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Fermat and Wilson Quotients for p-Adic Integers

Ladislav Skula

Abstract: Using the p-adic limit, the notions of Fermat and Wilson quotients for composite moduli are transferred to those for p-adic integers. Some theorems on these quotients are presented which in particular are analogous to results of Eisenstein, Lerch, Friedmann and Tamarkine.

Key Words: Fermat quotient, Euler quotient, Wilson quotient, p-adic numbers.
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1. Introduction

Let p be a prime and a an integer not divisible by p . As it is well-known the *Fermat quotient of p with base a* is the integer

$$q(a,p) = \frac{a^{p-1} - 1}{p}.$$

The first general statements on this quotient are due to Eisenstein ([E], 1850):

(E1) If p is odd, then

$$2q(2,p) = \sum_{n=1}^{p-1} \left(\frac{1}{n} \right)^{n-1} \sim \left(\frac{1}{p} \right)^{p-1}$$

$$\left(= - \frac{1}{p} \pmod{p} \right).$$

(E2) If u, v are integers and $p \nmid uv$, then

$$q(uv,p) = q(u,p) + q(v,p) \pmod{p}$$

(the "logarithmic property").

(E3) // $u, v \in \mathbb{Z}$ and $p \nmid u$, then

$$q(u + pv,p) = q(u,p) - \frac{v}{u} \pmod{p}.$$

As a corollary of (E3) we have for integers $a, b, p \nmid ab$:

$$a \equiv b \pmod{p^2} \implies q(a, p) \equiv q(b, p) \pmod{p}.$$

Thus by (E2) we can consider the function $q(\cdot, p)$ as a homomorphism from the multiplicative group $((\mathbf{Z}/p^2\mathbf{Z})^*, \cdot)$ into the additive group $(\mathbf{Z}/p\mathbf{Z}, +)$ of the respective residue class rings.

According to Euler's well-known theorem generalizing Fermat's little theorem we can define for relatively prime integers $m \geq 2$ and a the *Euler quotient* (or the *generalized Fermat quotient for composite moduli*) of m with base a by

$$q(a, m) = \frac{a^{\varphi(m)} - 1}{m}.$$

For this quotient similar laws are satisfied as (E1) - (E3). In [ADS1] the Fermat quotient for composite moduli m is investigated in more detail. Some formulas presented there for the case $m = p^n$ directly invite to use a limit process and to transfer this notion to the p -adic case. This is established in Section 3 in greater detail by means of the projective limit. In Section 4 Lerch's expression of the Fermat quotient is transferred to the p -adic case and in Section 6 the Friedmann-Tamarkine congruence is presented for the Fermat quotient for p -adic integers.

Similarly, the notion of the Wilson quotient is transferred to the p -adic case by means of the p -adic limit in Section 5. Here a theorem (Theorem 5.7) is derived presenting this " p -adic" Wilson quotient by means of the p -adic limit of expressions containing certain Bernoulli numbers.

The reader is referred for the basic facts on p -adic numbers to the book [BS] and for the theory of projective systems to the book [K].

2. Notations and Fundamental Assertions

Throughout this paper we will use the following notations:

- p a prime,
- n a positive integer,
- \mathbf{Z} the ring of (rational) integers,
- $\mathbf{Z}(n)$ the additive group of the ring of residue classes mod p^n , thus $\mathbf{Z}(n) = (\mathbf{Z}/p^n\mathbf{Z}, +)$,
- $\mathbf{Z}(n)^*$ the multiplicative group of the invertible elements of the ring of residue classes mod p^n , thus $\mathbf{Z}(n)^* = ((\mathbf{Z}/p^n\mathbf{Z})^*, \cdot)$,
- φ_n the canonical (ring) homomorphism from the ring $(\mathbf{Z}/p^{n+1}\mathbf{Z}, +, \cdot)$ onto the ring $(\mathbf{Z}/p^n\mathbf{Z}, +, \cdot)$; this homomorphism will be also considered as (group) homomorphism from the group $\mathbf{Z}(n+1)$ onto the group $\mathbf{Z}(n)$ or from the group $\mathbf{Z}(n+1)^*$ onto the group $\mathbf{Z}(n)^*$,
- $(\mathbf{Z}_p, +, \cdot)$ the ring of p -adic integers with p -adic topology,
- \mathbf{Z}_p the additive group of the ring $(\mathbf{Z}_p, +, \cdot)$,
- \mathbf{Z}_p^* the multiplicative group of the invertible elements of the ring $(\mathbf{Z}_p, +, \cdot)$,
- ψ_n the canonical (ring) homomorphism from the ring $(\mathbf{Z}_p, +, \cdot)$ onto the ring $(\mathbf{Z}/p^n\mathbf{Z}, +, \cdot)$, also considered as the group homomorphism from the group

