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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 9 (2001), No. 1, 59--66

Persistent URL: `http://dml.cz/dmlcz/120571`

Fuzzy Functions in Fuzzy Logic with Fuzzy Equality

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Abstract: We are interested in the problem, under which conditions approximation of a function can be syntactically characterized in fuzzy logic in narrow sense with evaluated syntax (FLn). The approximation corresponds to the provability degree and we use a special case of disjunctive or conjunctive normal forms based on the fuzzy equality. We show that when assuming special properties of the latter, we obtain a finitary characterization, given a prescribed accuracy. The interpretation of the disjunctive normal form is precisely the Mamdani-Assilian formula used in fuzzy control.

Key Words: fuzzy logic in narrow sense, fuzzy equality, logical approximation

Mathematics Subject Classification: 03B52

1. Preliminaries

In this paper, we deal with fuzzy logic in narrow sense with evaluated syntax presented extensively in the book [6] (known also under the name "Pavelka logic"). In this section, we will briefly outline some of the concepts and notation. All what is left unexplained can be found in the above book.

The set L of truth values is supposed to form the Łukasiewicz MV-algebra

$$\mathcal{L}_t = \langle [0,1], \otimes, \oplus, \neg, 0, 1\rangle$$

where $\otimes$ is Łukasiewicz product ($a \otimes b = 0 \vee (a + b - 1)$), $\oplus$ is Łukasiewicz sum ($a \oplus b = 1 \wedge (a + b)$) and $\neg$ is negation ($\neg a = 1 - a$). The set of all the well-formed formulas for the language J is denoted by F_J and the set of all the closed terms by M_J. If T is a fuzzy theory then its language is denoted by $J(T)$.

Let $A(x_1, \dots, x_n)$ be a formula and $t_1, \dots, t_n$ be terms substitutable into A for the variables $x_1, \dots, x_n$, respectively. Then $A_{x_1,\dots,x_n}[t_1, \dots, t_n]$ is an instance of A resulting from it when replacing all the free occurrences of the variables $x_1, \dots, x_n$ by the respective terms $t_1, \dots, t_n$.

If $\mathcal{V}$ is a structure for the language J then $\mathcal{V}(t) = v \in V$ is an element being the interpretation of the term t. By alternative notation, given a free variable x then

This paper has been supported by Grant A1187901/99 of the GA AV ČR and the project ME468 of MŠMT ČR.

v/x is an element assigned to x in the structure $\mathcal{V}$. If $P \in J$ is a predicate symbol then its interpretation in $\mathcal{V}$ is denoted by $P_{\mathcal{V}}$ and similarly for functional symbols.

The satisfaction fuzzy relation of $A(x_1, \dots, x_n) \in F_J$ in $\mathcal{V}$ is

$$A_{\mathcal{V}} = \left\{ {}^{a}/\langle v_1, \dots v_n \rangle \middle| \, a = \mathcal{V}(A(v_1/x_1, \dots, v_n/x_n)), v_1, \dots, v_n \in V \right\} \underset{\sim}{\subseteq} V^n \tag{1}$$

where the notation ${}^{a}/v$ means that an element v taken from some universe belongs to the given fuzzy set in the degree $a \in L$.

The following theorem will often be used. Its proof can be found in [6].

Theorem 1. (equivalence) *Let T be a fuzzy theory, A be a formula and $B_1, \dots, B_n$ some of its subformulas. Let $T \vdash_{a_i} B_i \Leftrightarrow B_i'$, $i = 1, \dots, n$. Then there are $m_1, \dots, m_n$ such that*

$$T \vdash_b A \Leftrightarrow A', \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n} \tag{2}$$

where A' is a formula which is a result of replacement of the formulas $B_1, \dots, B_n$ in A by $B_1', \dots, B_n'$.

The language of fuzzy logic used in this section is supposed to contain the fuzzy equality predicate $\approx$ fulfilling the following logical axioms (cf. [6]):

(E1) ${}^{1}/(x \approx x)$

(E2)

$$^{1}/((x_1 \approx y_1) \Rightarrow (\dots \Rightarrow ((x_n \approx y_n) \Rightarrow \\ \Rightarrow (f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n)) \dots)$$

for every n-ary functional symbol f.

