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ALTERNATING CONNECTIVITY OF DIGRAPHS

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In this paper we shall consider the most general case of digraphs, i.e. digraphs in which loops and multiple edges are admitted.

We shall introduce some definitions.

A sequence $P = [u_1, e_1, v_1, h_1, u_2, e_2, v_2, h_2, u_3, \ldots, u_{n-1}, e_{n-1}, v_{n-1}, h_{n-1}, u_n]$ where $u_1, \ldots, u_n, v_1, \ldots, v_{n-1}$ are vertices of a digraph G and $e_1, \ldots, e_{n-1}, h_1, \ldots, h_{n-1}$ are edges of G is a $(+-)$-alternating path (shortly $(+-)$-path) in G from u_1 to u_n if and only if $e_i = \overrightarrow{u_i v_i}$, $h_i = \overrightarrow{u_{i+1} v_i}$ for $i = 1, \ldots, n-1$.

A sequence $P = [u_1, e_1, v_1, h_1, u_2, e_2, v_2, h_2, u_3, \ldots, u_{n-1}, e_{n-1}, v_{n-1}, h_{n-1}, u_n]$ where $u_1, \ldots, u_n, v_1, \ldots, v_{n-1}$ are vertices of a digraph G and $e_1, \ldots, e_{n-1}, h_1, \ldots, h_{n-1}$ are edges of G is a $(-+)$-alternating path (shortly $(-+)$-path) in G from u_1 to u_n, if and only if $e_i = \overrightarrow{v_i u_i}$, $h_i = \overrightarrow{v_i u_{i+1}}$ for $i = 1, \ldots, n-1$.

An alternating path is either a $(+-)$-path or a $(-+)$-path.

The number n is called the length of the alternating path; thus any alternating path of the length n contains $2n$ edges and $2n + 1$ vertices.

If $u_i \neq u_j$ and $v_i \neq v_j$ for $i \neq j$, the alternating path P is called almost simple. If P is almost simple and moreover $u_i \neq v_j$ for any i and any j, the alternating path P is called simple.

An example of an almost simple $(+-)$-path which is not simple is on Fig. 1. An example of an almost simple $(-+)$-path which is not simple is on Fig. 2.

Two vertices a and b of a digraph G are called $(+-)$-alternatingly connected (shortly $(+-)$-connected), if and only if there exists a $(+-)$-path P from a to b, i.e. the path $[u_1, e_1, v_1, h_1, u_2, \ldots, u_{n-1}, e_{n-1}, v_{n-1}, h_{n-1}, u_n]$, which is a $(+-)$-path and $a = u_1$, $b = u_n$. Analogously we can define $(-+)$-connectivity of two vertices.

If any two vertices of a digraph G are $(+-)$-connected or $(-+)$-connected, we say that G is $(+-)$-connected or $(-+)$-connected respectively.

For the sake of simplicity of our considerations we shall consider also alternating paths of the length 0. Such a path consists only of one vertex u_1, we say that it connects u_1 with itself. It is at the same time a $(+-)$-path and a $(-+)$-path.

The relation of being $(+-)$-connected is then reflexive. If $P = [u_1, \ldots, u_k]$ is a $(+-)$-path, evidently also $[u_k, \ldots, u_1]$ is a $(+-)$-path, therefore the relation is also symmetric. If we have two $(+-)$-paths $[u_1, \ldots, u_k]$, $[u'_1, \ldots, u'_l]$ such that $u_k = u'_1$, the sequence $[u_1, \ldots, u_k, e'_1, \ldots, u'_l]$ is also a $(+-)$-path connecting u_1 with u'_l,

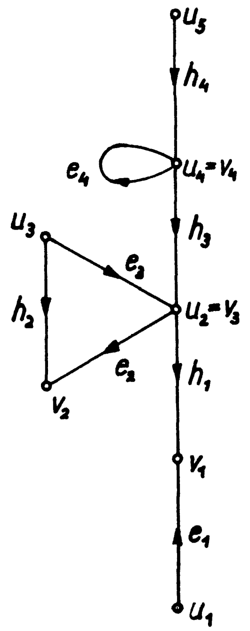

Fig. 1

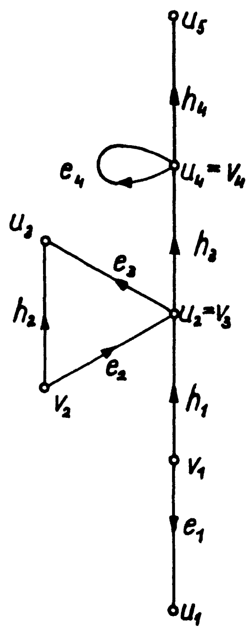

Fig. 2

therefore the relation of being $(+-)$-connected is transitive. This relation is an equivalence on the vertex set of G. Analogously also the relation of being $(-+)$-connected is an equivalence on this set.

