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ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

ČÁST MATEMATICKÁ

On sequences of integers containing no arithmetic progression.

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Erdös and Turán[1]) recently considered the following question: let $a_1 < a_2 < \ldots \leqq x$ be a sequence of positive integers containing no k consecutive members of an arithmetic progression, and denote by $r_k(x)$ the highest possible number of elements of such a sequence (a sequence with $r_k(x)$ elements may be called a maximum sequence). Erdös and Turán proved, by numerical arguments, that

$$\frac{r_3(x)}{x} < \frac{3}{8} + o(1), \tag{1}$$

but they were not able to show as little as

$$\frac{r_3(x)}{x} = o(1). \tag{2}$$

In the following I shall draw some immediate consequences from the theorem of van der Waerden[2]) which may throw some light on the problem.

1. It is easily to be seen that $\frac{r_k(x)}{x}$ converges; this follows from the evident fact that $r_k(mn) \leqq mr_k(n)$; put, namely,

$$\liminf \frac{r_k(x)}{x} = \varrho_k, \tag{3}$$

then for abitrary $\varepsilon > 0$, there exists n such that

$$\frac{r_k(n)}{n} \leqq \varrho_k + \varepsilon. \tag{4}$$

Hence, for $x > n$,

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1) Journal of the London Math. Soc. **11** (1936), 261—264.

2) Beweis einer Baudetschen Vermutung, Nieuw Archief voor Wiskunde **15** (1927), 212—216.

$$\frac{r_k(x)}{x} \leqq \frac{\frac{x}{n} r_k(n)}{x} + o(1) \leqq \varrho_k + \varepsilon + o(1), \tag{5}$$

i. e.

$$\limsup \frac{r_k(x)}{x} \leqq \liminf \frac{r_k(x)}{x} + \varepsilon \tag{6}$$

and

$$\frac{r_k(x)}{x} \to \varrho_k. \tag{7}$$

2. We have

$$\frac{r_k(n)}{n} > \varrho_k \tag{8}$$

for every n. Suppose, namely, this were not true, then $r_k(n)/n$ must assume its minimum for a certain value $n = n_0$ and

$$\frac{r_k(n_0)}{n_0} \leqq \varrho_k. \tag{9}$$

Now choose a sufficiently great m and a maximum sequence $a_1 < a_2 < \ldots$ for $x = n_0 m$. Denote by $A_1, A_2, \ldots, A_m$ the intervals $\langle 1, n_0 \rangle$, $\langle n_0 + 1, 2n_0 \rangle, \ldots$ Now

$$\frac{r_k(n_0)}{n_0} \leqq \frac{r_k(n_0 m)}{n_0 m} \leqq \frac{m r_k(n_0)}{n_0 m}, \tag{10}$$

hence

$$r_k(n_0 m) = m r_k(n_0), \tag{11}$$

which is only possible if every A_μ contains precisely $r_k(n_0)$ elements of the sequence $a_1, a_2, \ldots$ Define $A_\mu = A_\nu$, if the a's lying in A_μ are obtained by adding $n_0 (\mu - \nu)$ to the a's lying in A_ν. n_0 being fixed there is only a finite number of „different" A's. But from van der Waerden's theorem follows the existence of one interval, $\bar{A}$ say, which occurs among all intervalls $A_1, \ldots, A_m$ in an arithmetic progression of length k, if only m was chosen greater than a certain $m(n_0, k)$. This gives a contradiction because the first a's ocurring in the $\bar{A}$'s would form an arithmetic progression of length k.[3])

3. Consider also infinite sequences $b_1 < b_2 < \ldots$ Let $S(x)$ denote the number of $b_1 \leqq x$, then $\liminf \frac{S(x)}{x}$ and $\limsup \frac{S(x)}{x}$ are called the lower and the upper density of the sequence. There

[3]) Mr. Erdös draws my attention to the fact that van der Waerden's theorem may be avoided here. (8) follows from $r_k((k-1)n_0 + 1) \leqq \leqq (k-1) r_k(n_0)$ which can easily be proved directly.

will be a certain number σ_k such that all sequences with upper density $> \sigma_k$ contain an arithmetic progression of length k whereas to every $\varepsilon > 0$ there exists a sequence with upper density $\sigma_k - \varepsilon$ containing no arithmetic progression of length k. It is

$$\sigma_k \leqq \varrho_k \leqq \sigma_{k+1}. \tag{12}$$

The first inequality is trivial; the second may be proved in the following way: choose positive integers $x_1, x_2, \ldots$ such that

$$(i) \quad x_i > 2x_{i-1} + 1 \quad (i = 1, 2, \ldots),$$

$$(ii) \quad \lim_{i\to\infty} \frac{x_i}{x_{i-1}} = \infty.$$

To every x_i there exists a maximum sequence

$$a_{i1} < a_{i2} < \ldots < a_{ir_k(x_i)} \leqq x_i \tag{13}$$

not containing an arithmetic progression of length k; let a_{ij_i} be the first element of (13) $> 2x_{i-1} + 1$; drop the elements $a_{i1}, a_{i2}, \ldots, a_{ij_i-1} \leqq 2x_{i-1} + 1$ and with the remaining elements form the sequence

$$\begin{array}{ll} a_{11}, a_{12}, \ldots, a_{1r_k(x_1)}, & \text{(1st ,,group``)} \\ a_{2j_2}, a_{2j_2+1}, \ldots, a_{2r_k(x_2)}, & \text{(2nd ,,group``)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{ij_i}, a_{ij_i+1}, \ldots, a_{ir_k(x_i)}, & \text{(ith ,,group``)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array} \tag{14}$$

(14) evidently has an upper density $\geqq \varrho_k$, the number of elements $\leqq x_i$ being $\geqq r_k(x_i) - 2x_{i-1} - 1 = r_k(x_i) + o(x_i)$. An arithmetic progression contained in (14) can overleap at most one of the gaps between the single ,,groups``, because each gap is greater than the last element of the preceding ,,group``; consequently such a progression has at most $1 + (k - 1) = k$ elements, i. e. (14) contains no arithmetic progression of length $k + 1$. Hence $\varrho_k \leqq \sigma_{k+1}$.