(E3)

$$^{1}/((x_1 \approx y_1) \Rightarrow (\dots \Rightarrow ((x_n \approx y_n) \Rightarrow \\ \Rightarrow (P(x_1, \dots, x_n) \Rightarrow P(y_1, \dots, y_n)) \dots)$$

for every n-ary predicate symbol P.

The following lemma is a direct consequence of axioms (E1)–(E3).

Lemma 1. *Let T be a fuzzy predicate calculus with fuzzy equality. Then the following properties of the fuzzy equality are provable in T (in the degree 1):*

(a) Symmetry

$$T \vdash (x \approx y) \Rightarrow (y \approx x),$$

(b) transitivity

$$T \vdash ((x \approx y) \,\&\, (y \approx z)) \Rightarrow (x \approx z).$$

Let $A(x) := x \approx t$ where t is a closed term, which is assigned an element $v_0 = \mathcal{V}(t)$. Then its satisfaction fuzzy set in $\mathcal{V}$ is

$$A_{\mathcal{V}} = \left\{ {}^{a}/v \middle| \, a = \mathcal{V}(v/x \approx v_0/t), v \in V \right\}.$$

Fuzzy equality determines metrical properties of models as can be seen from the following lemma.

Lemma 2. *Let T be a fuzzy theory with a fuzzy equality $\approx$ and $\mathcal{V} \models T$ be its model. Put $\rho(u,v) = \mathcal{V}(\neg(u/x \approx v/y))$ for all $u, v \in V$. Then ρ is a pseudometrics on V for every $m \geq 1$.*

Proof. Obviously, ρ is a function $\rho : V^2 \to [0,1]$. Then axiom (E1) gives the property $\rho(u,u) = 0$ and Lemma 1 gives symmetry and the triangular inequality of ρ by the properties of Łukasiewicz MV-algebra. ◇

The pseudometrics ρ defined in a model $\mathcal{V}$ via fuzzy equality will be denoted by $\rho_{\approx}$. Thus, $\langle V, \rho_{\approx}\rangle$ is a pseudometric space. Consequently, all models of a fuzzy theory with fuzzy equality form a topological space with topology generated by $\rho_{\approx}$.

In the following definition, we introduce a kind of compactness of the fuzzy equality.

Definition 1. Let T be a consistent fuzzy theory with fuzzy equality $\approx$. We say that $\approx$ is totally bounded in T if to every a, $1 > a > 0$ there is $b > a$ and a finite number of closed terms $t_1, \dots, t_m$ such that

$$T \vdash_b (\forall x)(x \approx t_1 \vee \cdots \vee x \approx t_m). \tag{3}$$

Theorem 2. *Let T be a consistent fuzzy theory with fuzzy equality $\approx$. Then there is a consistent fuzzy theory T' being extension of T such that $\approx$ is totally bounded in T'.*

Proof. Let $\mathcal{V} \models T$ be a model. Then $\langle V, \rho_{\approx}\rangle$ is a pseudometric space. If the topological space generated by it is not compact, we pointwise compactify it to $\langle V^*, \rho_{\approx}\rangle$. In this case, we choose $v_0 \in V$ and put $P_{\mathcal{V}}(\infty) = P_{\mathcal{V}}(v_0)$ for all $P \in J(T)$ and similarly for functions.

Let $a \in (0,1)$. Find a rational $a' > a$, put $\varepsilon = \neg a'$ and in standard way using coverings of V by open balls $K(v, \varepsilon)$ where $v \in V$ let us find a countable subset $\bar{V} \subset V$. Let us extend $J(T)$ by a countable set of constants U and for every $u \in U$ use a bijection $f : U \to \bar{V}$ to put $\mathcal{V}(u) = f(u) = u$.