Theorem 1. *Let two vertices a, b of a digraph G be $(+-)$-connected. Then they are connected by an almost simple $(+-)$-path.*

Proof. If a and b are $(+-)$-connected, there exists a $(+-)$-path $P = [u_1, \ldots, u_n]$ such that $a = u_1$, $b = u_n$. If for $i \neq j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$ we have always $u_i \neq u_j$, $v_i \neq v_j$, the path P is almost simple. Let $u_i = u_j$ for some $i \neq j$. Let l and m be the least and the greatest respectively positive integer less than or equal to n such that $u_l = u_m = u_i$. We take a path $P_1 = [u_1, \ldots, h_{l-1}, u_l, e_m, \ldots, u_n]$ It is also a $(+-)$-path. In this path we have no vertex equal to u_l except for u_l itself and by this procedure no vertices were identified. Thus we have reduced the number of the vertices u_i occuring in P more than once. By this way we proceed until $u_i \neq u_j$ for all pairs i, j where $i \neq j$. Similarly we reduce the number of vertices v_i occuring in P more than once, until there are also no such vertices. The result is an almost simple $(+-)$-path.

A stronger assertion obtained by changing the expression "almost simple $(+-)$-path" by "simple $(+-)$-path" is not true. If we consider the almost simple $(+-)$-path on Fig. 1 as a digraph G, we see that u_1, u_n are not connected by a simple $(+-)$-path.

Theorem 1′. *Let two vertices a, b of a digraph G be $(-+)$-connected. Then they are connected by an almost simple $(-+)$-path.*

The proof of this theorem is dual to the proof of Theorem 1.

In the following we shall use the terms source and sink. A source of a digraph is a vertex which is not a terminal vertex of any edge of this digraph. A sink of a digraph is a vertex which is not an initial vertex of any edge of this digraph.

Theorem 2. *Let a digraph G without sources be $(+-)$-connected. Then it is also $(-+)$-connected.*

Proof. Let a, b be two vertices of G, $a \neq b$. As G is without sources, there exists at least one edge e incoming into a and at least one edge e' incoming into b. Let c and d be the initial vertex of e and e' respectively. As G is $(+-)$-connected, there exists a $(+-)$-path $P = [u_1, \ldots, u_k]$ in G such that $u_1 = c$, $u_k = d$. Consider the sequence $[a, e, u_1, \ldots, u_k, e', b]$. It is a $(-+)$-path between a and b. As the vertices a and b were chosen arbitrarily, we see that G must be $(-+)$-connected.

Theorem 2′. *Let a digraph G without sinks be $(-+)$-connected. Then it is also $(+-)$-connected.*

A digraph without sources and sinks which is $(+-)$-connected (and therefore also $(-+)$-connected) will be called alternatingly connected.

On Fig. 3 we see an example that a digraph obtained from a $(+-)$-connected digraph by omitting the sources need not be $(-+)$-connected. Analogously a digraph obtained from a $(-+)$-connected digraph by omitting the sinks need not be $(+-)$-connected.

As the relations of being $(+-)$-connected and of being $(-+)$-connected are equivalences, they decompose the vertex set V of G into equivalence classes. If $u \in V$, we shall denote by $C^{+-}(u)$ and by $C^{-+}(u)$ the class of all vertices which are $(+-)$-connected or $(-+)$-connected with u, respectively.

Theorem 3. *If u is a source of G, then $C^{-+}(u) = \{u\}$.*

Proof. The first edge of a $(-+)$-path comes to the first vertex of this path. As u is a source, it cannot be the first vertex of any $(-+)$-path of a non-zero length. Therefore u cannot be $(-+)$-connected with any other vertex than u itself.

Theorem 3′. *If u is a sink of G, then $C^{+-}(u) = \{u\}$.*

Theorem 4. *Let u be a vertex of a digraph G and $C^{+-}(u)$ be the equivalence class of the relation of being $(+-)$-connected which a belongs to. Let b, c be terminal vertices of two edges of G whose initial vertices are in $C^{+-}(u)$. Then b and c are $(-+)$-connected.*

