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BEHAVIOUR OF SIGN TEST AND ONE SAMPLE MEDIAN TEST AGAINST CHANGES IN THE MODEL

ALFONSO GARCÍA–PÉREZ

The sign test and the test based on the sample median are asymptotically equivalent and as a consequence, equivalent from a robustness point of view because the most important robust measures in hypotheses testing are asymptotic. However, as this paper proves, the behaviour of their power functions against changes in the model, inside a class of distribuctions, is appreciably different for finite samples sizes. A new definition of sensitivity of tests with respect to the type of the alternative is defined and, with it, we see that the one sample median test is less sensitive than the sign test.

1. INTRODUCTION

The Neyman–Pearson lemma for maximin tests between neighborhoods of probability measures which are dominated by 2-alternating capacities (see [14] and [15]) is, basically, the only finite-sample optimality result of robust statistics; related with it are [24], [2], [18] and also [4], [5] and [6].

Although the application of these results is not straightforward, the usage of second order approximations, such as Edgeworth expansion or saddlepoint techniques, can increase their possibilities of application. A good book on the first topic is [13] and, on the second one, [7] and the paper [23].

Other traditional solution is to use asymptotic approach. Nevertheless, equivalent tests from an (asymptotic) robustness viewpoint can have a remarkable different behaviour when we consider finite sample sizes.

Here, we prove that two of these tests, the sign test and the one sample median test, which have the same degree of robustness (for instance with the Rousseeuw and Ronchetti approach, [26] and [27], based on the Hampel influence function), have a different sensitivity in their power functions when we change the underlying distribution.

To prove this we shall consider tail orderings on distributions. This idea has been used extensively in mathematical statistics since Lehmann [21] introduced the stochastic ordering, particularly on hypotheses testing (see [3], [25] and [12]).

Here we shall use the tail ordering $<_t$ defined by Loh [22] which generalizes the classical tail orderings ([28], [19] and [1]).

The paper is orgenized as follows. After some preliminaries, in Section 2 we study the behaviour of sign test with respect tail ordering $<_t$. We do the same about the one sample median test in Section 3 and, finally, in Section 4 we get the main result comparing the behaviour of both tests.

1.1. Preliminaries

Let X be a random variable with distribution F_θ depending on a location parameter $\theta \in \Theta$.

In this paper we shall consider tests of the null hypothesis $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$, although the results can be extended to other kind of hypotheses.

We shall suppose that the distribution of X belongs to the class of distributions (see [12]) $\mathcal{F}^* = \{F_{\theta,b}$: $F_{\theta,b}$ is a distribution function **(a)** with density $f_{\theta,b}$ with respect to the Lebesgue measure, **(b)** a location in θ and scale in b family, being the scale parameter determined by the condition

$$f_{\theta,b}(\theta) = c, \tag{1}$$

with c a known constant, **(c)** symmetric in θ, **(d)** strictly increasing in a neighborhood of θ, **(e)** strongly unimodal$\}$, which includes as a subclass the Box–Tiao families with densities

$$f_\theta^\beta(x) = \frac{1}{b\,\Gamma\left(1+\frac{1+\beta}{2}\right) 2^{1+\frac{1+\beta}{2}}} \exp\left\{-\frac{1}{2}\left|\frac{x-\theta}{b}\right|^{\frac{2}{1+\beta}}\right\}, \qquad -1 < \beta \leq 1.$$

The main reason to consider these classes of distributions is that they are ordered with respect to the tail ordering $<_t$ introduced by Loh [22]. Namely, if $F, G \in \mathcal{F}^*$ and have the same symmetry center, θ,

$$F <_t G \Longleftrightarrow F_{\theta,b}(x) \geq G_{\theta,b'}(x), \quad \forall\, x > \theta$$

or

$$F <_t G \Longleftrightarrow F_{\theta,b}(x) \leq G_{\theta,b'}(x), \quad \forall\, x < \theta$$

being the uniform and double exponential distributions the extreme distributions of $\mathcal{F}^*$ class. For instance, it is

$$\text{Uniform} <_t \text{Normal} <_t \text{Logistic} <_t \text{Double Exponential}.$$

And, for Box–Tiao families,

$$\text{Uniform} <_t f^{\beta_1} <_t f^{\beta_2} <_t \text{Double Exponential}$$

if $-1 < \beta_1 < \beta_2 \leq 1$. Also $<_t$ is location and scale-free.

