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GEODESIC CURVATURE OF CURVES ON SURFACES IN EQUIAFFINE SPACE

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In this paper we shall investigate the fundamental properties of the geodesic curvature of curves on a surface in the equiaffine space and on the basis of these properties we shall show a geometric interpretation of the Pick invariant. We shall make use of the notation and terminology of [4].

In the real equiaffine 3-space let there be given a surface of the third class with no singular points by its parametric description

$$\mathbf{r} = \mathbf{r}(u^1, u^2), \quad [u^1, u^2] \in O\,. \tag{1}$$

On the surface (1) let a curve of the third class be described by the parametric equations

$$u^a = u^a(t), \quad t \in (t_1, t_2), \tag{2}$$

also with no singularities (in what follows the Latin indices assume the values 1,2). We suppose that no tangent of the curve (2) is an asymptotic tangent of the surface. We denote the affine arc of the curve (2) by s and we define it by the equation[1])

$$s = \int_{t_0}^{t} \sqrt{\left(\left|g_{ab}\frac{\mathrm{d}u^a}{\mathrm{d}t}\frac{\mathrm{d}u^b}{\mathrm{d}t}\right|\right)}\,\mathrm{d}t,\ t_0,\ t \in (t_1, t_2), \tag{3}$$

where g_{ab} is the fundamental tensor of the surface (1) (See [4], p. 149).

From equation (3) it follows that

$$\operatorname{sgn}[u^a(s)] \equiv \operatorname{sgn}\left(g_{ab}\frac{\mathrm{d}u^a}{\mathrm{d}s}\frac{\mathrm{d}u^b}{\mathrm{d}s}\right) = g_{ab}\frac{\mathrm{d}u^a}{\mathrm{d}s}\frac{\mathrm{d}u^b}{\mathrm{d}s}\,. \tag{4}$$

For the curvature vector of the curve (2) we take the vector $\mathrm{d}^2\mathbf{r}/\mathrm{d}s^2$. We shall consider it placed at the corresponding point of the curve (2). If we use equation (4) and Gauss equations (see [4], p. 153, equations (14) and foll.)

$$\partial_j\mathbf{r}_i = \overset{1}{\Gamma}{}^s_{ij}\mathbf{r}_s + g_{ij}\mathbf{N}\,, \tag{5}$$

[1]) Cf. [2] Vol. II, pp. 237, 238.

then we find easily that

$$\frac{d^2\boldsymbol{r}}{ds^2} = \frac{\overset{1}{\delta}}{ds}\frac{du^a}{ds}\boldsymbol{r}_a + \operatorname{sgn}\left[u^a(s)\right]\boldsymbol{N}. \tag{6}$$

Let us add that $\overset{1}{\delta}$ is the symbol for the absolute derivative in connection, by means of which we have written the Gauss equations (5). In equation (6) we have expressed the curvature vector of the curve (2) as the sum of two vectors, the first of which will be called the *vector of geodesic curvature* and the second *the vector of normal curvature*. Both vectors realize the resolution of the curvature vector into the tangent plane and the affine normal of the surface (1).

Now, let us consider the set of all curves on the surface (1) which are in contact with a non asymptotic tangent t at a given point of the surface and satisfy the condition imposed on the curve (2). For every such curve, as it follows from relation (4), the equation

$$\overset{1}{\nabla}_c g_{ab}\frac{du^a}{ds}\frac{du^b}{ds}\frac{du^c}{ds} + 2g_{ab}\left(\frac{\overset{1}{\delta}}{ds}\frac{du^a}{ds}\right)\frac{du^b}{ds} = 0, \tag{7}$$

holds at the point T; $\overset{1}{\nabla}$ is the symbol for the covariant derivative in connection, by means of which we have written the Gauss equations (5). By (6) we may consider (7) as a linear non-homogeneous equation for the geodesic curvature vector. Because the above mentioned set of curves on the surface (1) contains also the geodesic Γ, one of the solutions of equation (7) is the vector defined by the relation