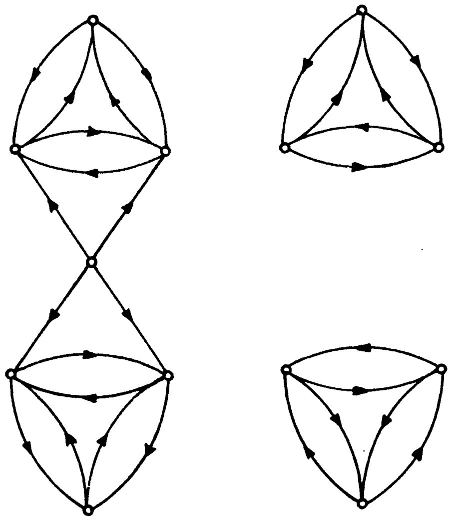

Fig. 3

Proof. Let b', c' be the vertices of $C^{+-}(a)$ such that $\overrightarrow{b'b}$, $\overrightarrow{c'c}$ are edges of G. The vertices b' and c' are $(+-)$-connected, therefore there exists a $(+-)$-path $[u_1, \ldots, u_k]$ such that $b' = u_1$, $c' = u_k$. Now take a sequence $[b, \overrightarrow{b'b}, u_1, \ldots, u_k, \overrightarrow{c'c}, c]$; this is a $(-+)$-path from b to c.

Theorem 4'. *Let u be a vertex of a digraph G and $C^{-+}(u)$ be the equivalence class of the relation of being $(-+)$-connected which a belongs to. Let b, c be initial vertices of two edges of G whose terminal vertices are in $C^{-+}(a)$. Then b and c are $(+-)$-connected.*

We say that two classes $C^{+-}(u)$, $C^{-+}(v)$ where u, v are vertices of G are associated to each other, if and only if there exists an edge in G whose initial vertex is in $C^{+-}(u)$ and whose terminal vertex is in $C^{-+}(v)$.

Theorem 5. *To each class $C^{+-}(u)$ which is not formed by a sink, exactly one class $C^{-+}(v)$ is associated.*

Theorem 5'. *To each class $C^{-+}(v)$ which is not formed by a source, exactly one class $C^{+-}(u)$ is associated.*

These theorems are immediate consequences of Theorems 4 and 4'.

If $C^{+-}(u)$ and $C^{-+}(v)$ are associated, we define

$$C(u, v) = C^{+-}(u) \cup C^{-+}(v), \quad C_1(u, v) = C^{+-}(u) \dot{-} C^{-+}(v),$$
$$C_0(u, v) = C^{+-}(u) \cap C^{-+}(v), \quad C_2(u, v) = C^{-+}(u) \dot{-} C^{+-}(v).$$

Evidently

$$C_0(u, v) \cup C_1(u, v) \cup C_2(u, v) = C(u, v),$$
$$C_0(u, v) \cap C_1(u, v) = C_0(u, v) \cap C_2(u, v) = C_1(u, v) \cap C_2(u, v) = \emptyset.$$

Evidently all vertices of $C(u, v)$ incident with loops are in $C_0(u, v)$.

Theorem 6. *For any four vertices a_1, b_1, a_2, b_2 for which $C^{+-}(a_1)$ is associated with $C^{-+}(b_1)$ and $C^{+-}(a_2)$ is associated with $C^{-+}(b_2)$ and $C(a_1, b_1) \neq C(a_2, b_2)$, we have*

$$C(a_1, b_1) \cap C(a_2, b_2) = [C_1(a_1, b_1) \cap C_2(a_2, b_2)] \cup [C_1(a_2, b_2) \cap C_2(a_1, b_1)].$$

Proof. We have

$$C_0(a_1, b_1) \cap C(a_2, b_2) = C^{+-}(a_1) \cap C^{-+}(b_1) \cap [C^{+-}(a_2) \cup C^{-+}(b_2)] =$$
$$= [C^{+-}(a_1) \cap C^{-+}(b_1) \cap C^{+-}(a_2)] \cup [C^{+-}(a_1) \cap C^{-+}(b_1) \cap C^{-+}(b_2)].$$

If $C^{+-}(a_1) = C^{+-}(a_2)$, we must have (according to Theorem 5) also $C^{-+}(b_1) = C^{-+}(b_2)$ and thus $C(a_1, b_1) = C(a_2, b_2)$. Analogously in the case when $C^{-+}(b_1) = C^{-+}(b_2)$. As we assume

$$C(a_1, b_1) \neq C(a_2, b_2),$$

we have

$$C^{+-}(a_1) \neq C^{+-}(a_2), \quad C^{-+}(b_1) \neq C^{-+}(b_2).$$

As $C^{+-}(a_1)$, $C^{+-}(a_2)$, $C^{-+}(b_1)$, $C^{-+}(b_2)$ are equivalence classes, this means

$$C^{+-}(a_1) \cap C^{+-}(a_2) = \emptyset, \quad C^{-+}(b_1) \cap C^{-+}(b_2) = \emptyset.$$