2. BEHAVIOUR OF SIGN TEST

Let $X_1, \ldots, X_n$ be a random sample of X. For testing $H_0 : \theta \leq \theta_0$ at level α against $H_1 : \theta > \theta_0$, the ordinary sign test rejects H_0 when the number, S, of plus signs among the n differences $X_i - \theta_0$ is $S \geq k_\alpha$, where k_α is the smallest integer which satisfies

$$\frac{1}{2^n} \sum_{x=k_\alpha}^{n} \binom{n}{x} \leq \alpha.$$

In order to achieve the α-level we shall consider only the natural levels (smaller than 0′5) for the sign test, i.e.,

$$\alpha = \frac{1}{2^n} \sum_{i=j}^{n} \binom{n}{i}, \qquad j = [\tfrac{n+3}{2}], \ldots, n.$$

2.1. The Sign Test and the Tail Ordering $<_t$

Although the sign test, ϕ_s, is a distribution-free hypothesis test, its power function (nondecreasing in θ)

$$\begin{aligned} \beta^F_{\phi_s}(\theta) &= \sum_{x=k_\alpha}^{n} \binom{n}{x} [1 - F_\theta(\theta_0)]^x \, [F_\theta(\theta_0)]^{n-x} \\ &= \frac{n!}{(k_\alpha - 1)! \, (n - k_\alpha)!} \int_0^{1-F_\theta(\theta_0)} x^{k_\alpha - 1} \, (1-x)^{n-k_\alpha} \, \mathrm{d}x \end{aligned}$$

is very sensitive to the supposed model F_θ.

For example, if the constant c which determines the scale parameter through condition (1) is $c = 1/2$ and also it is $\theta_0 = 0$ and $n = 5$, the power functions under uniform, normal, logistic and double exponential distributions are given in Figure 1 ($\alpha_1 = 0.1875$) and Figure 2 ($\alpha_2 = 0.03125$).

In both cases, the sensitivity of the power function can be noted. Moreover, the tail ordering between the distributions is preserved. This a general property we prove now.

Proposition 1. If $F, G \in \mathcal{F}^*$ and $F <_t G \implies \beta^F_{\phi_s}(\theta) \geq \beta^G_{\phi_s}(\theta), \ \forall\, \theta > \theta_0$.

Proof. The power function of the sign test, $\beta^F_{\phi_s}(\theta)$, is a Beta cumulative distribution function $\beta(k_\alpha\, , \, n - k_\alpha + 1)$ in $1 - F_\theta(\theta_0)$,

$$\beta^F_{\phi_s}(\theta) = B\left(1 - F_\theta(\theta_0)\right).$$

Since, for the same $\theta > \theta_0$, $F <_t G$ implies

$$F_{\theta,b}(x) \leq G_{\theta,b'}(x), \qquad \forall\, x < \theta$$

if $x = \theta_0$, it will be

$$1 - F_{\theta,b}(\theta_0) \geq 1 - G_{\theta,b'}(\theta_0), \qquad \forall\, \theta > \theta_0.$$

Because of the monotonicity of any cumulative distribution function we get the result. $\square$

