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Locally Coherent Algebras

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Abstract

An algebra $\mathcal{A}$ with 0 is locally coherent if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in \operatorname{Con} \mathcal{A}$, $[0]_\theta \subseteq \mathcal{B}$ whenever $\mathcal{B}$ contains at least one class of θ. We characterize varieties of such algebras by a Mal'cev condition and we show that these varieties are locally regular and satisfy LCUT. If a variety $\mathcal{V}$ is, moreever, permutable at 0, also the converse implication holds.

Key words: Coherence, local coherence, weak coherence, regularity, local regularity, permutability at 0.

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The concept of coherent algebra was introduced by D. Geiger [7] as follows: an algebra $\mathcal{A}$ is *coherent* if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in \operatorname{Con} \mathcal{A}$ we have

$$[b]_\theta \subseteq \mathcal{B} \text{ for some } b \in \mathcal{B} \text{ implies } [a]_\theta \in \mathcal{B} \text{ for each } a \in \mathcal{B}.$$

In othe words, $\mathcal{A}$ is coherent if every its subalgebra which contains at least one congruence class is a union of congruence classes. A variety $\mathcal{V}$ is *coherent* if each $\mathcal{A} \in \mathcal{V}$ has this property.

It was shown by D. Geiger that every coherent variety is both regular and permutable and W. Taylor showed in [9] that the converse does not hold. One new condition, the so called CUT, was introduced in [1] and it was shown that CUT is independent on regularity and permutability and, morever, a variety $\mathcal{V}$ is coherent if and only if $\mathcal{V}$ is CUT, regular and permutable.

Coherent varieties were investigated also by J Duda [6]. The concept of coherence was weakened in [2]. Let $\mathcal{A}$ be an algebra with a constant 0 (i.e. 0 is a nullary term function of $\mathcal{A}$ alias a unary term function with the constant value equal to 0). $\mathcal{A}$ is *weakly coherent* if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in \operatorname{Con} \mathcal{A}$ we have

$$[0]_\theta \subseteq \mathcal{B} \text{ implies } [a]_\theta \in \mathcal{B} \text{ for each } a \in \mathcal{B}.$$

A variety $\mathcal{V}$ with a constant 0 is *weakly coherent* if every $\mathcal{A}$ of $\mathcal{V}$ has this property.

Analogously as for coherent varieties, it was proven in [2] that every weakly coherent variety is permutable and weakly regular (but not vice versa). One new condition, the so called 0-CUT was introduced and it was shown that a variety $\mathcal{V}$ is weakly coherent if and only if $\mathcal{V}$ is 0-CUT, weakly regular and permutable. Hence, we can recognize a strong conection between coherency and regularity and weak coherency and weak regularity. Recently, the concept of local regularity was introduced in [3]:

An algebra $\mathcal{A}$ with 0 is *locally regular* if for every $\theta, \phi \in \operatorname{Con} \mathcal{A}$ it holds

$$\text{if } [a]_\theta = [a]_\phi \text{ for some } a \in \mathcal{B} \text{ then } [0]_\theta = [0]_\phi .$$

Varieties of locally regular algebras were characterized by a Mal'cev condition and some useful examples were presented in [3]. Of course, a variety $\mathcal{V}$ with 0 is regular if and only if $\mathcal{V}$ is both weakly regular and locally regular.

Hence, we can search for some "local" concept of coherence which will serve as a counterpart of local regularity in the sense mentioned above.

Definition 1 An algebra $\mathcal{A}$ with 0 is *locally coherent* if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in \operatorname{Con} \mathcal{A}$ it holds:

$$\text{if } [b]_\theta \subseteq \mathcal{B} \text{ for some } b \in \mathcal{B} \text{ then } [0]_\theta \subseteq \mathcal{B} .$$

A variety $\mathcal{V}$ with 0 is *locally coherent* if every $\mathcal{A} \in \mathcal{V}$ has this property.

