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Roman Bek; Zdeněk Pokorný; Milan Růžička

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PRINCIPLE CONCEPTS OF SYSTEMS FUZZIFICATION

Fuzzification of Systems for Technical and Medical Practice I

ROMAN BEK, ZDENĚK POKORNÝ, MILAN RŮŽIČKA

The paper describes motivation of large variable systems fuzzification the comprehensive language modelling of which, providing unique predictions of future events, may be for theoretical or practical reasons difficult or impossible. Therefore we deal with definitions of simpler subsystems which we fuzzify. Modelling language is modified to enable formation of correspondence between predictions and their certainty grades (measures).

In the article fundamental presumptions and means of such a modification of the language are introduced. Apart from notions of "fuzzy class" and "fuzzy relation" there are terms as multi-fuzzification of a class by means of language with time terms (chronology language).

The authors of this discourse have recently published in this journal papers [1] and [2] dealing with the definition and questions of large variable systems modelling and compartmental ones in particular. Modelling of those systems was formed in precise languages with predicate-logical classical (two-valued) base of higher degrees.

We would like to publish series of papers regarding the systems fuzzification questions of higher degrees and compartmental systems common in biology and medical sciences.

First we shall study a few general questions concerning the definition of "sharp", fuzzy systems related to their language modelling.

1. BASIC REASONS FOR FUZZIFICATION OF LARGE SYSTEMS

Required optimal solution of various practical tasks in technical and non-technical spheres, for example in medical sciences branches, supposes a definition of considered systems on a given ontological sphere and their exact language modelling.

This modelling enables a formation of requested future events which occur as consequences of coincidentally existing conditions and proposed work operations. Language model of presumed processes envolves logical operation deduction of sentences on predicted consequences from those which describe circumstances and operations in physical sciences. In technical problems there are mostly mathematical language models and so especially for solving of difficult, for instance constructive, tasks if there is time enough to find a convenient language, mathematical formulation of assumptions and deduction of logical consequences. Even in biological and social sciences mathematical language modelling plays an increasing role.

A subject of language modelling may be a system defined on a suitable ontological sphere (technical, biological, social etc.). The semantic counterpart of propositions and logical consequences of a language model are input and output events of the system. The semantic counterpart of general, law-like sentences of the model are objective regularities in ontological sphere, which we can coincidentally consider as essential relations of the system.

The definition of "sharp" system is connected with a series of unique decisions which entities from the sphere we take as inputs and outputs events respectively and which regularities of the sphere we select as substantial system relations. These decisions regard registration of corresponding entities into given sets, sets of inputs, outputs, input and output events and relations. Obviously these sets can be classified as classical ("sharp") ones. Language modelling of such a system is therefore based on classical (two-valued) logical base.

A system, defined like this, is a result of a remarkable simplification of a sphere under consideration. From epistomological viewpoint, we can take that for a network located between us and the sphere. Its task is to provide an easier orientation on this sphere, where, at the same time, a series of various systems can be defined, as to the character of this simplification. Some of them can be taken for subsystems of other systems.

An effort for more adequate predictions (less distinguishing from really proceeding occurrences) leads to a definition of larger systems with greater number of inputs, outputs and essential systems relations.

A process from smaller to larger systems is usually related to their dishomogenity: qualitative differences among individual elements and parts of the system and its relations are more remarkable.

This system extension has its limits. A description of a very large system is too complicated. Greater dishomogenity leads not only to more complicated general assumptions in predictive procedures, but also to special demands regarding syntax and semantics of modelling precise language.

Biological and social sciences are characterized by statistical character of studied occurrences and systems and by vagueness of their terms. Undoubtedly it is caused by descriptive foundations of these sciences. These terms can be considered (cf. Zadeh [6]) as linguistic variables or restrictions respectively, that means as fuzzy sets [7]. We give following examples from morphological and clinical branches.

