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ALMOST HERMITIAN MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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1. INTRODUCTION

A particularly interesting and simple class of Kähler manifolds is formed by those with constant holomorphic sectional curvature. These manifolds are natural analogues of Riemannian manifolds of constant sectional curvature.

There is a local classification of Kähler manifolds with constant holomorphic sectional curvature ([10], [13]): A Kähler manifold of constant holomorphic sectional curvature μ is locally isometric to a complex projective space $CP^n(\mu)$, a complex hyperbolic space $CD^n(\mu)$, or to a complex Euclidean space C^n.

Of course, the definition of holomorphic sectional curvature (namely $H(X) = \|X\|^{-4} R_{XJXXJX}$) makes sense for any almost Hermitian manifold. The purpose of this paper is to consider almost Hermitian manifolds of constant holomorphic sectional curvature which are not necessarily Kählerian. The study of such manifolds is much more complicated and interesting than in the Kähler case. For example, one has the six sphere $S^6(\mu)$ to contend with. We pose two questions for a given class L of almost Hermitian manifolds:

(A) Does the theorem of Schur hold for L? More precisely, suppose $M \in L$ with $\dim M \geqq 4$ and assume M has constant holomorphic sectional curvature $\mu(m)$ at each point $m \in M$. Must μ be a constant function?

(B) Classify (either locally or globally) all $M \in L$ which have constant holomorphic sectional curvature.

As mentioned above, the solution of (A) and (B) for the class K of Kähler manifolds is well-known. (See [10], [13], [24].) Moreover, for the class NK, (A) and (B) have recently been solved ([7], [22]). See also [19]. The only non-Kähler nearly Kähler manifolds of constant holomorphic sectional curvature are locally isometric to a six sphere $S^6(\mu)$.

Perhaps the next most interesting class to consider is the class H of Hermitian manifolds. It is easy to see that there are many additional spaces of constant holomorphic sectional curvature in H, such as all of the simply connected spaces of constant sectional curvature. (Let M be a simply connected Riemannian manifold with constant sectional curvature $\lambda \neq 0$. Let φ be a conformal diffeomorphism between M and a piece of Euclidean space. Then φ induces an almost complex structure on M. It is easily checked that M is Hermitian but not Kählerian. Since M has constant sectional curvature *a fortiori* it has constant holomorphic sectional curvature.)

The interesting thing about the class H is that the answer to (A) is no. We prove in § 5

Theorem. *Let* $\mathrm{d}s^2$ *be the usual metric on* C^n *and* $f: C^n \to C$ *any nonlinear holomorphic function. Then the metric* $(1 + \mathrm{Re} f(z))^{-2}\, \mathrm{d}s^2$ *on* C^n *does not satisfy* (A). *Thus* $(C^n, (1 + (\mathrm{Re} f(z))^{-2})\, \mathrm{d}s^2)$ *is a Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant.*

Because the answer to (A) is no, it is hopeless to attempt a classification in the class H. Instead we turn to other types of almost Hermitian manifolds. In § 3 and § 4 we consider a certain class QK_2 which contains both NK and K as subclasses. The manifolds in QK_2 satisfy a certain natural curvature condition. The class QK_2 is interesting because it contains many homogeneous almost Hermitian manifolds, namely all 3-symmetric spaces (see [6]).

The answer to (A) for the class QK_2 is yes. Furthermore we are almost, but not quite, able to solve (B). More precisely, we show that manifolds in QK_2 with *nonzero* constant holomorphic sectional curvature μ consist of manifolds locally isometric to $CP^n(\mu)$, $CD^n(\mu)$, or $S^6(\mu)$. The case when the holomorphic sectional curvature vanishes remains in doubt. There are certainly flat almost Hermitian manifolds other than the usual Kähler structure on C^n. (See for example [20], [21].) However, whether or not there exist manifolds in QK_2 other than C^n with zero holomorphic sectional curvature remains an intriguing question.