We have

$$[C^{+-}(a_1) \cap C^{-+}(b_1) \cap C^{+-}(a_2)] \cup [C^{+-}(a_1) \cap C^{-+}(b_1) \cap C^{-+}(b_2)] =$$
$$= [\emptyset \cap C^{-+}(b_1)] \cup [C^{+-}(a_1) \cap \emptyset] = \emptyset \cup \emptyset = \emptyset.$$

We have proved that

$$C_0(a_1, b_1) \cap C(a_2, b_2) = \emptyset.$$

Analogously we can prove

$$C(a_1, b_1) \cap C_0(a_2, b_2) = \emptyset.$$

Thus

$$C(a_1, b_1) \cap C(a_2, b_2) = [C_0(a_1, b_1) \cup C_1(a_1, b_1) \cup C_2(a_1, b_1)] \cap C(a_2, b_2) =$$
$$= [C_0(a_1, b_1) \cap C(a_2, b_2)] \cup [C_1(a_1, b_1) \cap C(a_2, b_2)] \cup [C_2(a_1, b_1) \cap C(a_2, b_2)] .$$

The first term in this union is $\emptyset$. Take the second term.

$$C_1(a_1, b_1) \cap C(a_2, b_2) = C_1(a_1, b_1) \cap [C_0(a_2, b_2) \cup C_1(a_2, b_2) \cup C_2(a_2, b_2)] =$$
$$= [C_1(a_1, b_1) \cap C_0(a_2, b_2)] \cup [C_1(a_1, b_1) \cap C_1(a_2, b_2)] \cup [C_1(a_1, b_1) \cap C_2(a_2, b_2)] .$$

We have

$$C_1(a_1, b_1) \cap C_0(a_2, b_2) = \emptyset$$

because $C_1(a_1, b_1)$ is a subset of $C(a_1, b_1)$ which has an empty intersection with $C_0(a_2, b_2)$,

$$C_1(a_1, b_1) \cap C_1(a_2, b_2) = \emptyset$$

because $C_1(a_1, b_1) \subset C^{+-}(a_1)$, $C_1(a_2, b_2) \subset C^{+-}(a_2)$ and $C^{+-}(a_1)$, $C^{+-}(a_2)$ are different equivalence classes. Thus

$$C_1(a_1, b_1) \cap [C_0(a_2, b_2) \cup C_1(a_2, b_2) \cup C_2(a_2, b_2)] = C_1(a_1, b_1) \cap C_2(a_2, b_2) .$$

The third term

$$C_2(a_1, b_1) \cap C(a_2, b_2) = C_2(a_1, b_1) \cap [C_0(a_2, b_2) \cup C_1(a_2, b_2) \cup C_2(a_2, b_2)] =$$
$$= [C_2(a_1, b_1) \cap C_0(a_2, b_2)] \cup [C_2(a_1, b_1) \cap C_1(a_2, b_2)] \cup [C_2(a_1 b_1) \cap C_2(a_2, b_2)] .$$

Analogously we have

$$C_2(a_1, b_1) \cap C_0(a_2, b_2) = \emptyset , \quad C_2(a_1, b_1) \cap C_2(a_2, b_2) = \emptyset ,$$

thus

$$C_2(a_1, b_1) \cap [C_0(a_2, b_2) \cup C_1(a_2, b_2) \cup C_2(a_2, b_2)] = C_2(a_1, b_1) \cap C_1(a_2, b_2) .$$

From this the assertion of the theorem follows.

Thus if a vertex c is common to two different sets $C(a_1, b_1)$, $C(a_2, b_2)$, either all edges outgoing from c have terminal vertices in $C^{-+}(b_2)$ and all edges incoming into c have initial vertices in $C^{+-}(a_1)$, or all edges outgoing from c have terminal vertices in $C^{-+}(b_1)$ and all edges incoming into c have initial vertices in $C^{+-}(a_2)$. All sources in $C(a, b)$ are in $C^{+-}(a)$, all sinks in $C(a, b)$ are in $C^{-+}(b)$.

Now we can define the vertex degree of alternating connectivity and the edge degree of alternating connectivity analogously to the analogous concepts for the connectivity of undirected graphs.

Let a, b be two vertices of a digraph G. The vertex degree of $(+-)$-connectivity of the vertices a, b is the number $\omega_G^{+-}(a, b)$ defined as follows:

(α) If $a = b$, then $\omega_G^{+-}(a, b) = \infty$.

(β) If $a \neq b$ and there are no loops at a and b and no edge joins a and b, then $\omega_G^{+-}(a, b)$ is the minimal number of vertices which must be deleted from G in order that a, b might not be $(+-)$-connected in the resulting digraph.