For frame descriptions terms "short" – "long", "rounded" – "ellipsodical" – "filaminous", "flat" – cubic" – "cylindrical", network "dense" – "thin", colours "light" – "dark", "pale" – "vivid". Situs is denoted by so-called body region etc. Similarly in clinical sciences, anamnesis, subjective complaints and current state description are necessarily presented in vague terms regarding, for example, pain, (MUDr Jirásek distinguished 36 terms for pain quality), habitus, body holding, motions, height, meagerity, muscle tone, colour and humidity of skin, etc. Almost all symptoms, as palpatory, auditory and percutory signs, are denoted by vague terms. Descriptions of x-ray, ekg findings and others contain vague terms. Majority of quantitative labs findings are to compared with a norm, but this norm itself is not accurate. Even the notations of illness and health are vague.

Physiological branches have to start from facts given by a description. Apart from that, they have their own sources of inaccuracies, as for example: difficulties with preservation of experiment conditions, methodological modifications and their inaccurate description complications with result transfer on an isolated system (skin, organ etc.) for conditions in intact organism or between two biological systems, impossibility of direct parameters determinations. Quantification of a physiological system is often incomplete. We have to be only satisfied with a description of a response on a stimulus, as increasing or decreasing of a factor. These inaccuracies exceed often and practically possibilities of statistical registration.

In biological and medical sciences we meet so large dishomogenic systems that we do not even try to form their "sharp" definition. In this connection, let us remind L. A. Zadeh's sentence: "As the complexity of a system increases, our ability to make precise and yet significant statements its behaviour diminishes until a threshold is reached beyond which precision and significance become a most mutually exclusive characteristics." [6]

In an ideal case, a precise language model of a "sharp" system leads from true propositions to logical consequences, the truth of which is certain. In discussed circumstances we often have to be satisfied with limited grade of deducted consequences certainty. But we would like to know this grade of certainty (or certainty measure), because of decision formulation regarding prepared work operation. In such a case, "sharp" system can be replaced by a "fuzzy" one. If the system definition was based on the set theory concepts, then the corresponding sets are getting to be fuzzy sets. Precise language model which served to a description and system prediction, is now being generalized and completed by means for certainty grades (measures) calculation, values of related membership functions.

In the following part of this discourse we bring a few basic viewpoints and operations required for completion of exact language for describing fuzzy systems.

2. BASIC OPERATIONS WITH FUZZY SETS. SYMBOLS

In forthcoming reasoning we begin from three major grounds of systems fuzzification:

a) We are not quite sure if given objects belong to a given set (class) at a given moment, i.e. if they have certain property (hence we build up precise language on an extensional base).

b) We do not know exactly, if m-tuple of given objects belong to a certain relation at a moment and if these particular objects really belong to respective n-tuples. In further text we shall often use the term "to fuzzy". It is to be understood the replacement of a "sharp" system by a fuzzy one.

c) We are not certain if given moments belong to a given time interval.

A fuzzy class (set) we shall understand as (unique) mapping (transformation):

$$A^0 : U_L \to Q_{01}$$

where U_L is the language universe (in which we shall define system).

Q_{01} is interval of rational numbers $\langle 0, 1 \rangle$.

Thus the fuzzy set A^0 can be considered as a class of ordered pairs of the type:

$$\langle x_i, \mu_{A^0}(x_i) \rangle \,, \quad x_i \in U_L \,, \quad \mu_{A^0}(x_i) \in Q_{01}$$

The function μ_{A^0} is called "membership function" and for given argument x_i denotes certainty grade (measure) that x_i belongs to A^0.

Remark 2.1. In this article we shall not deal with general conception of fuzzy set theory, languages and automata. Hence the range of membership function does not have to be any concrete numerical interval, but any set L partially ordered by a relation of the type $\leqq$.

On the set L there are defined operations of the type:

$\cap, +, \cdot, \to$, and holds:

$(L, +, \cdot)$	is complete subring with element $1 \in L$ as the upper one,
$(L, \to)$	is implicative algebra,
$a \,.\, b \leqq c$	iff $a \leqq b \to c$,
$(L, \cap)$	is complete sublattice,
$+$	is idempotential operation.

By a specification of these requirements we can obtain an algebraic structure which is near to Lukasiewicz's logic:

instead of L we introduce closed interval $R_{01} = \langle 0, 1 \rangle$ partially ordered by the relation $\leqq$,

$a \cap b = \min\{a, b\}$
$a + b = \max\{a, b\}$
$a \,.\, b = \max\{0, a + b - 1\}$
$a \rightarrow b = \min\{1, 1 - a + b\}$ for $a, b \in R_{01}$.