2. CLASSES OF ALMOST HERMITIAN MANIFOLDS

We consider C^∞ almost Hermitian manifolds and use the notation of [1], [2], and [6]. In this paper we shall be concerned with five classes of almost Hermitian manifolds: K, NK, AK, QK, H. For reference, the defining conditions for these classes are as follows:

$$
\begin{aligned}
K &: \nabla_X(J)\, Y = 0\,, \\
NK &: \nabla_X(J)\, X = 0\,, \\
AK &: \mathrm{d}F = 0\,, \\
QK &: \nabla_X(J)\, Y + \nabla_{JX}(J)\, JY = 0\,, \\
H &: \nabla_X(J)\, Y - \nabla_{JX}(J)\, JY = 0\,, \quad X, Y \in X(M)\,.
\end{aligned}
$$

(The condition defining H is equivalent to the vanishing of the Nijenhuis tensor.)

Curvature identities are a key to understanding the geometry of these classes of almost Hermitian manifolds. In fact for each of the classes given above there exists a curvature identity (see [8]). However in this paper we shall be concerned with the following curvature identities:

$$(1)\quad R_{WXYZ} = R_{WXJYJZ},$$

$$(2)\quad R_{WXYZ} = R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ},$$

$$(3)\quad R_{WXYZ} = R_{JWJXJYJZ}.$$

For a given class L of almost Hermitian manifolds let L_i be the subclass of manifolds whose curvature operator satisfies identity (i).

Certain equalities occur among the various classes. We summarize the known results.

Theorem 2.1. *We have*

$$K_1 = K_2 = K_3 = K, \tag{2.1}$$

$$K = NK_1, \tag{2.2}$$

$$NK_2 = NK_3 = NK, \tag{2.3}$$

$$K = AK_1, \tag{2.4}$$

$$H_2 = H_3. \tag{2.5}$$

Proof. (2.1) is well-known, and (2.2), (2.3), (2.4) are proved in [4]. For (2.3), (2.4) see also [9], [15]. In [8] (2.5) is proved.

In [8] inclusions between the various classes are treated more fully. Moreover, in view of theorem (2.1) it suffices to consider the following classes: K, NK, AK_2, AK_3, AK, QK_1, QK_2, QK_3, QK, H_1, $H_2 = H_3$, H.

3. SCHUR'S THEOREM FOR THE CLASS QK_2

We say why QK_2 is a reasonable class in which to consider problems (A) and (B). In the first place we have $K \subset NK \subset QK_2$. Thus our results generalize all the known solutions to (A) and (B). Furthermore, although $QK_1 \subseteq QK_2$ and $AK_2 \subseteq QK_2$ our techniques do not yield better results for these two classes. Our techniques appear to be too weak to solve (A) and (B) for QK_3.

It will be convenient to define a tensor λ by $\lambda(WXYZ) = R_{WXYZ} - R_{WXJYJZ}$. We begin by generalizing a well known formula for Kähler manifolds of constant holomorphic sectional curvature.

Lemma 3.1. *Let M be any almost Hermitian manifold which satisfies curvature identity (2), and assume that M has pointwise constant holomorphic sectional curvature μ. Then*

$$(3.1)\quad R_{WXYZ} = \frac{\mu}{4}\{\langle W, Y\rangle \langle X, Z\rangle - \langle W, Z\rangle \langle X, Y\rangle + \langle JW, Y\rangle \langle JX, Z\rangle - - \langle JW, Z\rangle \langle JX, Y\rangle + 2\langle JW, X\rangle \langle JY, Z\rangle\} + + \frac{1}{4}\{2\lambda(WXYZ) - \lambda(WZXY) - \lambda(WYZX)\}.$$

Proof. Using (2) it is easy to check that λ satisfies the following identities:

$$(3.2)\qquad \lambda(WXYZ) = -\lambda(WXJYJZ) = \lambda(WJXYJZ) = \lambda(JWJXJYJZ).$$

Now by assumption

$$(3.3)\qquad R_{XJXXJX} = \mu\|X\|^4 \quad \text{for} \quad X \in \mathfrak{X}(M).$$

Let $X, Y \in \mathfrak{X}(M)$ be such that at a point $m \in M$ we have $\|X\| = \|Y\| = 1$, $\langle X, Y\rangle = = \langle JX, Y\rangle = 0$. We substitute $aX + bY$ for X in (3.3). Equating coefficients and using (2) we find

$$(3.4)\qquad R_{XYXY} = \frac{\mu}{4} + \frac{3}{4}\lambda(XYXY).$$

More generally put $U = aX + bJX + cY$, where $a^2 + b^2 + c^2 = 1$. From (3.2), (3.4) and (2) we find

$$(3.5)\quad R_{XUXU} = \frac{\mu}{4}\{\|X\|^2\|U\|^2 - \langle X, U\rangle^2 + 3\langle JX, U\rangle^2\} + \frac{3}{4}\lambda(XUXU).$$

In fact (3.4) also holds for $X, U \in \mathfrak{X}(M)$ of arbitrary norm. We linearize (3.5) and use the first Bianchi identity; after some calculation we obtain (3.1).