$$\frac{\overset{1}{\delta}}{ds}\frac{du^a}{ds} = k_g\frac{du^a}{ds}. \tag{8}$$

Let us call the scalar $|k_g|$ the *geodesic curvature of the geodesic Γ* at the point T or the *geodesic curvature of the surface* (1) *in the direction of the tangent t*. If we take into consideration that the general solution of a homogeneous equation corresponding to equation (7) is cv^a, $c \in (-\infty, \infty)$, where v^a is an arbitrarily chosen, but fixed, vector which defines a tangent direction at the point T on the surface (1) associated with the tangent t, then we may write the general solution of equation (7) in the form

$$w^a = k_g\frac{du^a}{ds} + cv^a, \quad c \in (-\infty, \infty).$$

Thus we have proved the following theorem:

The end-points of geodesic curvature vectors drawn at the point T on all curves upon the surface (1), *which are in contact with their common non-asymptotic tangent, lie on a straight line. This straight line is in the tangent plane at the* *point T, its distance from the point T is equal to the geodesic curvature of the surface* (1) *in the direction of the tangent t, and in the considered plane it defines a direction associated with the direction of the tangent t.*

Let us add that, in accordance with (3), the distance between two points ξ^a and η^a, which lie in the common tangent plane of the surface (1), is defined by the number

$$\triangle = \sqrt{(|g_{ab}(\xi^a - \eta^a)(\xi^b - \eta^b)|)}\,. \tag{9}$$

Substituting from (8) in (7) and making use of equations (3) and (4) we obtain after a simple arrangement the formula for the geodesic curvature of a surface in the given direction

$$|k_g| = \left|T_{abc}\frac{du^a}{ds}\frac{du^b}{ds}\frac{du^c}{ds}\right| = \frac{|T_{abc}\,du^a\,du^b\,du^c|}{|g_{ab}\,du^a\,du^b|^{3/2}}\,, \tag{10}$$

in which the tensor T_{abc} is the so-called *Darboux tensor* of the surface (1) (see [4], pp. 158–160, equations (24), (26) and foll.) given by the relation

$$T_{abc} = -\tfrac{1}{2}\overset{1}{\nabla}_a g_{bc}\,.$$

The direction which satisfies the equation

$$T_{abc}\,du^a\,du^b\,du^c = 0 \tag{11}$$

is called the *Darboux direction.* If follows from equations (10) and (11) that the geodesic curvature of a surface is zero exactly in those non-asymptotic directions which are the Darboux directions. Let us say that a point on the surface (1) is a *regular* point if, at this point, the tensor T_{abc} is not zero, all Darboux directions are mutually different and no Darboux direction is an asymptotic direction.

In the following considerations we shall study, by means of the *Darboux indicatrix,* the dependence of the geodesic curvature at a given point on its corresponding tangent direction. We shall consider only the regular points of the surface. The case that the point of the surface is not regular, is a trivial one. We shall mention it at the end of this paper.

The equation $R = |k_g|^{-1}$, so far as it is valid, defines a scalar R which we shall call the *radius of the geodesic curvature of the geodesic* Γ *at the point* T or *of the surface* (1) *in the direction of the tangent t.* Let us consider the pencil of the tangents to the surface (1) at its given point T. Let us lay off in both directions from T the third root of the corresponding radius of geodesic curvature on each tangent of the pencil which lies neither in an asymptotic direction nor in a Darboux direction of the surface. We call the set of points obtained in this way the *Darboux indicatrix.* Let us denote ξ^a the coordinates of a current point of the Darboux indicatrix. Evidently

$$\xi^a = \pm\sqrt[3]{(R)}\,\frac{du^a}{ds}\,. \tag{12}$$

Thus we see easily that the Darboux indicatrix is described by the equation

$$|T_{abc}\xi^a\xi^b\xi^c| = 1 . \tag{13}$$

Let us make an additional agreement that we shall add to the Darboux indicatrix also the points described by equation (13) and belonging to the asymptotic directions of the surface. Then it follows from equation (13) that the Darboux indicatrix consists of two mutually symetrically conjugate cubics the common asymptotes of which lie in the Darboux directions of the surface.