For the following reasonings we perform further specifications: as the range of membership function we accept closed interval of rational numbers $Q_{01} = \langle 0, 1 \rangle$, because for empirical investigation of membership function of basic entities of fuzzy systems, rational numbers are fully sufficient. Further demand is computerization of membership function values for derived entities.

We assign operations $a + b$ and $a \,.\, b$ to those of union $(A \cup B)$ and intersection $(A \cap B)$ of classes, where a is a value of membership function of fuzzy set A^0 for given argument, b is a value of that for fuzzy set B^0 for the same argument, $a + b$, $a \,.\, b$ are respective values of membership functions of fuzzy sets $A \overset{\circ}{\cup} B$, $A \underset{\circ}{\cap} B$ for the same argument:

$a = \mu_{A^0}(x)$, $b = \mu_{B^0}(x)$ for $x \in U_L$,
$a + b = \max\{a, b\} = \mu_{(A \overset{\circ}{\cup} B)}(x)$
$a \,.\, b = \min\{a, b\} = \mu_{(A \underset{\circ}{\cap} B)}(x)$

Further we associate operation subtraction from the unit $(1 - a)$ with class A complement formation as follows:

$a = \mu_{A^0}(x)$ as formerly,

with the negation of the sentence $x \in A^0$ (i.e. $x \notin A^0$) corresponds value $(1 - a) = (1 - \mu_{A^0}(x))$.

Remark 2.2. When considering a, b as true values of sentences of the type:

"Element $x \in U_L$ belongs to A"

"Element $x \in U_L$ belongs to B", respectively,
we can take operations $a + b$, $a \,.\, b$, $(1 - a)$ for arithmetical once, which define functors of multi-valued logic.
(This specification corresponds with non-generalized conception of fuzzy set from [4])

When looking for fuzzy algebraic interpretation of sentences of chronology language envolving also time terms, we often meet possibility of coincidental fuzzification of time interval and successive multi-fuzzification of a given class.

Let a simple sentence have a form: "An object x has at a moment t_i property A". ("The object x belongs to a class A at a moment t_i"). Symbolically: $\langle x, t_i \rangle \in A$.

The class A can be fuzzified once. Let a cause of fuzzification be a fact that we cannot verify with a full certainty, if the object x has really the property A at the moment t_i. To denote explicitly the first fuzzification of the class A, we shall now write the letter A with number 01 and the membership function of once fuzzified class will be abbreviated like this:

$$\mu_{A^{01}}(x, t_i) .$$

In further explanation A^{0n} is to be understood as n-times successively fuzzified class A.

Time interval Δt, i.e. class of moments ordered by the relation "to happen earlier, than" can be fuzzified once, because we are not able to verify completely sentences of the type "moment t_i belongs to time interval Δt". Hence Δt^0 will be a fuzzy set (interval) of moments and we consider that as transformation

$$\Delta t^0 : \tau \to Q_{01}$$

where τ is the class of all moments and forms a base of time structure from the ontology of applied precise language. We fuzzify time intervals only once and therefore we omit from symbolic abbreviation number 1 in the circle over Δt. The membership function of fuzzy interval Δt^0 is denoted by $\mu_{\Delta t^0}(t_i)$. We sometimes consider membership of a moment t_i in subinterval $\Delta t' \subset \Delta t$ limited by duration time of given characteristic for a given object.

Membership uncertainty regarding time we meet when studying biological rythms, determining a childbirdth delivery or investigating successive combination of patogennic impulses.

Let us have now a further simple sentence of the type "n-tuple of objects $x_1, \ldots$ $\ldots, x_n$ occurs at a moment t_i in a relation R" abbreviated by

$$\langle x_1, \ldots, x_n, t_i \rangle \in R$$

The class R of ordered $(n + 1)$-tuples (on the last place in each is always time term) can be fuzzified twice successively. Let us notice that the sentence "Object x_j belongs to the field of $(n + 1)$-member relation R at a moment t_i" $(n \geqq 2)$ may lead to uncertainty of the following type: as known, x_j belongs to the field R at t_i, if it belongs to at least one from $(n + 1)$-tuples, with the last term t_i, which is an element of the class R.