\mathbf{Z}_p onto the group $\mathbf{Z}(n)$ or from the group \mathbf{Z}_p^* onto the group $\mathbf{Z}(n)^*$, thus for $\alpha = \sum_{i=0}^{\infty} a_i p^i \in \mathbf{Z}_p$ ($a_i \in \mathbf{Z}, 0 \leq a_i < p$) we have $\psi_n(\alpha) = \sum_{i=0}^{n-1} a_i p^i + p^n \mathbf{Z}$, $v(\alpha)$ the p -adic exponent of a p -adic integer α , $\lim_{k \rightarrow \infty} \alpha_k$ the p -adic limit for p -adic integers α_k , similarly all topological notions (continuity, convergence, infinite series, etc.) concern the p -adic topology, $q(a, p^n)$ the Fermat quotient of (composite moduli) p^n (the Euler quotient of p^n) with base a ($a \in \mathbf{Z}, p \nmid a$) (see, e.g. [ADS1]), thus $q(a, p^n) = \frac{a^{p^n-1(p-1)} - 1}{p^n}$.

Since for each p -adic integer α not divisible by p we have

$$\alpha^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n},$$

the p -adic number $q(\alpha, p^n) = \frac{\alpha^{p^{n-1}(p-1)} - 1}{p^n}$ is p -adic integer. In this way the former function $q(\cdot, p^n)$ is extended to all p -adic integers not divisible by p .

Proposition 2.1. (a) *The function $q(\cdot, p^n)$ is a uniformly continuous mapping from \mathbf{Z}_p^* into \mathbf{Z}_p .*

If we assume that $\alpha, \beta \in \mathbf{Z}_p^$, then we have:*

$$(b) \quad q(\alpha, p^n) \equiv q(\beta, p^n) \pmod{p^n}$$

provided that $\alpha \equiv \beta \pmod{p^{n+1}}$,

$$(c) \quad q(\alpha\beta, p^n) \equiv q(\alpha, p^n) + q(\beta, p^n) \pmod{p^n}.$$

Proof. For $\alpha, \beta \in \mathbf{Z}_p^*$ there exists $\gamma \in \mathbf{Z}_p$ such that $\alpha^{p^{n-1}(p-1)} - \beta^{p^{n-1}(p-1)} = (\alpha - \beta)\gamma$, hence

$$v(q(\alpha, p^n) - q(\beta, p^n)) \geq v(\alpha - \beta) - n.$$

This proves part (a). Part (b) is obvious and part (c) follows from (b) and the logarithmic property for the Fermat quotient of p^n . □

Notation. Let $A = a + p^{n+1}\mathbf{Z} \in \mathbf{Z}(n+1)^*$, $a \in \mathbf{Z}, p \nmid a$. Put $q_n(A) = q(a, p^n) + p^n \mathbf{Z} \in \mathbf{Z}(n)$. Using Proposition 2.1 (b), (c) we get that q_n is a group homomorphism from the group $\mathbf{Z}(n+1)^*$ into the group $\mathbf{Z}(n)$.

Proposition 2.2. (a) *Let p be an odd prime or $p = 2$ and $n = 1$. Then q_n is surjective and for $A \in \mathbf{Z}(n+1)^*$ we have $q_n(A) = 0$ if and only if $A^{p-1} = 1$.*

(b) *Let $p = 2$ and $n \geq 2$. Then $q_n(\mathbf{Z}(n+1)^*) = 2\mathbf{Z}(n)$ and for $A \in \mathbf{Z}(n+1)^*$ we have $q_n(A) = 0$ if and only if $A = \pm 1$.*

(The symbols 0 and 1 denote the zero element and the unity in the rings of residue classes mod p^n and mod p^{n+1} , respectively.)

Proof. In case (a) we have, by property (E3), $q(1 + p, p^n) \equiv -1 \pmod{p}$, hence $p \nmid q(1 + p, p^n)$. Using the logarithmic property of the Fermat quotient of p^n and the existence of a primitive root mod p^n we get that q_n is surjective and $q_n(A) = 0$ for an element $A \in \mathbf{Z}(n+1)^*$ if and only if $A^{p-1} = 1$.

For $n \geq 2$ we have $q(5, 2^n) = 2k$ for an odd integer k . Let $a \in \mathbf{Z}, 2 \nmid a$. Then there exists an integer x such that $0 \leq x \leq 2^{n-1} - 1$ and $a \equiv \pm 5^x \pmod{2^{n+1}}$, which implies $q(a, 2^n) \equiv xq(5, 2^n) \pmod{2^n} = 2kx$, and we are done. □

Proposition 2.3. *With exception of the case $p = 2$ and $n = 1$ we have*

$$q(\alpha, p^{n+1}) \equiv q(\alpha, p^n) \pmod{p^n}$$

for each $\alpha \in \mathbf{Z}_p^*$.