One can easily check the following:

Observation *A variety $\mathcal{V}$ with 0 is coherent if and only if $\mathcal{V}$ is weakly coherent and locally coherent.*

Locally coherent varieties can be characterized by a Mal'cev condition:

Theorem 1 *For a variety $\mathcal{V}$ with 0, the following are equivalent:*

(1) $\mathcal{V}$ is locally coherent;

(2) there exist an n-ary term s ($n \geq 1$) and binary terms $t_1, \dots, t_n$ such that the following identities hold in $\mathcal{V}$:

$$t_i(0, y) = y \quad \text{for } i = 1, \dots, n, \qquad x = s(t_1(x, y), \dots, t_n(x, y)) .$$

Proof (1) $\Rightarrow$ (2): Let $\mathcal{A} = F_v(x, y)$ be a free algebra of $\mathcal{V}$ with two free generators x, y and let $\theta = \theta(x, 0) \in \operatorname{Con} \mathcal{A}$. Take C $= [y]_\theta$ and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ generated by the set C. Then $[y]_\theta \subseteq \mathcal{B}$ and, by (1) we have

$[0]_\theta \subseteq \mathcal{B}$. Since $x \in [0]_\theta$, it gives $x \in \mathcal{B}$, i.e. there exist elements $c_1, \ldots, c_n$ of $\mathcal{B}$ and an n-ary term s ($n \geq 1$) with

$$x = s(c_1, \ldots, c_n).$$

Since $c_i \in F_v(x, y)$, there exist binary terms $t_1, \ldots, t_n$ such that $c_i = t_i(x, y)$ whence

$$x = s(t_1(x, y), \ldots, t_n(x, y)).$$

Moreover, $t_i(x, y) = c_i \in \mathrm{C} = [y]_{\theta(x,0)}$ which immediately implies

$$t_i(0, y) = y \quad \text{for } i = 1, \ldots, n.$$

(2) $\Rightarrow$ (1): Let $\mathcal{A} \in \mathcal{V}$, $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, $\theta \in \operatorname{Con} \mathcal{A}$ and $b \in \mathcal{B}$. Suppose $[b]_\theta \subseteq \mathcal{B}$. If $x \in [0]_\theta$ then $\langle x, 0 \rangle \in \theta$ and hence

$$t_i(x, b) \theta t_i(0, b) = b,$$

i.e. $t_i(x, b) \in [b]_\theta \subseteq \mathcal{B}$. By (2) we conclude

$$x = s(t_1(x, b), \ldots, t_n(x, b)) \in \mathcal{B}$$

proving (1). Thus $\mathcal{V}$ is locally coherent. □

Example 1 Let $\mathcal{V}$ be a variety of type (2,0) where the binary operation is denoted by $+$ and the nullary one by 0 and let $\mathcal{V}$ salisfies the identities

$$(x + y) + y = x \qquad \text{and} \qquad 0 + y = y.$$

Then $\mathcal{V}$ is locally coherent. Namely, we can set $n = 2$ and $t_1(x, y) = x + y$, $t_2(x, y) = y$ and $s(z_1, z_2) = z_1 + z_2$. Then, of course, $t_1(0, y) = y = t_2(0, y)$ and $s(t_1(x, y), t_2(x, y)) = (x + y) + y = x$.

Theorem 2 *Every locally coherent variety is locally regular.*

Proof By (2) of Theorem 1 we have

$$s(x, \ldots, x) = s(t_1(0, x), \ldots, t_n(0, x)) = 0.$$

Take

$$q_i(y, x) = t_i(x, y) \quad (i = 1, \ldots, n)$$

and

$$p_1(z_1, \ldots, z_n, v_1, \ldots, v_n, x, y) = s(z_1, \ldots, z_n).$$

Then

$$p_1(q_1(x, y), \ldots, q_n(x, y), x, \ldots, x, x, y) = s(t_1(y, x), \ldots, t_n(y, x)) = y,$$
$$p_1(x, \ldots, x, q_1(x, y), \ldots, q_n(x, y), x, y) = s(x, \ldots, x) = 0.$$

By Theorem 2 in [3], $\mathcal{V}$ is locally regular. □

The following example shows that local regularity is essentialy weaker condition than local coherency:

Example 2 By Corollary 2.1 in [5], every uniquely complemented lattice is locally regular. Consider the four-element lattice as shown in Fig. 1.