Let uncertainty be based on unability to decide with full reliability, if x_j belongs to the given $(n + 1)$-tuple (this $(n + 1)$-tuple is an ordered class itself).

In diagnostics, for example, we are not sure, if determined signs "belong together". We do not know, how remarkably proceeding processes are dependent on one another.

Let us call the k-th $(n + 1)$-tuple, belonging to R, the k-th context of the R, abbreviated by $(\text{cont})_{k,R}$. This context is a class which can be fuzzified. Its membership function will be $\mu_{(\text{cont})^0_{k,R}}(x_j, t_i)$.

We can formulate n-tuple of membership functions of the above mentioned form fo n given objects $x_1, \ldots, x_n$ with respect to moment t_i and these tuples are connected with certainty grade (measure) that objects $x_1, \ldots, x_n$ belong to the k-th $(n + 1)$-tuple at t_i belonging to R. For the whole $(n + 1)$-tuple we can calculate then the value of "total" membership function:

$$\mu_{(\text{cont})^0_{k,R}}(x_1, \ldots, x_n, t_i) =$$

$$= \min \{\mu_{(\text{cont})^0_{k,R}}(x_j, t_i) \mid \text{for all } x_j, 1 \leqq j \leqq n, \langle x_1, \ldots, x_n, t_i \rangle \in R\} .$$

Further: the relation R itself is a class of ordered $(n + 1)$-tuples of the type $\langle x_1, \ldots, x_n, t_i \rangle$. When verifying sentence $\langle x_1, \ldots, x_n, t_i \rangle \in R$ we may not be quite sure if this $(n + 1)$-tuple completely belong to the class R.

For example, we are not certain, if present symptoms set belongs to one or other nosological unit. We do not have certainty regarding alternatives of state prediction or therapy determination. In such a case, we shall write corresponding membership function $\mu_{R01}(x_1, \ldots, x_n, t_i)$.

Let us consider now two types of fuzzification. Related total membership function (for twice fuzzified class R) can be calculated:

$$\mu_{R02}(x_1, \ldots, x_n, t_i) = \min \{\mu_{R01}(x_1, \ldots, x_n, t_i), \mu_{(\text{cont})^0_{k,R}}(x_1, \ldots, x_n, t_i)\}$$

where $\langle x_1, \ldots, x_n, t_i \rangle$ is the k-th element of R.

Relations and attributes of higher degrees can be successively fuzzified also from some other viewpoint:

Let ${}^{(s)}R$ be a relation among distinct objects and have degree $s > 1$. Objects within this relation can be elements of the universe U of "sharp" system being fuzzified or they can be relations of this system with the degree utmost $(s - 1)$ and time term:

$$\langle {}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i \rangle \in {}^{(s)}R$$

For simplicity reason, we consider attribute as unary relation and denote an element of the universe by symbol ${}^{(0)}R$.

As we have just seen, the first two fuzzification of the relation ${}^{(s)}R$ are motivated by uncertainty whether $(n + 1)$-tuple

$$\langle {}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i \rangle$$

belong to the class ${}^{(s)}R$ and if its individual members ${}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n$ belong to this $(n + 1)$-tuple. The membership function of twice fuzzified relation ${}^{(s)}R$ is

$$\mu_{(s)R02}({}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i) .$$

The second two fuzzifications of the relation ${}^{(s)}R$ are motivated by uncertainty whether respective ordered classes of lower degrees relations belong at a moment t_i to corresponding classes

$${}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n$$

and if particular members of these classes belong there.

The total membership function can be obtained from membership functions values of these relations by means of operation min from mentioned initial values. Obviously, the fuzzification process of the relation ${}^{(s)}R$ can continue if $s > 1$.