Proof. Since $2|q(\alpha, 2^n)$ for each odd α and $n \geq 2$, we get the proposition from Proposition 4.1 of ([ADS1]) for $\alpha \in \mathbf{Z}$ ($p \nmid \alpha$). Using Proposition 2.1 (b) we obtain the general case. \square

Immediately from this proposition we get:

Proposition 2.4. *If p is odd or $p = 2$ and $n \geq 2$, then the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{Z}(n+1)^* & \xrightarrow{q_n} & \mathbf{Z}(n) \\ \varphi_{n+1} \uparrow & & \uparrow \varphi_n \\ \mathbf{Z}(n+2)^* & \xrightarrow{q_{n+1}} & \mathbf{Z}(n+1) \end{array}$$

3. Fermat Quotient for p -Adic Integers

Let \mathcal{I} be the set of all positive integers in the case where p is odd and the set of all integers ≥ 2 in the case $p = 2$. For $m, n \in \mathcal{I}$, $m \geq n$ denote by φ_n^m the (group) homomorphism $\varphi_n^m = \varphi_n \circ \varphi_{n+1} \circ \dots \circ \varphi_m$. Then $\{\mathcal{I}, \mathbf{Z}(n), \varphi_n^m\}$ is a projective system whose projective limit is given by the family of (group) homomorphisms $\{\psi_n : \mathbf{Z}_p \rightarrow \mathbf{Z}(n) | n \in \mathcal{I}\}$. According to Proposition 2.4 the following diagram is commutative for each $n \in \mathcal{I}$:

$$\begin{array}{ccc} & & \mathbf{Z}(n) \\ & \nearrow q_n \circ \psi_{n+1} & \uparrow \varphi_n \\ \mathbf{Z}_p^* & & \\ & \searrow q_{n+1} \circ \psi_{n+2} & \\ & & \mathbf{Z}(n+1) \end{array}$$

Using the properties of the projective limit we can state:

Theorem 3.1. *There exists a unique continuous homomorphism q from the group \mathbf{Z}_p^* into the group \mathbf{Z}_p such that the following diagram is commutative for each $n \in \mathcal{I}$:*

$$\begin{array}{ccc} \mathbf{Z}(n+1)^* & \xrightarrow{q_n} & \mathbf{Z}(n) \\ \psi_{n+1} \uparrow & & \uparrow \psi_n \\ \mathbf{Z}_p^* & \xrightarrow{q} & \mathbf{Z}_p \end{array}$$

Definition 3.1 The mapping q in Theorem 3.1 will be called the *Fermat quotient for the p -adic integers*, or simply *p -adic Fermat quotient*.

Using the definition of q and Proposition 2.2 we can derive the following theorem:

Theorem 3.2. (a) For $p \neq 2$ the mapping q is surjective and for $\alpha \in \mathbf{Z}_p^*$ we have $q(\alpha) = 0$ if and only if $\alpha^{p-1} = 1$.

(b) If $p = 2$, then $q(\mathbf{Z}_2^*) = 2\mathbf{Z}_2$ and for $\alpha \in \mathbf{Z}_2^*$ we have $q(\alpha) = 0$ if and only if $\alpha = \pm 1$.

Theorem 3.3. (a) We have for each $\alpha \in \mathbf{Z}_p^*$ and each $n \in \mathcal{I}$

$$q(\alpha) \equiv q(\alpha, p^n) \pmod{p^n}.$$

(b) The sequence of mappings $\{q(\cdot, p^n)\}_{n=1}^\infty$ converges uniformly to the mapping q .

(c) The mapping q is uniformly continuous.

Proof. According to Proposition 2.3 there exists $\lim_{\nu \rightarrow \infty} q(\alpha, p^\nu)$ for each $\alpha \in \mathbf{Z}_p^*$, which will be denoted by $f(\alpha)$. By Proposition 2.1 (c) f is a homomorphism from the group \mathbf{Z}_p^* into the group \mathbf{Z}_p .

Assume that $n \in \mathcal{I}$ and $\alpha \in \mathbf{Z}_p^*$. Then there exists an integer $m \geq n$ such that $v(f(\alpha) - q(\alpha, p^m)) \geq n$. Using Proposition 2.3 we get

$$v(f(\alpha) - q(\alpha, p^n)) \geq \min\{v(f(\alpha) - q(\alpha, p^m)), v(q(\alpha, p^m) - q(\alpha, p^n))\} \geq n,$$

from which we obtain that the sequence $\{q(\cdot, p^n)\}_{n=1}^\infty$ converges uniformly to f and $f(\alpha) \equiv q(\alpha, p^n) \pmod{p^n}$.

Since the p -adic valuation v is non-Archimedean ([BS], Chapt. 1, Sec. 4, Ex. 4) and $q(\cdot, p^n)$ are uniformly continuous (Proposition 2.1 (a)), the mapping f is uniformly continuous.