Define a tensor P by

$$(3.6)\qquad P(VWXYZ) = \underset{VWX}{\mathfrak{S}}\{-3\,\nabla_V(\lambda)(WXYZ) + \underset{XYZ}{\mathfrak{S}}\nabla_V(\lambda)(WXYZ)\},$$

for $V, W, X, Y, Z \in \mathfrak{X}(M)$. Here $\mathfrak{S}$ denotes the cyclic sum. We shall compute the expression $P(JWWXWX) + P(JWWJXWJX)$ in two different ways.

Lemma 3.2. *For $M \in QK_2$ and $W, X \in \mathfrak{X}(M)$ we have*

$$(3.7)\qquad P(JWWXWX) + P(JWWJXWJX) = 0.$$

Proof. Using standard formulas, the covariant derivative of λ is found to be

$$(3.8)\quad \nabla_V(\lambda)(WXYZ) = \nabla_V(R)_{WXYZ} - \nabla_V(R)_{WXJYJZ} - R_{WX\nabla_V(J)YJZ} - R_{WXJY\nabla_V(J)Z}\,.$$

From the second Bianchi identity and (3.8) we obtain

$$(3.9)\qquad \mathop{\mathfrak{S}}_{VWX} \nabla_V(\lambda)(WXYZ) = -\mathop{\mathfrak{S}}_{VWX} \{R_{WX\nabla_V(J)YJZ} + R_{WXJY\nabla_V(J)Z}\}.$$

Substituting (3.9) into (3.6), we find

(3.10)

$$P(VWXYZ) = \mathop{\mathfrak{S}}_{VWX} \{(3 - \mathop{\mathfrak{S}}_{XYZ})(R_{WX\nabla_V(J)YJZ} + R_{WXJY\nabla_V(J)Z}) - \mathop{\mathfrak{S}}_{XYZ} \nabla_V(R)_{WXJYJZ}\} =$$
$$= \mathop{\mathfrak{S}}_{VWX} \{2R_{WX\nabla_V(J)YJZ} - R_{WZ\nabla_V(J)XJY} - R_{WY\nabla_V(J)ZJX} +$$
$$+ 2R_{WXJY\nabla_V(J)Z} - R_{WZJX\nabla_V(J)Y} - R_{WYJZ\nabla_V(J)X} - \mathop{\mathfrak{S}}_{XYZ} \nabla_V(R)_{WXJYJZ}\}.$$

Let $V = JW$, $Y = W$ and $Z = X$ in (3.10). Using the fact that M is quasi-Kählerian we get

$$(3.11)\quad P(JWWXWX) = -2\{R_{WXJ\nabla_W(J)WJX} + R_{JWX\nabla_W(J)WJX}\} +$$
$$+ 3\{R_{WJW\nabla_X(J)XJW} - R_{WXJWJ\nabla_W(J)X} - R_{WJW\nabla_W(J)WJX}\} +$$
$$+ R_{WXW\nabla_W(J)X} - 2R_{JWXJW\nabla_W(J)W} +$$
$$+ \nabla_W(R)_{XJWJXJW} + \nabla_W(R)_{XWJXJW}.$$

In (3.11) we replace X by JX and add the resulting equation to (3.11). Again using the fact that $M \in \mathrm{QK}$ and (3), it follows that

$$(3.12)\quad P(JWWXWX) + P(JWWJXWJX) = 3\{R_{WXW\nabla_W(J)X} - R_{WXJWJ\nabla_W(J)X} -$$
$$- R_{WJXJW\nabla_W(J)X} - R_{JWXJW\nabla_W(J)X}\}.$$

Now (3.7) is immediate from (3.12) and (2).