Let us call the principal direction such tangent direction of the surface on which the geodesic curvature attains its non-zero *extreme value*. We may easily verify that the function $z = |k_g|^{-1/3}$ attains its all non-zero extreme values exactly in those directions which are the principal directions. So we may describe the principal direction of the surface as such direction to which there corresponds a point on the Darboux indicatrix (we shall call it the *vertex of the Darboux indicatrix*) at which the non-zero distance from the point T upon the surface (1) attains its extreme value. Let us suppose that the vertex of the Darboux indicatrix exists and lies in the direction that is defined by the non-zero vector v^c. Let us denote the distance between the vertex and the point T on the surface (1) by R. The set of all points in the tangent plane the distance of which from the point T is equal to the number R is a curve homothetic to the first indicatrix and is described by the equation

$$|g_{ab}\xi^a\xi^b| = R^2 . \tag{14}$$

The indicatrix (14) passes through the vertex of the Darboux indicatrix and it is geometrically evident that both indicatrices have a common tangent at the considered vertex which is, by (14), parallel to the vector $\tilde{v}^m$, defined by the relation

$$\tilde{v}^m = \varepsilon_c{}^m v^c . \tag{15}$$ [2]

From equation (13) it follows that

$$T_{abm}v^a v^b \tilde{v}^m = 0 . \tag{16}$$

If we introduce the notation

$$S_{abc} = T_{abm}\varepsilon_c{}^m$$

(see [4], p. 163, equation (7)) we may write (16) in the form

$$S_{abc}v^a v^b v^c = 0 . \tag{17}$$

The tensor S_{abc} is the so-called *Segre tensor*. The direction which satisfies equation

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[2]) The numbers ε_{ab} are the components of the so-called *discriminant tensor*. The following relations hold:

$$\varepsilon_{11} = \varepsilon_{22} = 0 , \quad \varepsilon_{12} = -\varepsilon_{21} = \sqrt{(|g_{11}g_{22} - (g_{12})^2|)} .$$

(17) is called the *Segre direction*. The Darboux directions are mutually associated with the Segre directions (see [4], p. 164). From this result it follows that at a regular point on the surface no Segre direction is an asymptotic or Darboux direction and that all Segre directions are mutually different. Hence we have the following theorem:

At a regular point on the surface the principal directions are exactly those directions which are the Segre directions.

From these equations it follows that, at a regular point on the surface, there exist either exactly one real Darboux direction and exactly one real principal direction, or exactly three real Darboux directions and three real principal directions.

Instead of the tangent plane of the surface (1) let us take, to simplify the following considerations, its complex extension. In the usual way let us extend all concepts which we have used up to now. The distance between two complex points is calculated by means of equation (9), the Darboux indicatrix is the set of all complex points which satisfy equation (13), the principal direction is the direction which is neither asymptotic nor the Darboux direction and is the solution of equations (17) or (15) and (16) and so on. We find easily that the following theorem is true:

At each regular point on a surface there exist exactly three principal directions. These directions are mutually different and they are the Segre directions.

Now we shall state fully the meaning of the concept of the Darboux indicatrix. Through the point T on the surface (1) let us lay a straight line which lies in the principal direction of the surface. Every point of this straight line which is also on the Darboux indicatrix is called the *vertex of the Darboux indicatrix*. It is evident that to each principal direction of the surface there correspond infinitely many vertices. We shall now calculate the distance between the vertices and the point of contact T of the tangent plane. To simplify the calculation we shall make use of asymptotic parameters on the surface[3]) and we shall introduce the following notation for the components of the tensor g_{ab}, g^{ab}, T_{abc}, $T_{ab}{}^{c}$:

$$g_{12} = g_{21} = F\,, \quad g^{12} = g^{21} = \frac{1}{F}\,,$$

$$T_{111} = A\,, \quad T_{222} = B\,,$$

$$T_{11}{}^{2} = \frac{A}{F}\,, \quad T_{22}{}^{1} = \frac{B}{F}\,.$$

We can easily verify[4]) that all other components of these tensors are zeros and that the well-known Pick invariant J, where $J = -\frac{1}{2}T_{ijk}T^{ijk}$, can be expressed by the