It is easy to see (Proposition 2.1 (b)) that for each $n \in \mathcal{I}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{Z}(n+1)^* & \xrightarrow{q_n} & \mathbf{Z}(n) \\ \psi_{n+1} \uparrow & & \uparrow \psi_n \\ \mathbf{Z}_p^* & \xrightarrow{f} & \mathbf{Z}_p \end{array}$$

The result follows from the uniqueness of q . □

In the following theorem we use the symbol \log for the p -adic logarithm and we apply *Leopoldt's* formula ([Lp],(0))

$$\log H = \lim_{n \rightarrow \infty} \frac{H^{p^n} - 1}{p^n}$$

to the p -adic integer $H = \alpha^{p-1}$, where $\alpha \in \mathbf{Z}_p^*$:

Theorem 3.4. If $\alpha \in \mathbf{Z}_p^*$, then

$$q(\alpha) = \frac{\log \alpha^{p-1}}{p}.$$

4. Lerch's Expression for the Fermat Quotient

In his paper [Lr1] in 1905, Lerch presented the following expression for the Fermat quotient of an odd prime with base a ($a \in \mathbf{Z}, p \nmid a$):

$$(L1) \quad aq(a, p) \equiv \sum_{x=1}^{p-1} \frac{1}{x} \left[\frac{ax}{p} \right] \pmod{p}.$$

This form was generalized by Lerch in [Lr2] (1906) for Fermat quotients of composite moduli m ($m \in \mathbf{Z}, m \geq 2$) for base a ($a \in \mathbf{Z}, (m, a) = 1$):

$$(L2) \quad aq(a, m) = a \frac{a^{\varphi(m)} - 1}{m} \equiv \sum_{x=1}^m \frac{1}{x} \left[\frac{ax}{m} \right] \pmod{m}$$

(see [ADS1], Theorem 2.3 and Historical remarks, p.34).

To state an analogous formula for the Fermat quotient q we will define for a p -adic number $\xi = \sum_{i=-m}^{\infty} x_i p^i$ ($x_i \in \mathbf{Z}, 0 \leq x_i \leq p-1, m \in \mathbf{Z}, m \geq 0$) the integral part $[\xi]_p$ of ξ with respect to p by

$$[\xi]_p = \sum_{i=0}^{\infty} x_i p^i \in \mathbf{Z}_p.$$

Clearly, if $\omega \in \mathbf{Z}$, then $\left[\frac{\omega}{p^n} \right]_p = \left[\frac{\omega}{p^n} \right]$.

Theorem 4.1. *If $\alpha \in \mathbf{Z}_p^*$, then*

$$\alpha q(\alpha) = \lim_{\substack{\nu \rightarrow \infty \\ p \nmid x}} \sum_{x=1}^{p^\nu} \frac{1}{x} \left[\frac{\alpha x}{p^\nu} \right]_p.$$

Proof. Assume that $\alpha \in \mathbf{Z}_p^*$, $\beta \in \mathbf{Z}$ and $\alpha \equiv \beta \pmod{p^{2n}}$. Using Proposition 2.1 (b) and (L2) we get

$$\alpha q(\alpha, p^n) \equiv \sum_{\substack{x=1 \\ p \nmid x}}^{p^n} \frac{1}{x} \left[\frac{\beta x}{p^n} \right] \pmod{p^n}.$$

Since $\alpha x \equiv \beta x \pmod{p^{2n}}$ for each rational integer x , there exists $\gamma = \gamma(x) \in \mathbf{Z}_p$ such that $\frac{\alpha x}{p^n} = \frac{\beta x}{p^n} + p^n \gamma$, therefore $\left[\frac{\alpha x}{p^n} \right]_p \equiv \left[\frac{\beta x}{p^n} \right] \pmod{p^n}$ and

$$\alpha q(\alpha, p^n) \equiv \sum_{\substack{x=1 \\ p \nmid x}}^{p^n} \frac{1}{x} \left[\frac{\alpha x}{p^n} \right]_p \pmod{p^n}.$$

The result follows. □

Notation. For integers N, k ($N \geq 1, p \nmid N, 0 \leq k \leq N - 1$), put

$$s(k, N, n) = \sum_{\substack{\frac{p^n}{N}k < x < \frac{p^n}{N}(k+1) \\ p \nmid x}} \frac{1}{x}.$$

Then Lerch's formula (L2) for Fermat quotient of p^n for base N can be expressed in the following way:

$$Nq(N, p^n) \equiv \sum_{k=0}^{N-1} ks(k, N, n) \pmod{p^n}.$$

Thus we can state:

Theorem 4.2. *If N is a positive integer ($p \nmid N$), then*

$$Nq(N) = \lim_{\nu \rightarrow \infty} \sum_{k=0}^{N-1} ks(k, N, \nu).$$

Corollary 4.3. *Let $N \in \{1, 2, 3, 4, 6\}$, $0 \leq k \leq N - 1$ ($k \in \mathbf{Z}$). Then there exists $\lim_{\nu \rightarrow \infty} s(k, N, \nu) = s(k, N)$ and we have*

- (a) $s(0, 1) = 0,$
- (b) $s(1, 2) = -s(0, 2) = 2q(2),$
 $s(3, 4) = -s(0, 4) = 3q(2),$
 $s(1, 4) = -s(2, 4) = q(2),$ for $p \neq 2,$
- (c) $s(2, 3) = -s(0, 3) = \frac{3}{2}q(3),$
 $s(1, 3) = 0,$ for $p \neq 3,$
- (d) $s(5, 6) = -s(0, 6) = 2q(2) + \frac{3}{2}q(3),$
 $s(1, 6) = -s(4, 6) = 2q(2),$
 $s(3, 6) = -s(2, 6) = 2q(2) - \frac{3}{2}q(3),$ for $p \geq 5.$

Proof. The result follows from the congruence

$$s(k, N, n) \equiv -s(N - 1 - k, N, n) \pmod{p^n}$$

and from Theorem 4.2. □

Remark. For $N = 5$ or $N \geq 7$ ($N \in \mathbf{Z}$) the question which sequences

$$\{s(k, N, \nu)\}_{\nu=1}^{\infty}$$

are convergent ($0 \leq k \leq N - 1$) remains an open problem.