Lemma 3.3. *Suppose $M \in \mathrm{QK}_2$ and that M has pointwise constant holomorphic sectional curvature μ. Then we have*

$$(3.13)\quad P(JWWXWX) + P(JWWJXWJX) =$$
$$= 2(JW\mu)\{\|W\|^2\|X\|^2 + \langle W, X\rangle^2 + \langle JW, X\rangle^2\} +$$
$$+ 4\|W\|^2\{\langle W, JX\rangle X\mu - \langle W, X\rangle JX\mu\} -$$
$$- 6\mu\{\langle W, X\rangle\langle\nabla_W(J)W, X\rangle + \langle W, JX\rangle\langle\nabla_W(J)W, JX\rangle\}.$$

Proof. We write (3.1) in the form

$$(3.14)\qquad 4R_{WXYZ} - 3\lambda(WXYZ) + \mathop{\mathfrak{S}}_{XYZ} \lambda(WXYZ) =$$
$$= \mu\{\langle W, Y\rangle\langle X, Z\rangle - \langle W, Z\rangle\langle X, Y\rangle + \langle JW, Y\rangle\langle JX, Z\rangle -$$
$$- \langle JW, Z\rangle\langle JX, Y\rangle + 2\langle JW, X\rangle\langle JY, Z\rangle\}.$$

We take the covariant derivative of (3.14) and use the second Bianchi identity. The result is

$$\begin{aligned}(3.15)\quad P(VWXYZ) = \underset{VWX}{\mathfrak{S}} \{&(V\mu)(\langle W, Y\rangle \langle X, Z\rangle - \langle W, Z\rangle \langle X, Y\rangle + \\ &+ \langle JW, Y\rangle \langle JX, Z\rangle - \langle JW, Z\rangle \langle JX, Y\rangle + \\ &+ 2\langle JW, X\rangle \langle JY, Z\rangle) + \mu[\langle \nabla_V(J)\, W, Y\rangle \langle JX, Z\rangle + \\ &+ \langle JW, Y\rangle \langle \nabla_V(J)\, X, Z\rangle - \langle \nabla_V(J)\, W, Z\rangle \langle JX, Y\rangle - \\ &- \langle JW, Z\rangle \langle \nabla_V(J)\, X, Y\rangle + 2\langle \nabla_V(J)\, W, X\rangle \langle JY, Z\rangle + \\ &+ 2\langle JW, X\rangle \langle \nabla_V(J)\, Y, Z\rangle]\}\,.\end{aligned}$$

In (3.15) we let $V = JW$, $Y = W$ and $Z = X$ to obtain

$$\begin{aligned}(3.16)\quad P(JWWXWX) = &(JW\mu)\{\|W\|^2\, \|X\|^2 - \langle W, X\rangle^2 + 3\langle JW, X\rangle^2\} - \\ &- 4(X\mu)\, \|W\|^2 \langle JW, X\rangle + 4(W\mu) \langle JW, X\rangle \langle W, X\rangle - \\ &- 9\mu\langle W, JX\rangle \langle \nabla_W(J)\, W, JX\rangle + \\ &+ 3\mu\langle \nabla_W(J)\, W, X\rangle \langle W, X\rangle + 3\mu\langle \nabla_X(J)\, X, W\rangle\, \|W\|^2\,.\end{aligned}$$

In (3.16) we replace X by JX and add the resulting equation to (3.16). The result is (3.13).

We can now prove the main result of this section.

Theorem 3.4. *Suppose $M \in QK_2$ with $\dim M \geqq 4$, and that M has pointwise constant holomorphic sectional curvature μ. Then μ is a constant function.*

Proof. Since $\dim M \geqq 4$ for each $m \in M$ and $W \in X(M)$ there exists $X \in X(M)$ such that W, JW, X, JX are mutually orthogonal at m. From lemmas 3.2 and 3.3 we find $(W\mu)\, \|W\|^2\, \|X\|^2 = 0$. Hence the result follows.

4. THE CLASSIFICATION

Now that we have proved the technical lemmas of § 3, we can effect a classification. In particular we have the following theorem, the proof of which is surprisingly simple.