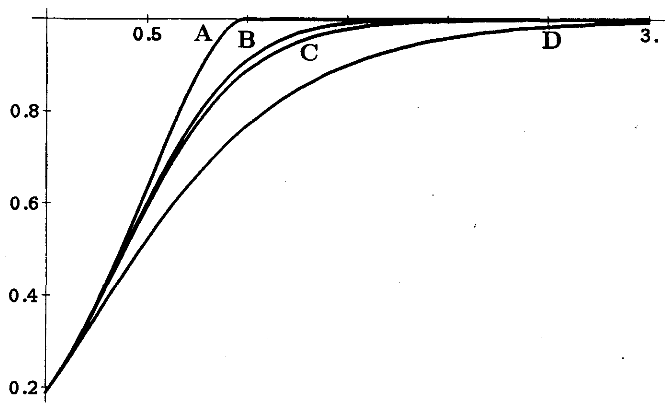

Fig. 1. $\mathbf{A} = \beta_{\phi_s}^{\mathrm{Unif}}$; $\mathbf{B} = \beta_{\phi_s}^{\mathrm{Nor}}$; $\mathbf{C} = \beta_{\phi_s}^{\mathrm{Log}}$; $\mathbf{D} = \beta_{\phi_s}^{\mathrm{De}}$.

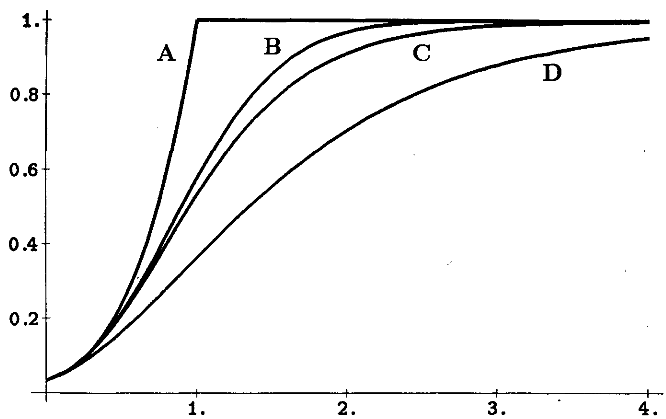

Fig. 2. $\mathbf{A} = \beta_{\phi_s}^{\mathrm{Unif}}$; $\mathbf{B} = \beta_{\phi_s}^{\mathrm{Nor}}$; $\mathbf{C} = \beta_{\phi_s}^{\mathrm{Log}}$; $\mathbf{D} = \beta_{\phi_s}^{\mathrm{De}}$.

3. BEHAVIOUR OF THE ONE SAMPLE MEDIAN TEST

For testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$, at level $\alpha < 0.5$ and samples of size n odd, the one sample median test, $\phi_m(M)$, is defined as

$$\phi_m(M) = \begin{cases} 1 & \text{if } M > k_n \\ 0 & \text{otherwise} \end{cases}$$

where M is the sample median and k_n such that

$$F_{n;\theta_0}(k_n) = P_{\theta_0}\{M \leq k_n\} = 1 - \alpha.$$

This test is uniformly most powerful if $F_\theta \in \mathcal{F}^*$ when $n = 1$ and has good robustness properties for finite sample sizes ([8], [9]). Also (see [10] and [11]) its p-value has an asymptotic normal distribution which allows us to get its influence function and qualitative robustness in the sense of Lambert ([16] and [17]).

Here we shall study the behaviour of its power funtion (nondecreasing in θ)

$$\begin{aligned} \beta^F_{\phi_m}(\theta) &= \sum_{x=\frac{n+1}{2}}^{n} \binom{n}{x} [1 - F_\theta(k_n)]^x [F_\theta(k_n)]^{n-x} \\ &= \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \int_0^{1-F_\theta(k_n)} x^{(n-1)/2} (1-x)^{(n-1)/2} \, \mathrm{d}x \end{aligned}$$

against changes in the model.

3.1. The One Sample Median Test and the Tail Ordering $<_t$

All the numerical results we have got untill now, confirm for ϕ_m the same monotonicity property that Proposition 1 established for the sign test. Nevertheless, by now we have only a proof for Box–Tiao families; for these we first need a lemma.