For value calculation of membership functions of classes (relations) the following algorithm can be formed:

2.1. RULES.

1. Let a relation of the type ${}^{(s)}R$ be class of ordered $(n + 1)$-tuples

$$\langle {}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i \rangle , \quad s \geqq 1 , \quad n \geqq 1 .$$

The first two fuzzifications of ${}^{(s)}R$ lead (as we have already seen) to fuzzy class ${}^{(s)}R^{02}$ with the membership function:

$$\mu_{{}^{(s)}R^{02}}({}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i) =$$
$$= \min \{\mu_{{}^{(s)}R^{01}}({}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i), \mu_{(\mathrm{cont})^0_{k}, {}^{(s)}R}({}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i)\} ,$$

where $\langle {}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n, t_i \rangle$ is the k-th element of ${}^{(s)}R$.

2. Let a relation of the type ${}^{(s_j)}R_j$ be class of ordered $(j_m + 1)$-tuples

$$\langle {}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}, t_i \rangle .$$

The first two fuzzifications of this relation lead to fuzzy set ${}^{(s_j)}R_j^{02}$ with the membership function

$$\mu_{{}^{(s_j)}R_j{}^{02}}({}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}, t_i) =$$
$$= \min \{\mu_{{}^{(s_j)}R_j{}^{01}}({}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}, t_i), \mu_{(\mathrm{cont})^0_{j,k}\,{}^{(s_j)}R_j}({}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}, t_i) \mid$$
$$\mid \text{for all } {}^{(s_j)}R_j, \langle {}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}, t_i \rangle \text{ is the } k\text{-th element of } {}^{(s_j)}R_j\} .$$

3. Are all elements of the type ${}^{(s_1)}R_1, \ldots, {}^{(s_n)}R_n$ coincidentally those from the system U universe, i.e. $s_1 = s_2 = \ldots = s_n = 0$?

4. The second fuzzification of ${}^{(s)}R$ is final.

   Let respective membership function identify final value for max fuzzified ${}^{(s)}R$.

5. Take successively:

$$s_j = s_1$$
$$\vdots$$
$$s_j = s_n$$

6. Do for these s_j successively rule no. 2 for all their elements, i.e. for ordered $(j_m + 1)$-tuples of the type:

$$\langle {}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{jm}, t_i \rangle .$$

   Calculate min from these membership functions values.

7. Are all elements of the type

$${}^{(s_{j,1})}R_{j,1}, \ldots, {}^{(s_{j,m})}R_{j,m}$$

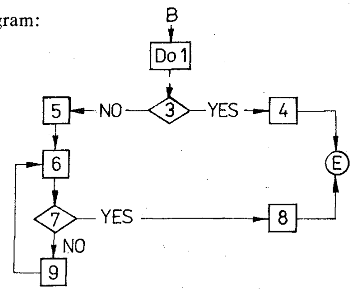
   coincidentally those of the universe U (i.e. $s_{j,1} = s_{j,2} = \ldots = 0$)?

8. The last fuzzification of ${}^{(s)}R$ is final.
   Determine min from all so far obtained membership functions values as final value for max fuzzified ${}^{(s)}R$.
9. Put respectively:

$$s_j = s_{j,1}$$
$$\vdots$$
$$s_j = s_{j,m}$$

   for all $s_{j,1}, \ldots, s_{j,m}$ greater than zero.

   Development diagram:



It is obvious, that a relation of the type ${}^{(s)}R$ can be fuzzified max $2s$-times. For denomination simplicity we shall write only $\mu_{{}^{(s)}R^{0}}$ instead of $\mu_{{}^{(s)}R^{02s}}$. Objects attributes (i.e. unary relations of the first degree ${}^{(1)}R$) we fuzzify only once.

E. 2.1. Example.

Let in "sharp conception" be defined following relations:

$$ {}^{(1)}R = \{\langle x, t_i\rangle, \langle y, t_j\rangle\}\,, \quad {}^{(2)}R = \{\langle {}^{(1)}R, y, t_j\rangle\}\,, $$
$$ {}^{(3)}R = \{\langle {}^{(1)}R, {}^{(2)}R, x, t_i\rangle, \langle {}^{(1)}R, {}^{(2)}R, y, t_j\rangle\}\,. $$