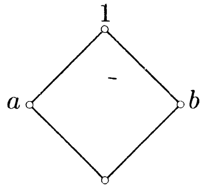

Fig. 1

Then L is locally regular. On the other hand, consider $\theta \in \operatorname{Con} L$ given by the partition $\{0, b\}$, $\{a, 1\}$ and a sublatice S of L given by $S = \{0, a, 1\}$. Then $[a]_\theta = \{a, 1\} \subseteq S$ but $b \notin S$ and $b \in [0]_\theta$, i.e. $[0]_\theta \not\subseteq S$ thus L is not locally coherent.

This motivated our effort to find out a condition which should be add to local regularity to obtain a condition equivalent with local coherency.

Definition 2 An algebra $\mathcal{A}$ with 0 has LCUT if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, each $\theta \in \operatorname{Con} \mathcal{A}$ an every n-ary polynomial φ over $\mathcal{A}$

$$\text{if } [y]_\theta \subseteq \mathcal{B} \text{ and } \varphi(y, \dots, y) = 0 \text{ for some } y \in \mathcal{B} \text{ then } \varphi([y]_\theta) \subseteq \mathcal{B}.$$

A variety $\mathcal{V}$ with 0 has LCUT if each $\mathcal{A} \in \mathcal{V}$ has this property.

Of course, by $\varphi(C)$ we mean the set $\{\varphi(c_1, \dots, c_n); c_i \in C\}$.

Theorem 3 *Every locally coherent variety has LCUT.*

Proof Let $\mathcal{V}$ be a locally coherent variety, $\mathcal{A} \in \mathcal{V}$ and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let $\theta \in \operatorname{Con} \mathcal{A}$ and φ be an n-ary polynomial over $\mathcal{A}$. Suppose $[y]_\theta \subseteq \mathcal{B}$ and $\varphi(y, \dots, y) = 0$ for some $y \in \mathcal{B}$. By local coherence, also $[0]_\theta \subseteq \mathcal{B}$. Moreover, $\varphi([y]_\theta)$ must be contained in some congruence class of θ; since $\varphi(y, \dots, y) = 0$, this class is $[0]_\theta$, thus $\varphi([y]_\theta) \subseteq [0]_\theta \subseteq \mathcal{B}$ and $\mathcal{A}$ has LCUT. □

There is a natural question under what condition the local regularity and LCUT imply local coherency. To answer this question, we must recall the following concept (see e.g. [8]):

An algebra $\mathcal{A}$ with 0 is *permutable at* 0 if $[0]_{\theta \cdot \phi} = [0]_{\phi \cdot \theta}$ holds for every two congruences $\theta, \phi \in \operatorname{Con} \mathcal{A}$. A variety $\mathcal{V}$ with 0 is *permutable at* 0 if every $\mathcal{A} \in \mathcal{V}$ has this property. The following result was proven in [8].

Lemma 1 *A variety $\mathcal{V}$ with 0 is permutable at 0 if and only if there exists a binary term $b(x, y)$ such that the identities*

$$b(x, x) = 0 \qquad \textit{and} \qquad b(x, 0) = x$$

hold in $\mathcal{V}$.

Let $\mathcal{A} = (A, F)$ be an algebra with 0 and R be a reflexive and compatible relation on $\mathcal{A}$ (recall that R is compatible on $\mathcal{A}$ if R is a subalgebra of $\mathcal{A} \times \mathcal{A}$). Denote by $\theta(R)$ the congruence of $\mathcal{A}$ generated by R, i.e. $\theta(R)$ is the least congruence on $\mathcal{A}$ with $R \subseteq \theta(R)$. Furher, denote by

$$[0]_R = \{x \in A;\ \langle 0, x\rangle \in R\}\,.$$

Lemma 2 *For a variety $\mathcal{V}$ with 0, the following conditions are equivalent:*

(a) $\mathcal{V}$ is permutable at 0;