(γ) If $a \neq b$ and there are some loops at a or b or some edges joining a and b, then

$$\omega_G^{+-}(a, b) = \omega_{G_0}^{+-}(a, b) + \min(\mu(a, b), \nu(b)) + \min(\mu(b, a), \nu(a))$$

where G_0 is the digraph obtained from G by deleting all loops at a and b and all edges joining a and b, $\mu(a, b)$ is the number of edges going from a to b, $\mu(b, a)$ is the number of edges going from b to a, $\nu(a)$ is the number of loops at a, $\nu(b)$ is the number of loops at b in G.

The vertex degree of $(-+)$-connectivity of the vertices a, b is the number $\omega_G^{-+}(a, b)$ defined as follows:

(α) If $a = b$, then $\omega_G^{-+}(a, b) = \infty$.

(β) If $a \neq b$ and there are no loops at a and b and no edge joins a and b then $\omega_G^{-+}(a, b)$ is the minimal number of vertices which must be deleted from G in order that a, b might not be connected in the resulting digraph.

(γ) If $a \neq b$ and there are some loops at a or b or some edges joining a and b, then

$$\omega_G^{-+}(a, b) = \omega_{G_0}^{-+}(a, b) + \min(\mu(a, b), \nu(a)) + \min(\mu(b, a), \nu(b))$$

where G_0 is the digraph obtained from G by deleting all loops at a and b and all edges joining a and b, $\mu(a, b)$ is the number of edges going from a to b, $\mu(b, a)$ is the number of edges going from b to a, $\nu(a)$ is the number of loops at a, $\nu(b)$ is the number of loops at b in G.

We shall explain the meaning of the expression

$$\min(\mu(a, b), \nu(b)) + \min(\mu(b, a), \nu(a)).$$

This is the maximal number of edge-disjoint almost simple $(+-)$-paths between a and b in G not containing any vertex except for a and b. Any of such paths consists either of an edge going from a to b and one loop at b, or of an edge going from b to a and one loop at a. Analogously

$$\min(\mu(a, b), \nu(a)) + \min(\mu(b, a), \nu(b))$$

is the maximal number of edge-disjoint almost simple $(-+)$-paths between a and b in G not containing any vertex except for a and b.

Now again let a, b be two vertices of a digraph G. The edge degree of $(+-)$-connectivity of the vertices a, b is the number $\sigma_G^{+-}(a, b)$ defined as follows:

(α) If $a = b$, then $\sigma_G^{+-}(a, b) = \infty$.

(β) If $a \neq b$, then $\sigma_G^{+-}(a, b)$ is the minimal number of edges which must be deleted from G in order that a, b might not be $(+-)$-connected in the resulting digraph.

Analogously the edge degree of $(-+)$-connectivity is defined.

When studying degrees of alternating connectivity we can help ourselves by a certain bipartite undirected graph assigned to a digraph.

Let G be a digraph with vertices $a_1, \ldots, a_n$. The bipartite undirected graph $\hat{G}$ assigned to G will be constructed as follows. The vertices of $\hat{G}$ are $b_1, \ldots, b_n, c_1, \ldots, c_n$ (pairwise different). In $\hat{G}$ there exists an edge $b_i c_j$ ($1 \leqq i \leqq n$, $1 \leqq j \leqq n$) if and only if the edge $\overrightarrow{a_i a_j}$ exists in G. No two of the vertices $b_1, \ldots, b_n$ and no two of the vertices $c_1, \ldots, c_n$ are joined by an edge. To each $(+-)$-path from a_i to a_j in G there corresponds (in one-to-one manner) a path between b_i and b_j in $\hat{G}$. To each $(-+)$-path between a_i and a_j in G there corresponds (again in one-to-one manner) a path between c_i and c_j in $\hat{G}$. To simple alternating paths in G there correspond simple paths in $\hat{G}$.

In this way the results concerning the edge degree of $(+-)$-connectivity and of $(-+)$-connectivity can be obtained directly from the corresponding results for connectivity in undirected graphs. We shall express two important results derived from the corresponding results in [3].

Theorem 7. *Let S_k^{-+} where k is a positive integer be the relation on the vertex set of a digraph G defined so that two vertices a, b are in S_k^{+-} if and only if $\sigma_G^{+-}(a, b) \geqq$ $\geqq k$. Then S_k^{+-} is an equivalence on this set.*

Theorem 7′. *Let S_k^{-+} where k is a positive integer be the relation on the vertex set of a digraph G defined so that two vertices a, b are in S_k^{-+} if and only if $\sigma_G^{-+}(a,b) \geqq$ $\geqq k$. Then S_k^{-+} is an equivalence on this set.*

Theorem 8. *Let a, b be two different vertices of a digraph G. The maximal number of edge-disjoint almost simple $(+-)$-paths between a and b in G is equal to $\sigma_G^{+-}(a, b)$.*

Theorem 8′. *Let a, b be two different vertices of a digraph G. The maximal number of edge-disjoint almost simple $(-+)$-paths between a and b in G is equal to $\sigma_G^{-+}(a, b)$.*

The vertex analoga of Theorems 7 and 7′ do not hold, so as the corresponding assertions about undirected graphs. The analoga of Theorems 8 and 8′ will be expressed here as conjecture (they cannot be proved by the mechanical use of the bipartite graph technique like Theorems 8 and 8′).