Lemma 4.4. *Let N be a positive integer, $p \nmid N$ and suppose there exists*

$$\lim_{n \rightarrow \infty} s(0, N, n) = \sigma.$$

Then

$$\sum_{\nu=0}^{\infty} \left(\sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x} \right) = \sigma.$$

Proof. Put

$$\sigma_{\nu} = \sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x}$$

for each non-negative integer ν . Then the n th partial sum of the series $\sum_{\nu=0}^{\infty} \sigma_{\nu}$ equals

$$\sum_{\nu=0}^{n-1} \sigma_{\nu} = \sum_{\substack{1 \leq x < p^n \\ p \nmid x}} \frac{1}{x} = s(0, N, n)$$

and the result follows. \square

Corollary 4.5.

(a) If $p \neq 2$, then

$$2q(2) = - \sum_{\nu=0}^{\infty} \left(\sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x} \right), 3q(2) = - \sum_{\nu=0}^{\infty} \left(\sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x} \right).$$

(b) If $p \neq 3$, then

$$3q(3) = -2 \sum_{\nu=0}^{\infty} \left(\sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x} \right).$$

(c) If $p \geq 5$, then

$$q(2^4 \cdot 3^3) = 4q(2) + 3q(3) = -2 \sum_{\nu=0}^{\infty} \left(\sum_{\substack{p^{\nu} < x < p^{\nu+1} \\ p \nmid x}} \frac{1}{x} \right).$$

5. Wilson Quotients for the p -Adic Case

Definition 5.1. Let $m \geq 2$ be an integer and $\varepsilon_m = -1$ if $m = 2, 4, p^\alpha$ or $2p^\alpha$ (p an odd prime and α a positive integer) and $\varepsilon_m = 1$ otherwise.

The integer

$$W(m) = \frac{1}{m} \left[\prod_{\substack{j=1 \\ (j,m)=1}}^m j - \varepsilon_m \right]$$

is called the *generalized Wilson quotient of m* (see [ADS2], Definition 2.1).

According to [ADS2], Propositions 3.1 and 3.2, we have

$$(5.1) \quad W(p^{n+1}) \equiv W(p^n) \pmod{p^{n-1}},$$

hence there exists $\lim_{n \rightarrow \infty} W(p^n)$.

Definition 5.2. Set

$$W = W_p = \lim_{n \rightarrow \infty} W(p^n)$$

and call the p -adic integer W_p the *Wilson quotient for the p -adic case*, or simply *p -adic Wilson quotient*.

Proposition 5.1. $v(W - W(p^n)) \geq n - 1$.

Proof. According to (5.1) we get for each integer $m > n$ the inequality $v(W(p^m) - W(p^n)) \geq n - 1$. There exists an integer $m > n$ such that $v(W - W(p^m)) \geq n - 1$, therefore

$$\begin{aligned} v(W - W(p^n)) &= v((W - W(p^m)) + (W(p^m) - W(p^n))) \geq \\ &\geq \min\{v(W - W(p^m)), v(W(p^m) - W(p^n))\} \geq n - 1 \end{aligned}$$

and we are done. □

Notation. For an integer $m \geq 2$ set

$$\sigma_1(m) = \sum_{\substack{a=1 \\ (a,m)=1}}^m q(a, m), \quad \sigma_2(m) = \sum_{\substack{a=1 \\ (a,m)=1}}^m \sum_{\substack{b=a+1 \\ (b,m)=1}}^m q(a, m)q(b, m).$$

Further let

$$\bar{\varepsilon}_p = \begin{cases} -1 & \text{if } p \text{ is odd} \\ 1 & \text{if } p = 2 \end{cases} \quad (= \varepsilon_{p^3}) \text{ and } c(n) = \varphi(p^n) = p^{n-1}(p - 1).$$

As usual the n -th *Bernoulli number* will be denoted by B_n ($B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \dots$).

In the following proposition the congruence mod m in Proposition 2.1 of [ADS2] is extended to mod m^2 using the same method of the proof.