Lemma 1. If $F^{\beta_1}, F^{\beta_2} \in \mathcal{F}^*$ are two Box–Tiao families with densities f^{β_1} and f^{β_2}, and with k_n^1 and k_n^2 as critical points for ϕ_m, then

$$\text{if } \beta_1 < \beta_2 \implies f^{\beta_1}_{\theta_0}(k^1_n) > f^{\beta_2}_{\theta_0}(k^2_n).$$

P r o o f. The density function of a Box–Tiao familiy can be written as

$$f^\beta_\theta(x) = c(\beta) \exp\left\{ -\frac{1}{2} \left| \frac{x-\theta}{b(\beta)} \right|^{\frac{2}{1+\beta}} \right\}.$$

Because distributions belong to $\mathcal{F}^*$, condition (1) implies $c(\beta) = c$, $\forall\, \beta$; thus we can write

$$f^\beta_{\theta_0}(k_n) = c \exp\left\{ -\frac{1}{2} \left[w(\beta)\right]^2 \right\}$$

where $w(\beta)$ is the function

$$w(\beta) = \left[\frac{k_n(\beta) - \theta_0}{b(\beta)}\right]^{1/(1+\beta)} = [u(\beta)]^{1/(1+\beta)}$$

($k_n(\beta) > \theta_0$, $\forall\, \beta$, because $\alpha < 0.5$). Last equality is used as notation.

Then, to prove the lemma, it is enough to prove that function $w(\beta)$ is increasing.

Let us observe that $w(\beta)$ is differentiable because the functions

$$b(\beta) = \frac{1}{c\,\Gamma((3+\beta)/2)\; 2^{(3+\beta)/2}}$$

and $k_n(\beta)$, defined as the inverse of the distribution function of the corresponding Box–Tiao family in $y = B^{-1}(1-\alpha)$, are differentiable.

We have

$$1 - \alpha = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \int_0^{F^{\beta}_{\theta_0}(k_n)} x^{(n-1)/2}\,(1-x)^{(n-1)/2}\,\mathrm{d}x$$

where

$$F^{\beta}_{\theta_0}(k_n) = 0.5 + r(\beta) \int_0^{w(\beta)} \exp\left\{-\frac{1}{2}\,z^2\right\}\,(1+\beta)\,z^{\beta}\,\mathrm{d}z$$

and $r(\beta)$ the function

$$r(\beta) = \frac{1}{\Gamma((3+\beta)/2)\,2^{(3+\beta)/2}}.$$

Then, if $h(y)$ is the function — increasing and differentiable in (0,1) —

$$h(y) = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \int_0^{y} x^{(n-1)/2}\,(1-x)^{(n-1)/2}\,\mathrm{d}x$$

we can write that

$$1 - \alpha = h\left(0.5 + r(\beta) \int_0^{w(\beta)} \exp\left\{-\frac{1}{2}\,z^2\right\}\,(1+\beta)\,z^{\beta}\,\mathrm{d}z\right).$$

On differentiating with respect β, it will be

$$0 = h'\left(0.5 + r(\beta) \int_0^{w(\beta)} \exp\left\{-\frac{1}{2}\,z^2\right\}\,(1+\beta)\,z^{\beta}\,\mathrm{d}z\right)$$

$$\cdot\left[r'(\beta) \int_0^{w(\beta)} e^{-z^2/2}\,(1+\beta)\,z^{\beta}\,\mathrm{d}z + r(\beta)\right.$$

$$\left.\cdot\left(\exp\left\{-\frac{1}{2}[w(\beta)]^2\right\}\,[w(\beta)]^{\beta}\,(1+\beta)\,w'(\beta) + \int_0^{w(\beta)} e^{-z^2/2}\,z^{\beta}\,[1 + (1+\beta)\log z]\,\mathrm{d}z\right)\right]$$

and then,

$$w'(\beta) = \frac{-\displaystyle\int_0^{w(\beta)} e^{-z^2/2}\, z^\beta \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\log z\right)\right] \mathrm{d}z}{\exp\left\{-\frac{1}{2}[w(\beta)]^2\right\}\, [w(\beta)]^\beta\,(1+\beta)},$$

derivative that will be positive when and only when the integral

$$\int_0^{w(\beta)} e^{-z^2/2}\, z^\beta \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\log z\right)\right] \mathrm{d}z \tag{2}$$

is negative.