(b) for each $\mathcal{A} \in \mathcal{V}$ and every reflexive and compatible relation R on $\mathcal{A}$,

$$[0]_R = [0]_{\theta(R)}\,.$$

Proof $(a) \Rightarrow (b)$: Of course, $R \subseteq \theta(R)$ implies $[0]_R \subseteq [0]_{\theta(R)}$. Suppose $x \in [0]_{R^{-1}}$. Then $\langle 0, x\rangle \in R^{-1}$, i.e. $\langle x, 0\rangle \in R$ and, by (a) and Lemma 1,

$$\langle 0, x\rangle = \langle b(x,x), b(x,0)\rangle \in R$$

whence $x \in [0]_R$. We have $[0]_{R^{-1}} \subseteq [0]_R$.

Now suppose $y \in [0]_{R\cdot R}$. Then $\langle 0, y\rangle \in R \cdot R$, i.e. there is $z \in A$ with $\langle 0, z\rangle \in R$ and $\langle z, y\rangle \in R$. Applying (a) and Lemma 1 once more, we have

$$\langle 0, y\rangle = \langle b(z, b(z,0)), b(y, b(z,z))\rangle \in R$$

giving $y \in [0]_R$. We have $[0]_{R\cdot R} \subseteq [0]_R$.

Together, it implies $[0]_{\theta(R)} \subseteq [0]_R$.

$(b) \Rightarrow (a)$: Let $\mathcal{A} \in \mathcal{V}$ and $\phi, \psi \in \operatorname{Con} \mathcal{A}$. Clearly $\phi \cdot \psi$ and $\psi \cdot \phi$ are reflexive and compatible relations and $\theta(\phi \cdot \psi) = \theta(\psi \cdot \phi)$. By (b) we conclude

$$[0]_{\phi\cdot\psi} = [0]_{\theta(\phi\cdot\psi)} = [0]_{\theta(\psi\cdot\phi)} = [0]_{\psi\cdot\phi}$$

thus $\mathcal{A}$ is permutable at 0. □

Now, we can state our result:

Theorem 4 *Let $\mathcal{V}$ be a variety permutable at 0. The following are equivalent:*

(1) $\mathcal{V}$ is locally coherent;

(2) $\mathcal{V}$ has LCUT and is locally regular.

Proof (1) $\Rightarrow$ (2) by Theorem 2 and Theorem 3.

Prove (2) $\Rightarrow$ (1): Let $\mathcal{A} \in \mathcal{V}$, $\phi \in \operatorname{Con} \mathcal{A}$ and $C = [b]_\phi$ for some $b \in \mathcal{A}$. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ and $C \subseteq \mathcal{B}$. Consider the minimal congruence containing $C \times C$, i.e. $\theta(C \times C)$. It si trivial to show that

$$\theta(C \times C) = \theta(\{b\} \times C)$$

and $\theta(C \times C)$ has the class C. By local rngularity, we have

$$[0]_\phi = [0]_{\theta(\{b\}\times C)}\,.$$

Let $x \in [0]_\phi$. Then $\langle 0,x\rangle \in \theta(\{b\}\times C)$. Since Con $\mathcal{A}$ is compactly generated, there exist $c_1,\ldots,c_n \in C$ with

$$\langle 0,x\rangle \in \theta(b,c_1)\vee\cdots\vee\theta(b,c_n) = \theta(\langle b,c_1\rangle,\ldots,\langle b,c_n\rangle).$$

Since $\mathcal{V}$ is permutable at 0, we apply Lemma 2 to obtain

$$\langle 0,x\rangle \in R(\langle b,c_1\rangle,\ldots,\langle b,c_n\rangle).$$

By [4], there exists an n-ary polynomial φ over $\mathcal{A}$ with

$$0=\varphi(b,\ldots,b), \qquad x=\varphi(c_1,\ldots,c_n).$$

In account of LCUT, we conclude $\varphi(C)\subseteq\mathcal{B}$ thus also $x=\varphi(c_1,\ldots,c_n)\in\mathcal{B}$. Hence $[0]_\phi\subseteq\mathcal{B}$, i.e. $\mathcal{A}$ is locally coherent. □

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