Let (empiricaly obtained) membership functions values be given:

$$\mu_{{}^{(1)}R^{01}}(x, t_i) = 0{\cdot}85 \qquad \mu_{{}^{(1)}R^{01}}(y, t_j) = 0{\cdot}82$$
$$\mu_{(\mathrm{cont})^{01}, {}^{(2)}R}({}^{(1)}R, t_i) = 0{\cdot}75 \qquad \mu_{(\mathrm{cont})^{01}, {}^{(2)}R}(x, t_i) = 0{\cdot}8$$
$$\mu_{{}^{(2)}R^{01}}({}^{(1)}R, x, t_i) = 0{\cdot}7$$
$$\mu_{(\mathrm{cont})^{01}, {}^{(3)}R}({}^{(1)}R, t_i) = 0{\cdot}71\,, \qquad \mu_{(\mathrm{cont})^{01}, {}^{(3)}R}({}^{(2)}R, t_i) = 0{\cdot}82$$
$$\mu_{(\mathrm{cont})^{01}, {}^{(3)}R}(x, t_i) = 0{\cdot}86$$
$$\mu_{{}^{(3)}R^{01}}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}6$$
$$\mu_{(\mathrm{cont})^{02}, {}^{(3)}R}({}^{(1)}R, t_j) = 1\,, \qquad \mu_{(\mathrm{cont})^{02}, {}^{(3)}R}({}^{(2)}R, t_j) = 0{\cdot}69$$
$$\mu_{(\mathrm{cont})^{02}, {}^{(3)}R}(y, t_j) = 1\,,$$
$$\mu_{{}^{(3)}R^{01}}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}65$$

Now we can calculate further values of membership functions.

${}^{(1)}R$ can be obviously fuzzified only once.

For ${}^{(2)}R$: this relation contains only one triple which is therefore its first and last context and membership function values of this context will be:

$$\mu_{(\text{cont})^{01},{}^{(2)}R}({}^{(1)}R, x, t_i) = 0{\cdot}75$$

The relation ${}^{(2)}R$ now we fuzzify for the second time:

$$\mu_{{}^{(2)}R^{02}}({}^{(1)}R, x, t_i) = 0{\cdot}7$$

The relation ${}^{(2)}R$ can be fuzzified for the third time (in the context ${}^{(1)}R$ will be fuzzified)

$$\mu_{{}^{(2)}R^{03}}({}^{(1)}R, x, t_i) = 0{\cdot}7\,.$$

${}^{(2)}R$ cannot be further fuzzified, since all its parts are already max fuzzified. For the part ${}^{(1)}R$ is unary relation of the first degree, so has been fuzzified only once and hence ${}^{(1)}R^{01} = {}^{(1)}R^{02}$. For the same reason ${}^{(2)}R^{03} = {}^{(2)}R^{04}$.

For ${}^{(3)}R$: this relation envolves only two quadruples, hence, we consider two comprehensive contexts.

Membership function values for both comprehensive contexts:

$$\mu_{(\text{cont})^{01},{}^{(3)}R}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}71$$

$$\mu_{(\text{cont})^{02},{}^{(3)}R}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}69$$

The relation ${}^{(3)}R$ can be fuzzified for the second time:

$$\mu_{{}^{(3)}R^{02}}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}6$$

$$\mu_{{}^{(3)}R^{02}}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}65$$

${}^{(3)}R$ can be gradually fuzzified for the third time (${}^{(1)}R$ and ${}^{(2)}R$ will be now fuzzified in contexts for the first time):

$$\mu_{{}^{(3)}R^{03}}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}6$$

$$\mu_{{}^{(3)}R^{03}}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}65$$

${}^{(3)}R$ is fuzzifiable for the fourth time (${}^{(1)}R$ cannot be fuzzified anymore, but ${}^{(2)}R$ can be still fuzzified once more):

$$\mu_{{}^{(3)}R^{04}}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}6$$

$$\mu_{{}^{(3)}R^{04}}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}65$$

${}^{(3)}R$ can be fuzzified for the fifth time (in context we consider ${}^{(2)}R$ to be fuzzified three times):

$$\mu_{{}^{(3)}R^{05}}({}^{(1)}R, {}^{(2)}R, x, t_i) = 0{\cdot}6$$

$$\mu_{{}^{(3)}R^{05}}({}^{(1)}R, {}^{(2)}R, y, t_j) = 0{\cdot}65$$

Further ${}^{(3)}R$ cannot be fuzzified.