The function

$$e^{-z^2/2}\, z^\beta \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\log z\right)\right]$$

is negative untill the point $z_0 = \exp\{-1/(1+\beta) - r'(\beta)/r(\beta)\}$, from which it is always positive. If β in (2) is such that $w(\beta) \leq z_0$, the integral will be negative.

If $w(\beta) > z_0$, we have a negative area (integral till z_0) plus a positive area, but proving that

$$\int_0^{\infty} e^{-z^2/2}\, z^\beta \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\log z\right)\right] \mathrm{d}z < 0$$

the positive area will never exceed the negative one, concluding that $w'(\beta) > 0$ and then proving the lemma.

Easily we get

$$\frac{r'(\beta)}{r(\beta)} = -\frac{1}{2}\left[\log 2 + \mathrm{Digamma}\left(\frac{3+\beta}{2}\right)\right]. \tag{3}$$

And also that

$$\int_0^{\infty} e^{-z^2/2}\, z^\beta \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\log z\right)\right] \mathrm{d}z < 0$$

is equivalent to

$$\int_0^{\infty} e^{-y/2}\, y^{(\beta-1)/2} \left[1+(1+\beta)\left(\frac{r'(\beta)}{r(\beta)}+\frac{1}{2}\log y\right)\right] \mathrm{d}y < 0$$

i.e.,

$$\Gamma\left(\frac{1+\beta}{2}\right) + (1+\beta)\frac{r'(\beta)}{r(\beta)}\Gamma\left(\frac{1+\beta}{2}\right) + (1+\beta)\left[\Gamma'\left(\frac{1+\beta}{2}\right) + \frac{1}{2}\Gamma\left(\frac{1+\beta}{2}\right)\log 2\right] < 0$$

replacing now $r'(\beta)/r(\beta)$ by (3), the last expresion will be equivalent to

$$2 + (1+\beta)\left[2\,\mathrm{Digamma}\left(\frac{1+\beta}{2}\right) - \mathrm{Digamma}\left(\frac{3+\beta}{2}\right)\right] < 0$$

inequality (Figure 3) that follows from the properties of the digamma function (see [20], pages 5 to 8). □

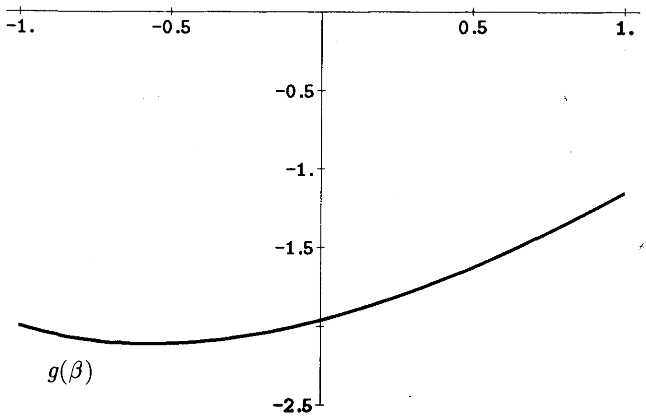

Fig. 3. $g(\beta) = 2 + (1+\beta)\,[2\text{Digamma}((1+\beta)/2) - \text{Digamma}((3+\beta)/2)]$.

Lemma 2. Let F and G be two Box–Tiao families. Then, it holds that

$$\text{If} \quad F <_t G \quad \Longrightarrow \quad F_\theta(k_n^F) \leq G_\theta(k_n^G), \quad \forall\, \theta > \theta_0.$$

Proof. If it is $\theta \geq k_n^F > \theta_0$, then it will be

$$F_\theta(k_n^F) \leq G_\theta(k_n^F).$$

And because it is $k_n^F \leq k_n^G$, it will be

$$F_\theta(k_n^F) \leq G_\theta(k_n^F) \leq G_\theta(k_n^G)$$

getting the inequality. (For all $F, G \in \mathcal{F}^*$, not necessarily Box–Tiao families.)