Theorem 4.1. *Let $M \in QK_2$ have pointwise constant holomorphic sectional curvature $\mu \neq 0$. Then M is nearly Kählerian.*

Proof. If $\dim M = 2$, then M is automatically Kählerian. Thus we may assume $\dim M \geqq 4$. Then μ is constant, and so from lemmas 3.2 and 3.3 we have

$$(4.1)\qquad \mu\{\langle \nabla_W(J)\, W, X\rangle \langle W, X\rangle + \langle \nabla_W(J)\, W, JX\rangle \langle W, JX\rangle\} = 0\,.$$

In (4.1) we replace X by $W + \nabla_W(J)\,W$. Note that W, JW, $\nabla_W(J)\,W$ and $J\,\nabla_W(J)\,W$ are mutually perpendicular. Therefore (4.1) becomes

$$\mu\|\nabla_W(J)\,W\|^2\,\|W\|^2 = 0\,. \tag{4.2}$$

Hence we obtain the theorem.

In [7] nearly Kähler manifolds of constant holomorphic sectional curvature are classified. Therefore combining [7] and theorem 4.1 we have the following classification for QK_2.

Theorem 4.2. *Let $M \in QK_2$ have pointwise constant holomorphic sectional curvature $\mu \neq 0$. Assume* $\dim M \geqq 4$. *Then M is locally isometric to one of the following spaces:*

(1) *A complex hyperbolic space $CD^n(\mu)$;*
(2) *A complex projective space $CP^n(\mu)$;*
(3) *The sphere $S^6(\mu)$.*

Furthermore from theorem 4.2 and [7] we have a global classification theorem.

Theorem 4.3. *Let $M \in QK_2$ have pointwise constant holomorphic sectional curvature $\mu \neq 0$. If* $\dim M \geqq 4$ *and M is complete, then M is isometric to one of the following spaces:*

(1) *$CD^n(\mu)/\Gamma$ where Γ is a discrete group;*
(2) *$CP^n(\mu)$;*
(3) *$S^6(\mu)$.*

The hypothesis in theorem 4.1 that $\mu \neq 0$ is very curious indeed. This hypothesis is not needed for the classification of nearly Kähler manifolds of constant holomorphic sectional curvature. There are two unsolved questions here:

1. Do there exist flat quasi-Kähler manifolds which are not Kählerian?
2. Do there exist nonflat quasi-Kähler manifolds with zero holomorphic sectional curvature?

Note that (1) is false for the classes AK and NK, and (2) is false for the class NK.

Corollary 4.4. *Let M be a 3-symmetric space with positive definite metric, and suppose M has constant holomorphic sectional curvature $\mu \neq 0$. Then M is isometric to one of the spaces listed in theorem* 4.3.

5. HERMITIAN MANIFOLDS WITH POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

If M and M^0 are conformally equivalent almost Hermitian manifolds, then either both M and M^0 are Hermitian, or neither is. This is obvious because the integrability condition for an almost complex structure does not depend on a metric.

Theorem 5.1. *Let M be a contractible even dimensional Riemannian manifold of constant sectional curvature K. Then $M \in H$ and has constant holomorphic sectional curvature $\mu = K$. If* $\dim M \geqq 4$ *and* $K \neq 0$, *then* $M \notin QK$ (*in fact M is not semi-Kählerian, see* [2]).

Proof. Everything is obvious, except perhaps for the last statement. This is proved in [2].

Next we modify this technique to show that Schur's theorem fails for the class H. It will be necessary to recall some relations between the connections and curvature tensors of conformally related metrics $\langle , \rangle$ and $\langle , \rangle^0$ on the same manifold M. There exists a C^∞ real valued function σ such that $\langle , \rangle^0 = e^{2\sigma}\langle , \rangle$. We define a vector field grad σ by $\langle \operatorname{grad} \sigma, Z \rangle = Z\sigma$ for $Z \in X(M)$. Also define a symmetric tensor ψ_σ by

$$\psi_\sigma(X, Y) = (\nabla_X Y)\,\sigma - XY\sigma + (X\sigma)(Y\sigma)\,.$$

Well-known calculations yield relations between the connections ∇, ∇^0 and between the curvature tensors R, R^0. In our notation these are

$$\nabla^0_X Y = \{\nabla_X Y + (X\sigma)\,Y + (Y\sigma)\,X - \langle X, Y\rangle \operatorname{grad} \sigma\}^0\,,$$

$$\begin{aligned}(5.1)\quad R^0_{WXYZ} = e^{2\sigma}\{R_{WXYZ} &- \|\operatorname{grad}\sigma\|^2 (\langle W, Y\rangle \langle X, Z\rangle - \langle W, Z\rangle \langle X, Y\rangle + \\ &+ \psi_\sigma(W, Y)\langle X, Z\rangle - \psi_\sigma(W, Z)\langle X, Y\rangle + \\ &+ \psi_\sigma(X, Z)\langle W, Y\rangle - \psi_\sigma(X, Y)\langle W, Z\rangle)\}\,.\end{aligned}$$

Now assume that $\langle , \rangle$ is almost Hermitian with respect to the almost complex structure J. Then so is $\langle , \rangle^0$. The following lemma is a special case of (5.1).