Conjecture 1. *Let a, b be two different vertices of a digraph G. The maximal number of vertex-disjoint (up to a and b) almost simple $(+\,-)$-paths between a and b in G is equal to $\omega_G^{+-}(a, b)$.*

Conjecture 1′. *Let a, b be two different vertices of a digraph G. The maximal number of vertex-disjoint (up to a and b) almost simple $(-\,+)$-paths between a and b in G is equal to $\omega_G^{-+}(a, b)$.*

However, some analoga of Theorems 8 and 8′ can be obtained as theorems, if we introduce the concept of simple alternating connectivity.

We say that two vertices a and b of a digraph G are simply $(+\,-)$-connected or simply $(-\,+)$-connected, if and only if there exists a simple $(+\,-)$-path or simple $(-\,+)$-path respectively joining a and b in G.

The relation of being simply $(+\,-)$-connected is evidently reflexive and symmetric, but in general not transitive. Thus its investigation is not of such an interest as the investigation of alternating connectivity. The same holds also for the relation of being simply $(-\,+)$-connected.

Let a, b be two vertices of a digraph G. The vertex degree of simple $(+\,-)$-connectivity of the vertices a, b is the number $\tilde{\omega}_G^{+-}(a, b)$ defined as follows:

(α) If $a = b$, then $\tilde{\omega}_G^{+-}(a, b) = \infty$.

(β) If $a \neq b$, then $\tilde{\omega}_G^{+-}(a, b)$ is the minimal number of vertices which must be deleted from G in order that a, b might not be simply $(+\,-)$-connected in the resulting digraph.

The vertex degree of simple $(-\,+)$-connectivity of the vertices a, b is the number $\tilde{\omega}_G^{-+}(a, b)$ defined as follows:

If $a = b$, then $\tilde{\omega}_G^{-+}(a, b) = \infty$.

If $a \neq b$, then $\tilde{\omega}_G^{-+}(a, b)$ is the minimal number of vertices which must be deleted from G in order that a, b might not be simply $(-\,+)$-connected in the resulting digraph.

Now we may express a theorem.

Theorem 9. *Let a, b be two different vertices of a digraph G. The maximal number of vertex-disjoint (up to a and b) simple $(+\,-)$-paths between a and b in G is equal to $\tilde{\omega}_G^{+-}(a, b)$.*

Proof. We shall use the bipartite graph $\hat{G}$ and the results of [4]. The vertex set of the graph G can be decomposed into pairwise disjoint two-element sets $\{b_i, c_i\}$ for $i = 1, \ldots, n$. There is a one-to-one correspondence between simple $(+\,-)$-paths joining vertices a_i and a_j in G and simple paths joining vertices b_i and b_j in $\hat{G}$ which have at most one vertex in common with any pair of vertices $\{b_k, c_k\}$ for $k = 1, \ldots, n$. Deleting a vertex a_i in G corresponds to deleting the pair $\{b_i, c_i\}$ in G. Therefore this theorem follows from Theorem 5 of [4].

Theorem 9'. *Let a, b be two different vertices of a diagraph G. The maximal number of vertex-disjoint (up to a and b) simple $(-+)$-paths between a and b in G is equal to $\tilde{\omega}_G^{-+}(a, b)$.*

Now we shall investigate closed alternating paths, i.e. such alternating paths in which $u_1 = u_k$ (the first vertex coincides with the last). If in a closed alternating path $u_i \neq u_j$ (except for $u_1 = u_k$) and $v_i \neq v_j$ for $i \neq j$, we call this path an alternating almost-circuit. It is easy to show that we need not distinguish $(+-)$-almost-circuits and $(-+)$-almost-circuits and that generally a closed $(+-)$-path is also a closed $(-+)$-path.

If two edges e_1 and e_2 of a digraph G lie on the same almost-circuit or are identical, we write $e_1 \bigcirc_{alt} e_2$ (analogously to König's relation $\bigcirc$ of [2]).

Theorem 10. *The relation $\bigcirc_{alt}$ is an equivalence on the edge set of the digraph G.*

Proof. Alternating almost-circuits in G correspond to circuits in $\hat{G}$ in one-to-one manner. Thus if $k_1 \bigcirc_{alt} k_2$ in G, then the corresponding edges in $\hat{G}$ are in the relation $\bigcirc$ which is known from [2] and inversely. This relation is an equivalence, thus the same holds for $\bigcirc_{alt}$.