Sometimes determination of membership function values of vague terms arises from definition, but that is based on individual or collective experience with applications of these terms in given situations.

At basic cases it is possible to find functional value of membership function using empirical way. It seems to be necessary to stress that a value of membership function of a given class for an argument — so the certainty grade (measure), that a sentence about elementshood of an object with respect to a certain class is true — cannot be replaced by a probability value. Specific algebraic structure, having been defined on the interval of rational numbers Q_{01} and serving for precise language model of fuzzified systems, differs from algebraic structure of probability theory.

Values of membership functions for some (defined) classes can be determined, as we have already shown. We shall use formulas which will be further introduced in connection with definitions of corresponding entities from the systems theory.

An important part of these formulas play operations min, max and that of complementation, which are uniquely associated in our specification with respective operations and sentential functions:

$$(A(x) \wedge B(x)) \ldots \mu_{A \cap B}(x) = \min \{\mu_{A^o}(x), \mu_{B^o}(x)\} ,$$
$$(A(x) \vee B(x)) \ldots \mu_{A \cup B}(x) = \max \{\mu_{A^o}(x), \mu_{B^o}(x)\} ,$$
$$\sim A(x) \qquad \ldots \mu_{\bar{A}^o}(x) = 1 - \mu_{A^o}(x)$$
$$(A(x) \rightarrow B(x)) \ldots \min \{1, 1 - \mu_{A^o}(x) + \mu_{B^o}(x)\} .$$

Formula with universally bounded variable with finite domain can be replaced by finite conjunction of sentential functions and value of corresponding membership function is obtainable by operation min:

$$\underset{x \in U}{\forall x} A(x) \ldots \min \{\mu_{A^o}(x) \mid x \in U, \ U \text{ is finite}\}$$

Using similar way, we can substitute the formula with existentially bounded variable by disjunction of sentential functions and value of membership function calculate by means of operation max:

$$\underset{x \in U}{\exists x} A(x) \ldots \max \{\mu_{A^o}(x) \mid x \in U, \ U \text{ is finite}\} .$$

In future reasoning we shall simplify formulas with successive operations min and max, as for example:

$$\min \{\ldots, \min \{- - -\}\} = \min \{\ldots, - - -\}$$
$$\max \{- - -, \max \{\ldots\}\} = \max \{- - -, \ldots\} .$$

In some instances we may meet a few parenthesis. Expression of the type

$$\text{oper}_1 \{\mu_{A^o}(x), \text{oper}_2 \{\mu_{R^o}(x, y) \mid y \text{ satisfies a condition } C_1\} \mid$$
$$\mid x \text{ satisfies a condition } C_2\}$$

(where oper_1 and oper_2 are operations min and max respectively) is to be understood as follows:

the operation_1 is applied on the range of the function μ_{A^0} for all x satisfying a condition C_2 and to this range belong also results of the operation_2 when applied on the range of the function μ_{R^0} for all x satisfying the condition C_2 and all y satisfying the condition C_1.

Conditions for class formation mentioned beyond the vertical line | will be mostly formulated in natural language with common symbols. These conditions will deal with "sharp" entities. That is well satisfying common usage that we first select from reality a "sharp" system and afterwards we make a fuzzification.

In the following reasonings we shall deal with cardinalities of fuzzy classes. These cardinalities definitions we have taken from S. Gottwald's conception. As to that, every fuzzy class (set) A^0 can be divided into subclasses of the type $A^{0(i)}$, $i \in (0, 1\rangle$ as to i values of membership function μ_{A^0} for individual elements $x \in U_L$:

$$A^0 = \cup A^{0(i)}, \quad i \in (0, 1\rangle .$$

For abbreviation simplicity we omit at individual subclasses for respective elements corresponding value of membership function, which is already mentioned in index of the type i and instead of term $A^{0(i)}$ we use expression $A^{(i)}$:

$$A^{(i)} = \{x \mid \mu_{A^0}(x) = i, \; i \in (0, 1\rangle\} .$$

Thus cardinality of fuzzy class A^0 is defined in this way:

$$\text{card}_W (A^0) = \{\text{card}\,(A^{(i)}), \; i \in (0, 1\rangle\} .$$

E. 2.2. Example.

Let a fuzzy set A^0, after omitting all elements with zero value of membership function (abbreviated by $|A^0|$), be defined as follows:

$$|A^0| = \{\langle x_1, 0{\cdot}5\rangle, \langle x_2, 0{\cdot}3\rangle, \langle x_2, 0{\cdot}4\rangle, \langle x_4, 0{\cdot}5\rangle, \langle x_5, 0{\cdot}9\rangle, \langle x_6, 0{\cdot}3\rangle, \langle x_7, 0{\cdot}8\rangle, \langle x_8, 0{\cdot}5\rangle, \langle x_9, 1\rangle, \langle x_{10}, 0{\cdot}4\rangle\}$$

Sets of levels:

$$A^{(0,3)} = \{x_2, x_6\}, \quad A^{(0,4)} = \{x_3, x_{10}\}, \quad A^{(0,5)} = \{x_7\}, \quad A^{(0,9)} = \{x_5\}$$
$$A^{(1)} = \{x_9\} .$$

Generalized cardinality:

$$\text{card}_W (A^0) = \{\text{card}\,(A^{(0,3)}), \text{card}\,(A^{(0,4)}), \text{card}\,(A^{(0,5)}), \text{card}\,(A^{(0,8)}), \text{card}\,(A^{(0,9)}), \text{card}\,(A^{(1)})\} = \{2_{0,3}, 2_{0,4}, 3_{0,5}, 1_{0,8}, 1_{0,9}, 1_1\} .$$

We shall also meet operation of addition of fuzzy cardinalities. The sum of two fuzzy classes cardinalities can be defined like this:

$$A\,(+)^0\,B = \left(\mathrm{card}\,(A^{(i)} \cup B^{(i)})\right), \quad i \in (0, 1\rangle\,,$$

what is perfectly in accordance with common definition of addition from the set theory.

E. 2.3. Example.

Let a fuzzy class A^0 be the same as in E.2.2, and fuzzy set B^0 defined as follows:

$$B^0 = \{\langle x_{11}, 0{\cdot}3\rangle, \langle x_{12}, 1\rangle, \langle x_{13}, 0{\cdot}5\rangle, \langle x_{14}, 0{\cdot}3\rangle\}\,.$$

$$\mathrm{card}_W\,(B^0) = \{\mathrm{card}\,(B^{(0,3)}), \mathrm{card}\,(B^{(0,5)}), \mathrm{card}\,(B^{(1)})\} = \{2_{0,3}, 1_{0,5}, 1_1\}$$

the sum:

$$A\,(+)^0\,B = \{\mathrm{card}\,(A^{(0,3)} \cup B^{(0,3)}), \mathrm{card}\,(A^{(0,4)}), \mathrm{card}\,(A^{(0,5)} \cup B^{(0,5)}),$$
$$\mathrm{card}\,(A^{(0,8)}), \mathrm{card}\,(A^{(0,9)}), \mathrm{card}\,(A^{(1)} \cup B^{(1)})\} =$$
$$= \{4_{0,3}, 2_{0,4}, 4_{0,5}, 1_{0,8}, 1_{0,9}, 2_1\}\,.$$

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Doc. Dr. Roman Bek, DrSc., katedra aplikované matematiky strojní fakulty ČVUT (Department of Applied Mathematics, Faculty of Mechanical Engineering — Czech Technical University), Suchbátarova 6, 160 00 Praha 6. Czechoslovakia.

Doc. MUDr. Zdeněk Pokorný, CSc., Ústav pathologické fyziologie fakulty všeobecného lékařství UK (Institute of pathological physiology, Medical Faculty — Charles University), U nemocnice 5, 128 53 Praha 2. Czechoslovakia.

Dr. Milan Růžička, CSc., katedra matematiky strojní fakulty ČVUT (Department of Mathematics, Faculty of Mechanical Engineering — Czech Technical University), Karlovo nám. 13, 121 35 Praha 2. Czechoslovakia.