4. It follows from (12) that ϱ_k and σ_k are converging towards the same limit:

$$\lim_{k\to\infty} \varrho_k = \lim_{k\to\infty} \sigma_k = \varrho. \tag{15}$$

Also

$$0 \leqq \varrho \leqq 1. \tag{16}$$

Theorem: ϱ is either 0 or 1. This means e. g. that in order to prove $\varrho_k = 0$ it would suffice to prove the existence of a constant $c < 1$ (not depending on k) such that for all $k : \varrho_k \leqq c$. The argument is similar as in 2. Suppose namely

$$0 < \varrho < 1. \tag{17}$$

Then there exists a k with $\varrho_{k-1} - \varrho\varrho_k > 0$. Choose

(*i*) $\varepsilon > 0$ such that $\varepsilon < \frac{\varrho_{k-1} - \varrho\varrho_k}{4}$.

(*ii*) a sequence $b_1, b_2, \ldots$ with upper density $\geqq \varrho_{k-1}$ containing no arithmetic progression of length k,

(*iii*) n so great that every sequence of more than $(\varrho_k + \varepsilon)\, n$ integers $\leqq n$ contains an arithmetic progression of length k.

The intervals $\langle 1, n\rangle$, $\langle n + 1, 2n\rangle, \ldots$ are denoted by $B_1, B_2, \ldots$ Evidently there are at most 2^n „different" B's. The interval containing no b's at all is called the zero-interval Z, the others may be denoted by $A_1, A_2, \ldots, A_L$ $(L = 2^n - 1)$. The lower density of the Z's among the B's may be called ζ. Choose now

(*iv*) m such that

a) the number of Z's among the first m B's is $> (\zeta - \varepsilon)\, m$,

b) the number of b's $\leqq mn$ is $\geqq (\varrho_{k-1} - \varepsilon)\, mn$.

The last number must be, on the other hand, $\leqq (1-\zeta+\varepsilon)\, m\, (\varrho_k+\varepsilon)\, n$ (because the Z's do not contain any b's and the A's at most $(\varrho_k+\varepsilon)\, n$ from (*iii*)). Hence

$$(1 - \zeta + \varepsilon)\, (\varrho_k + \varepsilon) \geqq \varrho_{k-1} - \varepsilon, \tag{18}$$

$$\zeta \leqq \frac{(1 + \varepsilon)\, (\varrho_k + \varepsilon) - \varrho_{k-1} + \varepsilon}{\varrho_k + \varepsilon} < \frac{\varrho_k - \varrho_{k-1} + 4\varepsilon}{\varrho_k} < 1 - \varrho \tag{19}$$

from (*i*).

The upper density of the A's, consequently, is greater than ϱ. Choose now, by van der Waerden's theorem, $K(k, L)$ so great that, if we divide the numbers $1, 2, \ldots, K$ arbitrarily into $L = 2^n - 1$ classes, there can always be found in at least one of the classes an arithmetic progression of length k. As the A's have an upper density $> \varrho \geqq \varrho_K$, there can be found an arithmetic progression of A's (among the B's) of length K: $A_{\mu_1}, \ldots, A_{\mu_K}$. These form L classes of „equal" A's; consequently there exist k equal A's forming an arithmetic progression among the $A_{\mu_1}, \ldots, A_{\mu_K}$; they also form an arithmetic progression among all intervals $B_1, B_2, \ldots$ But this contradicts (*ii*) because the first b's contained in these A's would form an arithmetic progression of length k. Hence the theorem is proved.[4])

Prague, March 1937.

[4]) Mr. Erdős communicated to me a slightly different proof which makes use of van der Waerden's theorem only for the case of 2 classes.

O posloupnostech celých čísel, neobsahujících aritmetické posloupnosti.

(Obsah předešlého článku.)

Pro celá čísla $x > 0$, $k \geqq 3$ budiž $r_k(x)$ největší číslo m, mající tuto vlastnost: existuje množina m přirozených čísel nejvýše rovných x, neobsahující žádných k čísel, tvořících aritmetickou posloupnost. Potom existuje $\lim\limits_{x \to \infty} \frac{r_k(x)}{x} = \varrho_k$, $\lim\limits_{k \to \infty} \varrho_k = \varrho$ a platí tyto věty:

1. Pro každé přirozené n je $r_k(n) > \varrho_k n$.
2. Je buďto $\varrho = 0$ nebo $\varrho = 1$.

Druckfehlerberichtigung zum Aufsatz: K. Mack, Eine mit dem vollständigen Vierseit zusammenhängende Schließungsaufgabe (Časopis 67, S. 199—202).

Die Redaktion macht den Leser darauf aufmerksam, daß die Figur 1 mit der Figur 2 verwechselt ist.