As $\bigcirc_{alt}$ is an equivalence, it decomposes the edge set of G into equivalence classes. A subgraph of G formed by edges of one equivalence class together with their initial and terminal vertices is an analogy to the concept of lobe in undirected graphs.

But not always any two vertices of such a subgraph are alternatingly connected. Let A be such a subgraph. If C is an alternating almost-circuit contained in A_1 $C = [u_1, e_1, v_1, h_1, u_2, \ldots, u_{k-1}, e_{k-1}, v_{k-1}, h_{k-1}, u_1]$, $e_i = \overrightarrow{u_i v_i}$, $h_i = \overrightarrow{u_{i+1} v_i}$ for $i = 1, \ldots, k-1$, then u_i, u_j for $1 \leqq i \leqq k-1$, $1 \leqq j \leqq k-1$ are always $(+-)$-connected and v_i, v_j are $(-+)$-connected. From the definition of A it follows that if $\overrightarrow{ab}, \overrightarrow{cd}$ are in A, then a and c are $(+-)$-connected, b and d are $(-+)$-connected in A. Thus any two initial vertices of edges of A are $(+-)$-connected, any two terminal vertices of edges of A are $(-+)$-connected in A. Moreover, the vertex degree of $(+-)$-connectivity of any two initial vertices of edges of A is at least 2 and also the vertex degree of $(-+)$-connectivity of any two terminal vertices of edges of A is at least 2.

The set of all initial or terminal vertices of edges of A will be denoted by A^+ or A^- respectively.

Theorem 11. *Let A, B be two different subgraphs of a digraph G such that each of them consists of all edges of one equivalence class of the relation $\bigcirc_{alt}$ together with their initial and terminal vertices. Let A^+ and B^+ be the set of initial vertices of edges of A and B respectively. Let A^- and B^- be the set of terminal vertices of edges of A and B respectively. Then each of the intersections $A^+ \cap B^+$, $A^- \cap B^-$*

consists at most of one vertex, the intersections $A^+ \cap B^-$, $A^- \cap B^+$ may have an unlimited number of vertices.

Proof. Let a, b be two vertices of $A^+ \cap B^+$, $a \neq b$. The vertices a, b are $(+-)$-connected in A, therefore there exists an almost simple $(+-)$-connected in A, therefore there exists an almost simple $(+-)$-path C between a and b in A. They are $(+-)$-connected also in B, thus there exists an almost simple $(+-)$-path C' between a and b in B. Let a' be a common vertex of C and C' such that $a' \neq a$, $a' \in A^+ \cap B^+$ and there exists no vertex on C between a and a' belonging to $A^+ \cap B^+$. Such a vertex evidently must exist; it may be $a' = b$. Consequently the subpaths of C and C' between a and a' will be C_1, C'_1. Their union is evidently an alternating almost-circuit. Let k or k' be the edge of C or C' respectively beginning in a. Both the edges k and k' belong to the above mentioned alternating almost-circuit, thus $k \bigcirc_{\mathrm{alt}} k'$. But k belongs to A, k' belongs to B, $A \neq B$, which is a contradiction. Thus $A^+ \cap B^+$ can contain at most one vertex. The proof for $A^- \cap B^-$ is analogous.

Now let us have a digraph G consisting of the (pairwise different) vertices a_i, b_i, c_i for $i = 1, \ldots, k$ and of edges $\overrightarrow{a_i b_j}$, $\overrightarrow{b_i c_j}$ for any i, j, $1 \leqq i \leqq k$, $1 \leqq j \leqq k$. Evidently the edges $\overrightarrow{a_i b_j}$ form one equivalence class of $\bigcirc_{\mathrm{alt}}$, the edges $\overrightarrow{b_i c_j}$ form another equivalence class of this relation. Thus A has the vertices $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ and B has the vertices $b_1, \ldots, b_k$, $c_1, \ldots, c_k$ further $A^+ = \{a_1, \ldots, a_k\}$, $A^- = B^+ = \{b_1, \ldots, b_k\}$, $B^- = \{c_1, \ldots, c_k\}$. We have $A^- \cap B^+ = \{b_1, \ldots, b_k\}$; the number k was chosen arbitrarily. Analogously we can prove the assertion for $A^- \cap B^+$. The above defined digraph for $k = 5$ is on Fig. 4.