Now let $\theta_0 < \theta < k_n^F$.
$F_\theta(k_n^F) \leq G_\theta(k_n^G)$ is equivalent to

$$F\left(k_n^F - (\theta - \theta_0) - \theta_0\right) \leq G\left(k_n^G - (\theta - \theta_0) - \theta_0\right) \quad \text{if} \quad \theta_0 < \theta < k_n^F$$

i.e.,

$$F_{\theta_0}(k_n^F - x) \leq G_{\theta_0}(k_n^G - x), \quad \forall\ x \in (0, k_n^F - \theta_0).$$

Because k_n^F and k_n^G are critical points, it will be

$$F_{n;\theta_0}(k_n^F) = 1 - \alpha = G_{n;\theta_0}(k_n^G)$$

i.e.,

$$k_n^G = G_{n;\theta_0}^{-1} F_{n;\theta_0}(k_n^F) = G_{\theta_0}^{-1} F_{\theta_0}(k_n^F)$$

and so, $F_{\theta_0}(k_n^F) = G_{\theta_0}(k_n^G)$. Moreover, for Lemma 1 it is $f_{\theta_0}(k_n^F) > g_{\theta_0}(k_n^G)$, then there will exist an interval $(0, d)$, such that

$$F_{\theta_0}(k_n^F - x) \leq G_{\theta_0}(k_n^G - x), \quad \forall\ x \in (0, d).$$

If it is $d \geq k_n^F - \theta_0$ we have finished. Let us suppose it is $d < k_n^F - \theta_0$. Becuase of the continuity of functions F_{θ_0} and G_{θ_0}, in the extreme d we shall have the equality

$$F_{\theta_0}(k_n^F - d) = G_{\theta_0}(k_n^G - d)$$

that will be equivalent to

$$F_{n;\theta_0}(k_n^F - d) = G_{n;\theta_0}(k_n^G - d). \tag{4}$$

Considering now, as significance level α', one minus the common value (4),

$$1 - \alpha' = F_{n;\theta_0}(k_n^F - d) = G_{n;\theta_0}(k_n^G - d),$$

$k_n^F - d$ and $k_n^G - d$ would be the critical points, associated with F and G, for α', say $k_n^F(\alpha')$ and $k_n^G(\alpha')$, and then we should have again

$$F_{n;\theta_0}(k_n^F(\alpha')) = G_{n;\theta_0}(k_n^G(\alpha'))$$

and because of Lemma 1

$$f_{\theta_0}(k_n^F(\alpha')) > g_{\theta_0}(k_n^G(\alpha')).$$

Then, there would exist an interval to the left of $k_n^F(\alpha') = k_n^F - d$ in which

$$F_{\theta_0}(k_n^F - x) \leq G_{\theta_0}(k_n^G - x)$$

and then, d would not be the upper extreme of the interval in which we have the inequality. Hence, must be $d \geq k_n^F - \theta_0$. □

Proposition 2. Let F and G be two Box–Tiao families. Then, it holds that

$$\text{If} \quad F <_t G \quad \Longrightarrow \quad \beta_{\phi_m}^F(\theta) \geq \beta_{\phi_m}^G(\theta), \qquad \forall\, \theta > \theta_0.$$

P r o o f. Let $\theta > \theta_0$. The power function of median test, $\beta_{\phi_m}^F(\theta)$, is the cumulative distribution function of a Beta distribution $\beta((n+1)/2\,, (n+1)/2)$ in $1 - F_\theta(k_n^F)$

$$\beta_{\phi_m}^F(\theta) = B\left(1 - F_\theta(k_n^F)\right).$$

Using Lemma 2, if $F <_t G$ it will be $1 - F_\theta(k_n^F) \geq 1 - G_\theta(k_n^G)$, and now we shall get the result because of the monotonicity of any cumulative distribution function. □

4. JOINT BEHAVIOUR OF SIGN AND ONE SAMPLE MEDIAN TESTS

Because the sample median M is the Hodges–Lehmann estimator for θ associated with the sign test, the asymptotic behaviour of this test and the one sample median test will be the same when we use the traditional robustness measures which have an asymptotic character.