With respect to the above construction, to every $\varepsilon = \neg a'$ ($a' \in (0,1)$ rational) there is a finite set of constants $u_1, \dots, u_m \in U$ such that their corresponding interpretations $u_1, \dots, u_m$ form an ε-net in V. Let us put

$$\beta = \bigvee_{v \in V} \bigwedge_{j=1}^{m} \rho_{\approx}(v, u_j).$$

Because to every $v \in V$ there is u_j such that $\rho_{\approx}(v, u_j) < \varepsilon$, we obtain $\beta \leq \varepsilon$. Put $b = \neg\beta$. Then $b \geq a' > a$. Define a new fuzzy theory

$$T_{a'} = T \cup \{{}^{b}/(\forall x)(x \approx u_1 \vee \cdots \vee x \approx u_m)\}$$

and put

$$T' = \bigcup_{\substack{a' \in (0,1) \\ a' \text{ rational}}} T_{a'}.$$

We will demonstrate that T' is consistent and has the required property.

It follows from the above construction that to every $a \in (0,1)$ there is b, $b > a$ and a formula $(\forall x)(x \approx u_1 \vee \cdots \vee x \approx u_m)$, which is a special axiom of the fuzzy theory $T_{a'}$ in the degree $b \geq a'$ where a' is a rational number $a' > a$. Moreover,

$$\mathcal{V}((\forall x)(x \approx u_1 \vee \cdots \vee x \approx u_m)) = \bigwedge_{v \in V} \bigvee_{j=1}^{m} \neg \rho_{\approx}(v, u_j) = \neg \beta = b. \tag{4}$$

Thus, $\mathcal{V} \models T_{a'}$ and consequently, $\mathcal{V} \models T'$, i.e. T' is consistent. By the completeness theorem we conclude that

$$T' \vdash_b (\forall x)(x \approx u_1 \vee \cdots \vee x \approx u_m).$$

i.e. T' is the required extension of T. ◇

The following theorem has been proved in [5].

Theorem 3. *Let T be a fuzzy theory with totally bounded fuzzy equality $\approx$ and $A(x)$ be a formula. Then to every $0 < c < 1$ there are terms $t_1, \ldots, t_m$ and $d > c$ such that*

(a)

$$T \vdash_d (\exists x)A(x) \Leftrightarrow \bigvee_{j=1}^{m} A_x[t_j], \tag{5}$$

(b)

$$T \vdash_d (\forall x)A(x) \Leftrightarrow \bigwedge_{j=1}^{m} A_x[t_j]. \tag{6}$$

By this theorem, if a fuzzy theory contains a totally bounded fuzzy equality then every existential (universal) formula can be approximated by finite disjunction (conjunction) of instances of certain closed instances of the matrix of the given formula.

2. Fuzzy functions and fuzzy equality

F. Klawonn and R. Kruse in [3] have shown that the well known Mamdani-Assilian formula, which is used in fuzzy control, is closely related to the concept of fuzzy equality. I. Perfilieva has studied in [6], [8], [9] approximation properties of normal forms of fuzzy logic formulas when interpreted in special models. We will show using formal means of FLn that the Mamdani-Assilian formula, which is a special case of disjunctive normal form has an interpretation being a fuzzy relation, which approximates a fuzzy set of elements being fuzzy-equal to functional values of a given function. Similarly, we may also consider a special case of conjunctive normal form and demonstrate using formal means, how these forms are interrelated. Let us remark that our results are similar to those obtained by P. Hájek in [1].

By a fuzzy function, we understand a generalization of a classical function, which consists of the fuzzy set of all values *approximately equal* to the functional values of some crisp function $g(x)$. In logic, it is specified by a formula $F(x,y) := y \approx g(x)$, for which we can prove by the transitivity of $\approx$ that $T \vdash (F(x,y) \,\&\, F(x,y')) \Rightarrow y \approx y'$, and so the formula $F(x,y)$ determines a *fuzzy function*. Note that $F(x,y)$ is also extensional, i.e.

$$T \vdash F(x,y) \,\&\, y \approx y' \Rightarrow F(x,y').$$

Recall from [5] that the approximation problem leads to provability degree of equivalence of some formulas. Namely, if

$$T \vdash_d A(x_1, \dots, x_n) \Leftrightarrow B(x_1, \dots, x_n)$$

in some fuzzy theory T then it means that in every model $\mathcal{V} \models T$, the corresponding satisfaction fuzzy relations differ for not more than $\neg d = 1 - d$, i.e.

$$|A_{\mathcal{V}}(u_1, \dots, u_n) - B_{\mathcal{V}}(v_1, \dots, v_n)| \leq \neg d$$

holds for all $u_1, v_i \in V$, $i = 1, \dots, n$.