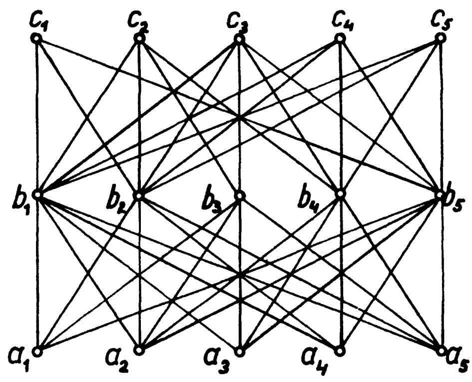

Fig. 4

Now we shall state another theorem which can be derived from the corresponding theorem about undirected graphs by using the bipartite graph technique.

Theorem 12. *If a, b are two vertices of a digraph G, we have*

$$\omega_G^{+-}(a, b) \leqq \sigma_G^{+-}(a, b), \quad \omega_G^{-+}(a, b) \leqq \sigma_G^{-+}(a, b).$$

In the end we shall investigate the minimal subgraphs of an alternatingly connected digraph with the property that they are alternatingly connected and contain all vertices of the given digraph. We shall call them alternatingly spanning subgraphs.

This concept is analogous to the concept of spanning tree in an undirected graph. We may again use the bipartite graph $\hat{G}$. As there is a one-to-one correspondence between edges of G and edges of $\hat{G}$, between simple $(+-)$-paths in G and simple paths from b_i to b_j (for some i and j) in $\hat{G}$, between simple $(-+)$-paths in G and simple paths from c_i to c_j in $\hat{G}$, we see that there is also a one-to-one correspondence between alternatingly spanning subgraphs of G and spanning trees of $\hat{G}$. The unique dominating sets for any spanning tree of the bipartite graph $\hat{G}$ are the sets $\{b_1, \dots, b_n\}$, $\{c_1, \dots, c_n\}$. Hence we see that n is the domination number of any such tree. We can construct any alternatingly spanning graph with n vertices in the following way.

Construction (C). Take an arbitrary undirected tree T with $2n$ vertices whose domination number is n. The vertex set of the tree T can be decomposed into two disjoint dominating subsets M, N each of which contains exactly n vertices. Choose an arbitrary one-to-one mapping φ of M onto N. Then identify any vertex $u \in M$ with the vertex $\varphi(u) \in N$. Let no edges be omitted at this procedure (thus loops could occur).

The result will be formulated as a theorem.

Theorem 13. *By the construction (C) always an alternatingly spanning graph with n vertices is created and any alternatingly spanning graph with n vertices can be obtained in this way.*

Proof follows from the above considerations.

Theorem 14. *Let G be a digraph with n vertices. Let $\mathbf{A}$ be the adjacency matrix of the digraph G, let $\mathbf{A}^*$ be the transposed matrix to A. Let $\mathbf{D} = \|d_{ij}\|$ be the square matrix of order n such that d_{ii} for $i = 1, \dots, n$ is the outdegree of the vertex a_i and $d_{ij} = 0$ for $i \neq j$. Let $\mathbf{D}^* = \|d^*_{ij}\|$ be the square matrix of order n such that d^*_{ii} for $i = 1, \dots, n$ is the indegree of the vertex a_i and $d^*_{ij} = 0$ for $i \neq j$. Then the number of alternatingly spanning subgraphs of G is equal to an arbitrary main minor of the matrix*

$$\left\| \begin{matrix} \mathbf{D} & -\mathbf{A} \\ -\mathbf{A}^* & \mathbf{D}^* \end{matrix} \right\|$$

Proof. The number of alternatingly spanning subgraphs of G is equal to the number of spanning trees of $\hat{G}$. This number is equal to an arbitrary main minor of the matrix $\hat{\mathbf{D}} - \hat{\mathbf{A}}$ where $\hat{\mathbf{D}}$ is the square matrix of the order $2n$, $\hat{\mathbf{D}} = \|\hat{d}_{ij}\|$, where $\hat{d}_{ii}$ for $i = 1, \dots, n$ is the degree of b_i (the outdegree of a_i), $\hat{d}_{ii}$ for $i = n + 1, \dots, 2n$ is the degree of c_{i-n} (the indegree of a_{i-n}), $\hat{d}_{ij} = 0$ for $i \neq j$, and $\hat{\mathbf{A}}$ is the adjacency matrix of G. If we denote the zero matrix of order n (the matrix whose elements are all zeros) by $\mathbf{O}_n$, we have evidently

$$\hat{\mathbf{D}} = \left\| \begin{matrix} \mathbf{D} & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{D}^* \end{matrix} \right\|, \quad \hat{\mathbf{A}} = \left\| \begin{matrix} \mathbf{O}_n & \mathbf{A} \\ \mathbf{A}^* & \mathbf{O}_n \end{matrix} \right\|.$$

The difference $\hat{\mathbf{D}} - \hat{\mathbf{A}}$ is the matrix of the theorem.

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