Proposition 5.2. For integers $m \geq 3$ we have

$$\varepsilon_m \varphi(m)W(m) + \binom{\varphi(m)}{2} mW(m)^2 \equiv \sigma_1(m) + m\sigma_2(m) \pmod{m^2}.$$

Proof. The result follows from observing that

$$\begin{aligned} \left(\prod_{\substack{j=1 \\ (j,m)=1}}^m j \right)^{\varphi(m)} &= (\varepsilon_m + mW(m))^{\varphi(m)} \equiv \varepsilon_m^{\varphi(m)} + \varphi(m)\varepsilon_m^{\varphi(m)-1}mW(m) + \\ &+ \binom{\varphi(m)}{2} \varepsilon_m^{\varphi(m)-2}m^2W(m)^2 \pmod{m^3} = \\ &= 1 + \varepsilon_m \varphi(m)mW(m) + \binom{\varphi(m)}{2} m^2(W(m))^2 \end{aligned}$$

and also, by the definition of $q(a, m)$,

$$\begin{aligned} \left(\prod_{\substack{j=1 \\ (j,m)=1}}^m j \right)^{\varphi(m)} &= \prod_{\substack{a=1 \\ (a,m)=1}}^m (1 + mq(a, m)) \equiv \\ &\equiv 1 + m\sigma_1(m) + m^2\sigma_2(m) \pmod{m^3}. \quad \square \end{aligned}$$

Proposition 5.3. $\lim_{n \rightarrow \infty} \sigma_1(p^n) = 0$.

Proof. Using Proposition 5.2 we get $v(\sigma_1(p^n)) \geq n - 1$ and the result follows. \square

Theorem 5.4. $\sum_{\substack{\nu=1 \\ p \nmid a}}^{\infty} \left(\sum_{a=p^{\nu-1}+1}^{p^{\nu}} q(a) \right) = 0$.

Proof. Let $n \geq 2$. According to Theorem 3.3 (a) we have for each integer a ($p \nmid a$) $v(q(a) - q(a, p^n)) \geq n$, therefore

$$\begin{aligned} v \left(\sum_{\substack{a=1 \\ p \nmid a}}^{p^n} q(a) - \sigma_1(p^n) \right) &= v \left(\sum_{\substack{a=1 \\ p \nmid a}}^{p^n} [q(a) - q(a, p^n)] \right) \geq \\ &\geq \min\{v(q(a) - q(a, p^n)) : 1 \leq a \leq p^n, a \in \mathbf{Z}, p \nmid a\} \geq n. \end{aligned}$$

The result follows from Proposition 5.3. \square

Proposition 5.5.

$$W_p = \bar{\varepsilon}_p \frac{p}{p-1} \lim_{n \rightarrow \infty} \left(\frac{\sigma_1(p^n)}{p^n} + \sigma_2(p^n) \right) = \frac{\bar{\varepsilon}_p}{p-1} \lim_{n \rightarrow \infty} \left(\frac{\sigma_1(p^n)}{p^{n-1}} + p\sigma_2(p^n) \right).$$

Proof. If we substitute for m the power p^n ($n \geq 3$) in the congruence of Proposition 5.2, we get

$$\bar{\varepsilon}_p(p-1)W(p^n) \equiv \frac{\sigma_1(p^n)}{p^{n-1}} + p\sigma_2(p^n) \pmod{p^{n-1}}.$$

Using Proposition 5.1 we get the result. \square

For the proof of Theorem 5.7 we will need the following lemma:

Lemma 5.6. *Let t be a positive integer and $n \geq 5$. Then*

$$\sum_{\substack{a=1 \\ p \nmid a}}^{p^n} a^{tc(n)} \equiv B_{tc(n)} p^n \pmod{p^{3n-1}}.$$

Proof. For the sake of simplicity put $c = c(n)$ and $m = p^n$. Then by a well-known identity for Bernulli numbers,

$$\sum_{a=1}^{m-1} a^{tc} = \frac{1}{tc+1} \sum_{k=0}^{tc} \binom{tc+1}{k} B_k m^{tc+1-k}.$$

Since for $0 \leq k \leq tc - 2$ the inequality $v(B_k m^{tc+1-k}) \geq 3n - 1$ is satisfied by the von Staudt-Clausen theorem, we get

$$\sum_{a=1}^{m-1} a^{tc} \equiv \frac{1}{tc+1} \left(\binom{tc+1}{1} B_{tc} p^n + \binom{tc+1}{2} B_{tc-1} p^{2n} \right) \pmod{p^{3n-1}}.$$

The integer $tc - 1$ is odd and greater than 3, hence $B_{tc-1} = 0$. If a is an integer divisible by p , then $v(a^{tc}) \geq tc \geq 2^{n-1} \geq 3n + 1$. The result follows. \square

Theorem 5.7.

$$W_p = -\bar{\epsilon}_p \frac{p}{2(p-1)} \lim_{n \rightarrow \infty} \frac{1}{p^n} \left(B_{2c(n)} - 4B_{c(n)} + \frac{3(p-1)}{p} \right).$$

Proof. Put $\gamma(n) = \sum_{\substack{a=1 \\ p \nmid a}}^{p^n} q(a, p^n)^2$ and $c = c(n)$. According to Lemma 5.6 we have for $n \geq 5$

$$\gamma(n) = \frac{1}{p^{2n}} \sum_{\substack{a=1 \\ p \nmid a}}^{p^n} (a^{2c} - 2a^c + 1) = \frac{1}{p^n} \left(B_{2c} - 2B_c + \frac{p-1}{p} \right) + x_n p^{n-1}$$

and

$$\sigma_1(p^n) = \frac{1}{p^n} \sum_{\substack{a=1 \\ p \nmid a}}^{p^n} (a^c - 1) = B_c - \frac{p-1}{p} + y_n p^{2n-1},$$

where x_n and y_n are (rational) integers.