We will be interested in the characterization of some function represented by a functional symbol $g(x)$. This can be obtained using two kinds of formulas:

$$(\exists u)((x \approx u) \wedge (y \approx g(u))), \tag{7}$$

$$(\forall u)((x \approx u) \Rightarrow (y \approx g(u))). \tag{8}$$

The relation between the formulas is characterized by the following theorem.

Theorem 4. *The following formulas are provable in every fuzzy theory T containing the fuzzy equality predicate:*

$$T \vdash (\exists u)(((x \approx u) \wedge (y \approx g(u))) \Rightarrow (\forall u)((x \approx u) \Rightarrow (y \approx g(u)))), \tag{9}$$

$$T \vdash (\forall u)((x \approx u)^2 \Rightarrow (\exists u)((x \approx u) \wedge (y \approx g(u))) \Leftrightarrow (\forall u)((x \approx u) \Rightarrow (y \approx g(u)))). \tag{10}$$

Proof. The provability of formulas can be checked semantically (by the completeness). For this, the transitivity of fuzzy equality and axiom (E2) should be used. ◇

Theorem 5. *Let T be a consistent fuzzy theory with a totally bounded fuzzy equality $\approx$. Then to every $0 < c < 1$ there are terms $t_1, \dots, t_m$ and $d > c$ such that*

(a)

$$T \vdash_d (\exists u)((x \approx u) \wedge (y \approx g(u))) \Leftrightarrow \bigvee_{j=1}^{m} ((x \approx t_j) \wedge (y \approx g(t_j))), \tag{11}$$

(b)

$$T \vdash_d (\forall u)((x \approx u) \Rightarrow (y \approx g(u))) \Leftrightarrow \bigwedge_{j=1}^{m} ((x \approx t_j) \Rightarrow (y \approx g(t_j))). \tag{12}$$

Proof. This is a corollary of Theorems 4 and 3. ◇

This theorem demonstrates that if we deal with a fuzzy theory containing the totally bounded fuzzy equality then we can approximate the general formulas (7) and (8) by finitary ones which do not contain quantifiers.

The formula

$$MA(x, y) := \bigvee_{j=1}^{m} ((x \approx t_j) \wedge (y \approx g(t_j)))$$

is a special version of disjunctive normal form and it will be called the *Mamdani-Assilian formula*. The second possibility is a conjunctive normal form

$$\bigwedge_{j=1}^{m} ((x \approx t_j) \Rightarrow (y \approx g(t_j))),$$

which has approximation properties analogous to the Mamdani-Assilian one in a sense precisely defined below.

Lemma 3. *Let T be a fuzzy theory with fuzzy equality. Then*

(a) $T \vdash (y \approx g(x)) \Leftrightarrow (\exists u)((x \approx u) \wedge (y \approx g(u)))$,

(b) $T \vdash (y \approx g(x)) \Rightarrow (\forall u)((x \approx u) \Rightarrow (y \approx g(u)))$.

(c) $T \vdash (\forall x)(\forall y)(\forall u)((x \approx u)^2 \Rightarrow ((\exists u)(x \approx u) \wedge (y \approx g(u))) \Leftrightarrow ((x \approx u) \Rightarrow (y \approx g(u))))$

Proof. (a) and (b) follow from the transitivity of $\approx$, possibility to introduce conservatively a new function symbol (cf. [4]) and the equivalence theorem. (c) can be verified semantically using the completeness theorem. ◇

Theorem 6. *Let T be a consistent fuzzy theory with a totally bounded fuzzy equality $\approx$. To every $0 < c < 1$ there are terms $t_1, \dots, t_m$ and $d > c$ such that*

(a)

$$T \vdash_d (y \approx g(x)) \Leftrightarrow \bigvee_{j=1}^{m} ((x \approx t_j) \wedge (y \approx g(t_j))), \tag{13}$$