[3]) If the points of the surface are elliptic we shall suppose that the functions (1) and (2) are analytic.

[4]) See also [4], pp. 177 and 178.

relation

$$(18) \qquad J = \frac{AB}{F^3}\,.$$

The non-zero vector (v, w) which lies in the principal direction is, according to (15) and (16), the solution of the equations

$$F(v\tilde{v} + w\tilde{w}) = 0\,, \quad Av^2\tilde{v} + Bw^2\tilde{w} = 0\,.$$

Hence it is easy to calculate that

$$(19) \qquad (v, w) = \left(c\sqrt[3]{B},\ c\sqrt[3]{A}\right),$$

where c runs through all complex numbers different from zero. By writing out the trigonometric forms of the numbers $\sqrt[3]{A}$, $\sqrt[3]{B}$ we may easily verify that, for $AB \neq 0$, the equations (19) define three mutually different principal directions. At a regular point we have always $AB \neq 0$. For $A = B = 0$ the tensor T_{abc} would become a zero tensor. For $A = 0$, $B \neq 0$ or $A \neq 0$, $B = 0$ the equation (19) defines exactly one direction. This direction is at the same time an asymptotic direction. Hence the following theorem is true:

At a point which is not a regular point of the surface (1) *either every direction is a Darboux direction or there exists exactly one Darboux direction which is at the same time an asymptotic direction*[5]).

Through a regular point on the surface (1) let us lay a straight line which lies in the principal direction of the surface and let us write its parametric equations in the form

$$(20) \qquad \xi = \sqrt[3]{(B)}\,t\,, \quad \eta = \sqrt[3]{(A)}\,t\,, \quad t \in K_1\,.$$

At is follows from (13) the equation

$$(21) \qquad A\xi^3 + B\eta^3 = \mathrm{j}\,,$$

where j assumes all values with $|\mathrm{j}| = 1$, describes a one-parametric system of cubics which constitute the Darboux indicatrix. Equations (20) and (21) define exactly all vertices of the Darboux indicatrix. An easy calculation gives this result:

$$\xi = \mathrm{j}(2A)^{-1/3}\,, \quad \eta = \mathrm{j}(2B)^{-1/3}\,, \quad |\mathrm{j}| = 1\,.$$

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[5]) It follows from (13) or (19), respectively, that in this last case every cubic of the Darboux indicatrix degenerates into three straight lines.

If we denote the distance between the examined vertex of the Darboux indicatrix and the point of contact T of the tangent plane by $\triangle$ we may write

$$\triangle = \sqrt{(|g_{ab}\xi^a\xi^b|)} = \sqrt[6]{\left(\left|\frac{2F^3}{AB}\right|\right)}. \tag{22}$$

It follows from (22) that the distance between each vertex of the Darboux indicatrix and the point T is constant. Let us call the geodesic curvature of the surface in the principal direction the *principal curvature* and let us denote it by k. Evidently,

$$\triangle = |k|^{-1/3}.$$

Thus we may write

$$k = \left|\frac{AB}{2F^3}\right|^{1/2}.$$

From this result and (18) it follows that

$$|J| = 2k^2.$$

Thus we have found the connection between the principal curvature of the surface and the Pick invariant of the surface. We may sum up the preceding investigations in this theorem:

At a regular point on the surface there exist three principal curvatures. These three curvatures assume the same value. The magnitude of the Pick invariant is equal to the double of the square of the principal curvature.

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