Further

$$\sigma_1(p^n)^2 = \gamma(n) + 2\sigma_2(p^n),$$

therefore

$$\begin{aligned} \frac{\sigma_1(p^n)}{p^n} + \sigma_2(p^n) &= \frac{1}{p^n} \left(B_c - \frac{p-1}{p} \right) + y_n p^{n-1} + \frac{\sigma_1(p^n)^2}{2} - \frac{\gamma(n)}{2} = \\ &= \frac{1}{2p^n} \left(2B_c - \frac{2(p-1)}{p} - B_{2c} + 2B_c - \frac{p-1}{p} \right) + (y_n - \frac{x_n}{2}) p^{n-1} + \frac{\sigma_1(p^n)^2}{2} = \\ &= \frac{-1}{2p^n} \left(B_{2c} - 4B_c + \frac{3(p-1)}{p} \right) + (y_n - \frac{x_n}{2}) p^{n-1} + \frac{\sigma_1(p^n)^2}{2}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} (y_n - x_n) p^{n-1} = 0$ and $\lim_{n \rightarrow \infty} \frac{\sigma_1(p^n)^2}{2} = 0$ (by Proposition 5.3), the proof is complete according to Proposition 5.5. \square

6. Friedmann-Tamarkine Congruence

In their paper [FT] (1909) *Friedmann* and *Tamarkine* proved for an odd prime p and an integer m ($3 \leq m \leq p-2$) the following congruence:

$$(FT) \quad \sum_{a=1}^{p-1} a^m q(a, p) \equiv -\frac{1}{m} B_m \pmod{p}.$$

Note that the congruences of this kind were given by *Lerch* ([Lr1], 1905) for special m .

$$(4), \quad m=0: \quad \sum_{a=1}^{p-1} q(a, p) \equiv W(p) \pmod{p},$$

$$(17), \quad m=1: \quad \sum_{a=1}^{p-1} a q(a, p) \equiv \frac{1}{2} \pmod{p},$$

$$(24), \quad m=2: \quad \sum_{a=1}^{p-1} a^2 q(a, p) \equiv -\frac{1}{12} \pmod{p}, (p \neq 3),$$

$$(18), \quad m = \frac{p-1}{2}: \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a q(a, p) \equiv 0 \pmod{p}, (p \equiv 3 \pmod{4}),$$

$$(21), \quad m = \frac{p-1}{2}: \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) q(a, p) \equiv 2B_{\frac{p-1}{2}} \pmod{p}, (p \equiv 1 \pmod{4}),$$

$$(22^1), \quad m = \frac{p+1}{2}: \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a q(a, p) \equiv 0 \pmod{p}, (p \equiv 1 \pmod{4}),$$

$$(22^2), \quad m = \frac{p+1}{2}: \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a q(a, p) \equiv C\ell(-p) \pmod{p}, (p \equiv 3 \pmod{4}),$$

where $C\ell(-p)$ is the number of divisor classes of the quadratic field $\mathbf{Q}(\sqrt{-p})$ and $C\ell(-p) \equiv -2B_{\frac{p+1}{2}} \pmod{p}$ for $p > 3$ ([BS], Chap. 5, Sec. 8, Problem 4).

The aim of this section is to transfer the congruence (FT) to the p -adic case for the Fermat quotient q . For $m=0$ the modified relation was expressed by Theorem 5.4.

Further we will assume that p is an odd prime and for a positive integer N set

$$S_N(n) = 1^N + 2^N + \cdots + (n-1)^N.$$

To prove the main theorem of this section we will state some lemmas.

Lemma 6.1. *If μ, ν are positive integers, $\nu \geq v(\mu + 1)$, then*

$$S_\mu(p^\nu) \equiv B_\mu p^\nu \pmod{p^{2\nu-1}}.$$

Proof. Put $x = v(\mu + 1)$. Since $v(B_k) \geq -1$ by the von Staudt-Clausen theorem, we have for $0 \leq k \leq \mu - 2$ (k an integer):

$$v\left(\frac{1}{\mu + 1} \binom{\mu + 1}{k} B_k p^{\nu(\mu+1-k)}\right) \geq -x - 1 + 3\nu \geq 2\nu - 1,$$

therefore, as in the proof of Lemma 5.6,

$$\begin{aligned} S_\mu(p^\nu) &= \frac{1}{\mu + 1} \sum_{k=0}^{\mu} \binom{\mu + 1}{k} B_k p^{\nu(\mu+1-k)} \equiv \\ &\equiv \frac{1}{\mu + 1} \left(\binom{\mu + 1}{\mu} B_\mu p^\nu + \binom{\mu + 1}{\mu - 1} B_{\mu-1} p^{2\nu} \right) \pmod{p^{2\nu-1}} \equiv \\ &\equiv B_\mu p^\nu \pmod{p^{2\nu-1}}. \end{aligned}$$

□

In Lemmas 6.2 and 6.3 we assume that m is a positive integer, $m \not\equiv 0 \pmod{p-1}$, ν is an integer, $\nu \geq v(m + 1) + 2$ and $M = m + p^{\nu-1}(p - 1)$.