For instance, because of the asymptotic efficacy of both tests is the same,

$$\mathrm{eff}(S,F) = \mathrm{eff}(M,F) = 2\,f(0), \tag{5}$$

the contribution of both tests to the asymptotic relative efficiency will be the same when we do comparisons with another test based on a statistic T,

$$\mathrm{ARE}(S,T) = \left[\frac{\mathrm{eff}(S,F)}{\mathrm{eff}(T,F)}\right]^2 = \left[\frac{\mathrm{eff}(M,F)}{\mathrm{eff}(T,F)}\right]^2 = \mathrm{ARE}(M,T)$$

and, of course, $\mathrm{ARE}(S,M) = 1$.

Also, because of (5), the asymptotic power of both tests will be the same. And, of course, the influence function (see [26] and [27]) of both tests will agree:

$$IF_{\phi_s}(x;T,F_\theta) = \frac{\mathrm{sign}(x-\theta)}{2\,f(\theta)} = IF_{\phi_m}(x;T,F_\theta).$$

Thus, the influence functions in the sense of Lambert [16] will also be the same.

4.1. Behaviour with Finite Samples

In contrast with the identical asymptotic behaviour that we saw in the last paragraph, the sensitivity to changes in the model of the two tests can be high if the sample size is finite.

For instance, for testing the null hypothesis $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$, with sample size $n = 5$ and significance level $\alpha = 0.1875$, the power function of the sign test with uniform and double exponential distributions (dotted curves) looks more sensitive than the corresponding ones to the one sample median test (solid curves) also with uniform and double exponential distributions (Figure 4) that, as we saw in Propositions 1 and 2, are the extreme distributions.

The new measure of robustness, we are going to define, takes into account these remarks. It considers only tests that reach power 1 with the aim to avoid tests like the trivial one $\phi(x) = \alpha \quad \forall\, x$, which is insensitive to any change of distribution although having a constant power α, and also because the two tests considered in this paper are more sensitive far away from θ_0 (for instance, ϕ_s has a power function with first derivative independent of the model in $\theta = \theta_0$).

Definition 1. Let Φ_α be the class of level α tests ϕ with power function continuous, increasing (in θ such that $\alpha < \beta_\phi(\theta) < 1$) and that reach power 1, at least when $\theta \to \infty$.

We shall call *sensitivity* of a test $\phi \in \Phi_\alpha$ *against changes in the model inside a class of distributions* $\mathcal{F}$ *for a given power* γ to

$$\Delta_\phi(\gamma) = \sup_{F,G\in\mathcal{F}} \left| \left(\beta_\phi^F\right)^{-1}(\gamma) - \left(\beta_\phi^G\right)^{-1}(\gamma) \right|$$

and we shall say that ϕ_1 has *tail-power more robust* than ϕ_2 against changes in models of $\mathcal{F}$ class if and only if there exists an interval $(\beta, 1)$ such that $\forall\, \gamma \in (\beta, 1)$ is $\Delta_{\phi_1}(\gamma) < \Delta_{\phi_2}(\gamma)$.

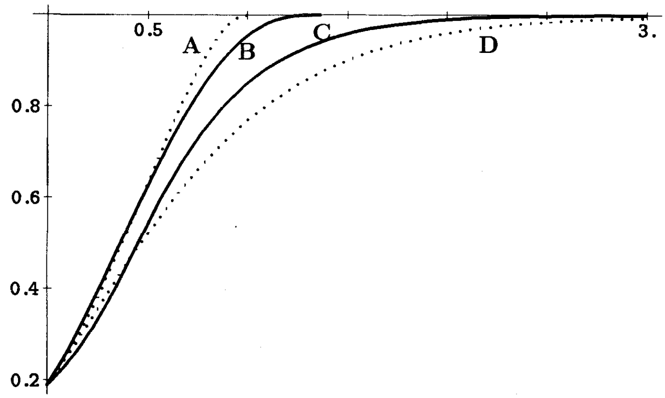

Fig. 4. $\mathbf{A} = \beta_{\phi_s}^{\mathrm{Unif}}$; $\mathbf{B} = \beta_{\phi_m}^{\mathrm{Unif}}$; $\mathbf{C} = \beta_{\phi_m}^{\mathrm{De}}$; $\mathbf{D} = \beta_{\phi_s}^{\mathrm{De}}$.