(b)

$$T \vdash_d (y \approx g(x)) \Rightarrow \bigwedge_{j=1}^{m} ((x \approx t_j) \Rightarrow (y \approx g(t_j))), \tag{14}$$

(c)

$$T \vdash_d \bigwedge_{j=1}^{m} ((x \approx t_j)^2 \Rightarrow ((y \approx g(x)) \Leftrightarrow ((x \approx t_j) \Rightarrow (y \approx g(t_j))))). \tag{15}$$

Proof. The theorem follows from Lemma 3, Theorem 5 and the equivalence theorem. ◇

According to this theorem, to every c we can find $d > c$ and the Mamdani-Assilian formula, which d-approximates the fuzzy function determined by the formula $y \approx g(x)$. Let us stress that this is a disjunction of conjunctions. If we want to replace it by a conjunction of implications, then the answer is given in items (b) and (c). Namely, both formulas become equivalent if we consider the square (in the sense of Łukasiewicz conjunction) of the assumption that only selected points in the domain of function g are considered. Note that this result is very close to that obtained by P. Hájek for basic logic in [1].

The main role in fuzzy control is played especially by the Mamdani-Assilian formula. Then our results provide the following: given a model $\mathcal{V} \models T$, let a function $g_{\mathcal{V}}$ be assigned to g. Then the fuzzy function $F_{\mathcal{V}}$ determined by $F(x, y)$ characterizes all elements $v \in V$, which are "close" to the functional values $g_{\mathcal{V}}(u)$ for all $u \in V$. If $\approx$ is totally bounded then, given some precision $1 > \varepsilon > 0$, we may approximate the fuzzy function $F_{\mathcal{V}}(x, y)$ by the fuzzy relation due to Mamdani-Assilian formula with the error

$$|F_{\mathcal{V}}(u, v) - MA_{\mathcal{V}}(u, v)| < \varepsilon$$

for all $u, v \in V$, i.e. the difference between the degree of truth that v is approximately equal to $g_{\mathcal{V}}(u)$ and its estimation using the Mamdani-Assilian formula is at most ε. It also follows from the latter that given u then any element v such that $MA_{\mathcal{V}}(u, v) > 0$ is good approximation of $g_{\mathcal{V}}(u)$. This means that it can be used as a result of a defuzzification procedure applied to a fuzzy set B_u defined by

$$B_u(v) = MA_{\mathcal{V}}(u, v), \qquad v \in V.$$

In other words, any function "passing through" the support of $MA_{\mathcal{V}}$ is suitable to approximate $g_{\mathcal{V}}$ with the accuracy ε and which has been derived on the basis of the Mamdani-Assilian formula MA using a defuzzification. The question, which of many possible defuzzification functions is the best one has been discussed by I. Perfilieva in [7].

By equivalence theorem, we can also prove the following corollary.

Corollary 1. *Let (13) hold and let*

$$T \vdash_a MA(x, y).$$

Then

$$T \vdash_b y \approx g(x) \text{ where } b \geq a \otimes d.$$

This corollary provides estimation of the provability degree (and thus, the precision of approximation) that an element y is close to the functional value of the approximated function g if we use the Mamdani-Assilian formula instead of the fuzzy function formula F. It follows from it that in every model $\mathcal{V} \models T$, if

$$MA_{\mathcal{V}}(u, v) \geq a$$

then

$$\rho_{\doteq}(v, g_{\mathcal{V}}(u)) < \neg a \oplus \varepsilon$$

where $\varepsilon = \neg c$ (recall that the provability degree d in (13) fulfils $d > c$).

3. Conclusion

In this paper, we have characterized an approximation of a function syntactically inside fuzzy logic in narrow sense with evaluated syntax. The latter makes possible to express also a prescribed accuracy inside the syntax. We have demonstrated a solution when based on the so-called totally bounded fuzzy equality, corresponds to totally bounded pseudometrics in models. The characterization has been obtained using either disjunctive or conjunctive normal forms. The former is precisely a formula, which in semantics is called the Mamdani-Assilian formula and which is used in most applications of fuzzy control.

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Received: November 20, 2001