, P), where $M = M_1 \times M_2$ and P is a ternary operation on M defined so that if $\{u_1, v_1, w_1\} \subseteq M_1$, $\{u_2, v_2, w_2\} \subseteq M_2$, then $P([u_1, u_2], [v_1, v_2], [w_1, w_2]) = [P_1(u_1, v_1, w_1), P_2(u_2, v_2, w_2)]$.

Theorem 13. *Let (M_1, P_1), (M_2, P_2) be two tree algebras, each with at least two elements. Then the direct product (M, P) of (M_1, P_1) and (M_2, P_2) is not a tree algebra.*

Proof. Let u_1, v_1 be two different elements of M_1 and let u_2, v_2 be two different elements of M_2. We have $P([u_1, u_2], [u_1, v_2], [u_2, v_2]) = [P_1(u_1, u_1, v_1), P_2(u_2, v_2, v_2)] = [u_1, v_2]$ and thus $[u_1, v_2] \in S([u_1, u_2], [v_1, v_2])$. Further $P([u_1, u_2], [v_1, u_2], [v_1, v_2]) = [P_1(u_1, v_1, v_1), P_2(u_2, u_2, v_2)] = [v_1, u_2]$ and thus also $[v_1, u_2] \in \in S([u_1, u_2], [v_1, v_2])$. Suppose that (M, P) is a tree algebra. Then $S([u_1, u_2], [v_1, v_2]) = S([u_1, u_2], [u_1, v_2]) \cup S([u_1, v_2], [v_1, v_2])$, $S([u_1, u_2], [u_1, v_2]) \cap \cap S([u_1, v_2], [v_1, v_2]) = \{[u_1, v_2]\}$. Hence either $[v_1, u_2] \in S([u_1, u_2], [u_1, v_2])$, or $[v_1, u_2] \in S([u_1, v_2], [v_1, v_2])$. But $P([u_1, u_2], [v_1, u_2], [u_1, v_2]) = [P_1(u_1, v_1, u_1), P_2(u_2, u_2, v_2)] = [u_1, u_2]$ and $P([u_1, v_2], [v_1, u_2], [v_1, v_2]) = [P_1(u_1, v_1, v_1), P_2(v_2, u_2, v_2)] = [v_1, v_2]$. Both $[u_1, u_2]$ and $[v_1, v_2]$ are different from $[v_1, u_2]$, which is a contradiction.

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Author's address: 460 01 Liberec 1, Komenského 2 (Katedra matematiky VŠST).