Lemma 5.2. *The holomorphic sectional curvatures K_{XJX} and K^0_{XJX} are related by*

$$(5.2)\quad K^0_{XJX} = e^{-2\sigma}\{K_{XJX} - \|\operatorname{grad}\sigma\|^2 + \|X\|^{-2}(\psi_\sigma(X, X) + \psi_\sigma(JX, JX))\}\,,$$

for $X \in X(M)$.

Corollary 5.3. *Suppose $M = C^n$ and $\langle , \rangle$ is the usual metric. Let* $\sigma = -\log(1 + s)$ *and assume $\|X\| = 1$, and that X is parallel. Then*

$$(5.3)\qquad K^0_{XJX} = -\|\operatorname{grad} s\|^2 + (1 + s)(X^2 s + (JX)^2 s)\,.$$

Proof. We have $K_{XJX} = 0$, $e^{-2\sigma} = (1 + s)^2$, $\|\operatorname{grad}\sigma\|^2 = (1 + s)^{-2}\|\operatorname{grad} s\|^2$, and $-X^2\sigma + (X\sigma)^2 = (1 + s)^{-1} X^2 s$. With these substitutions lemma 5.2 yields corollary 5.3.

Now we can prove the main theorem of this section.

Theorem 5.4. *Let $f: C^n \to C$ be any holomorphic function, and let $\langle , \rangle$ be the usual metric on C^n. Then* $(1 + \operatorname{Re} f(z))^{-2}\langle , \rangle$ *has pointwise constant holomorphic* *sectional curvature given by the formula*

$$K^0_{XJX} = -\ \|\text{grad}\ \text{Re} f(z)\|^2\,. \tag{5.4}$$

Proof. Since f is holomorphic, $\text{Re} f$ is harmonic in each variable. Therefore

$$(X^2 + (JX)^2)\ \text{Re} f(z) = 0 \tag{5.5}$$

whenever X is a coordinate vector field. Moreover by linearity, (5.5) holds for any parallel vector field X on C^n. Let $s = \text{Re} f(z)$. Then we obtain (5.4) from (5.3) and (5.5).

Although Schur's lemma fails for the class H there does exist a curvature identity.

Theorem 5.5. *Let M be a Hermitian manifold with constant holomorphic sectional curvature μ. Then the curvature operator of M satisfies*

$$\begin{aligned} R_{WXWX} + R_{JWJXJWJX} - R_{WJXWJX} - R_{JWXJWX} &= \\ = 2\mu\{-\langle W, X\rangle^2 + \langle JW, X\rangle^2\}\,. \end{aligned} \tag{5.6}$$

Proof. Let $X, Y \in X(M)$ be such that at a point $m \in M$ we have $\|X\| = \|Y\| = 1$, $\langle X, Y\rangle = 0$. Let $a^2 + b^2 = 1$. Substituting $aX + bY$ for X in (3.3) we find

$$R_{XJYXJY} + R_{JXYJXY} + 2R_{XJXYJY} - 2R_{XJYJXY} = 0\,. \tag{5.7}$$

Similarly

$$R_{XYXY} + R_{JXJYJXJY} + 2R_{XJXYJY} + 2R_{XYJXJY} = 0\,. \tag{5.8}$$

In [8] the following identity is proved for the class H

$$\begin{aligned} R_{ABCD} + R_{JAJBJCJD} - R_{JAJBCD} - R_{JABJCD} - R_{JABCJD} - \\ - R_{AJBJCD} - R_{AJBCJD} - R_{ABJCJD} = 0\,, \end{aligned} \tag{5.9}$$

for $A, B, C, D \in X(M)$. From (5.7), (5.8), (5.9) it follows that

$$R_{XYXY} + R_{JXJYJXJY} = R_{XJYXJY} + R_{JXYJXY}\,. \tag{5.10}$$

Now we use the method of the proof of lemma 3.1 to derive (5.6) from (5.10).

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