Lemma 6.2. $\frac{1}{p^\nu}(S_M(p^\nu) - S_m(p^\nu)) \equiv -p^{m-1}B_m \pmod{p^{\nu-1}}$.

Note that $\nu \geq v(m + 1) = v(M + 1)$, hence the numbers $\frac{1}{p^\nu}S_M(p^\nu)$ and $\frac{1}{p^\nu}S_m(p^\nu)$ are integers by Lemma 6.1.

Proof. We will use Kummer's congruence for the Bernoulli numbers modulo a prime power ([W], Corollary 12.3, p. 241):

If $m \equiv M \pmod{p^{\nu-1}(p - 1)}$, and $m \not\equiv 0 \pmod{p - 1}$, then

$$(1 - p^{m-1})\frac{B_m}{m} \equiv (1 - p^{M-1})\frac{B_M}{M} \pmod{p^\nu}.$$

Since $M - 1 \geq p^{\nu-1}(p - 1) \geq 3^{\nu-1} \cdot 2 \geq \nu$, we have $(1 - p^{M-1})\frac{B_M}{M} \equiv \frac{B_M}{M} \pmod{p^{\nu-1}}$, therefore

$$B_M \equiv (1 - p^{m-1})B_m \pmod{p^{\nu-1}}.$$

Using Lemma 6.1 we get

$$\frac{1}{p^\nu}(S_M(p^\nu) - S_m(p^\nu)) \equiv B_M - B_m \equiv -p^{m-1}B_m \pmod{p^{\nu-1}}. \quad \square$$

Lemma 6.3.

$$\sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m q(a, p^\nu) \equiv 0 \pmod{p^{\nu-2}}.$$

Proof. For an integer a divisible by p we have $v(q^M) \geq M > p^{\nu-1}(p-1) \geq 3^{\nu-1} \cdot 2 \geq 2\nu$, hence

$$\sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^M \equiv S_M(p^\nu) \pmod{p^{2\nu}}$$

and there exists $A \in \mathbf{Z}$ such that

$$(6.1) \quad \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^M = S_M(p^\nu) + A \cdot p^{2\nu}.$$

Since $S_m(p^\nu) = \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m + p^m \sum_{a=1}^{p^{\nu-1}-1} a^m$, we have

$$(6.2) \quad \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m = S_m(p^\nu) - p^m \zeta_m(p^{\nu-1}),$$

and using Lemma 6.1 we get $S_m(p^{\nu-1}) \equiv B_m p^{\nu-1} \pmod{p^{2\nu-3}}$. Therefore there exists a p -adic integer C such that

$$(6.3) \quad p^m S_m(p^{\nu-1}) = p^{m+\nu-1} B_m + p^{m+2\nu-3} C.$$

Summarizing (6.1) - (6.3) we obtain

$$\begin{aligned} \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m q(a, p^\nu) &= \frac{1}{p^\nu} \left(\sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^M - \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m \right) = \\ &= \frac{1}{p^\nu} (S_M(p^\nu) + A p^{2\nu} - S_m(p^\nu) + p^{m+\nu-1} B_m + p^{m+2\nu-3} C) \equiv \\ &\equiv \frac{1}{p^\nu} (S_M(p^\nu) - S_m(p^\nu)) + p^{m-1} B_m \pmod{p^{\nu-2}} \equiv \\ &\equiv 0 \pmod{p^{\nu-2}} \end{aligned}$$

according to Lemma 6.2. \square

Theorem 6.4. *If p is an odd prime and m a positive integer, $m \not\equiv 0 \pmod{p-1}$, then*

$$\sum_{\nu=1}^{\infty} \left(\sum_{\substack{a=p^{\nu-1}+1 \\ p \nmid a}}^{p^\nu} a^m q(a) \right) = 0.$$

Proof. For a positive integer ν put

$$A(\nu) = \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m q(a) \text{ and } B(\nu) = \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m q(a, p^\nu).$$

According to Theorem 3.3 (a) we have $v(A(\nu) - B(\nu)) \geq \nu$.

If $\nu \geq v(m+1) + 2$, then by Lemma 6.3 $v(B(\nu)) \geq \nu - 2$, therefore $v(A(\nu)) = v(A(\nu) - B(\nu) + B(\nu)) \geq \min\{v(A(\nu) - B(\nu)), v(B(\nu))\} \geq \nu - 2$ (for $\nu \geq v(m+1) + 2$).

This proves $\lim_{\nu \rightarrow \infty} A(\nu) = 0$ and the proof is complete. \square

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