Remark 1. A test ϕ_1 with tail-power more robust than ϕ_2 is not necessarily better (in terms of power) than ϕ_2. For example, let us think in a test ϕ_1 with a power function that increases very slowly to one and with nearly the same power function for all distributions in $\mathcal{F}$, and a test ϕ_2 that reaches power one very quickly for all distributions in $\mathcal{F}$ except for some few of them.

Considerations of efficiency are being studied in order to get optimal robust tests in the sense of find the most powerful inside the class of tests with bounded sensitivity, $\Delta_\phi(\gamma) \leq c$.

Definition 2. The sensitivity against changes in the model is easier to obtain in tests with *ordered power with respect a class of distributions*, i.e., in tests ϕ such that, if $\prec$ is a tail-ordering on distributions of a class $\mathcal{F}$, have the following property,

$$\text{If } F, G \in \mathcal{F} \text{ and } F \prec G \implies \beta_\phi^F(\theta) \geq \beta_\phi^G(\theta), \quad \forall\, \theta \in \Theta_1$$

(or $\beta_\phi^F(\theta) \leq \beta_\phi^G(\theta)\ \ \forall\, \theta \in \Theta_1$) where Θ_1 is the alternative hypothesis.

From Propositions 1 and 2 we get the next result.

Proposition 3. It holds
(a) ϕ_s has ordered power with respect to the $\mathcal{F}^*$ class and tail-ordering $<_t$, and
(b) ϕ_m has ordered power with respect to the Box–Tiao families and tail-ordering $<_t$.

We conclude the paper with a result that confirms the idea suggested by Figure 4.

Proposition 4. ϕ_m has tail-power more robust than ϕ_s against changes in models of Box–Tiao families.

Proof. Because $\phi_m, \phi_s \in \Phi_\alpha$ and have ordered powers with respect to the Box–Tiao families, given a power $\gamma \in (\alpha, 1)$, it will be

$$\Delta_{\phi_m}(\gamma) = \left(\beta_{\phi_m}^{DE}\right)^{-1}(\gamma) - \left(\beta_{\phi_m}^{U}\right)^{-1}(\gamma)$$

and

$$\Delta_{\phi_s}(\gamma) = \left(\beta_{\phi_s}^{DE}\right)^{-1}(\gamma) - \left(\beta_{\phi_s}^{U}\right)^{-1}(\gamma).$$

Because the first alternative, θ_1^m, for which $\beta_{\phi_m}^U(\theta_1^m) = 1$ is

$$\theta_1^m = k_n^U + \frac{1}{2c}$$

and the first θ_1^s in which $\beta_{\phi_s}^U(\theta_1^s) = 1$ is

$$\theta_1^s = \theta_0 + \frac{1}{2c}$$

it will be $\theta_1^s < \theta_1^m$ because $\alpha < 0.5$; then, there will exist an interval, $(\beta_1, 1)$, in which

$$\left(\beta_{\phi_s}^U\right)^{-1}(\gamma) < \left(\beta_{\phi_m}^U\right)^{-1}(\gamma), \quad \forall\, \gamma \in (\beta_1, 1).$$

Moreover, for $\theta > k_n^{DE}$, it is

$$\beta'_{\phi_m}(\theta) = \frac{n!\; c\, e^{c(k_n^{DE}-\theta_0)(n+1)}}{\left[\left(\frac{n-1}{2}\right)!\right]^2\, 2^{(n-1)/2}} \left[1 - \frac{1}{2} e^{-2c(\theta - k_n^{DE})}\right]^{(n-1)/2} \left[e^{-2c(\theta-\theta_0)}\right]^{(n+1)/2}$$

and

$$\beta'_{\phi_s}(\theta) = \frac{n!c}{(k_\alpha - 1)!(n-k_\alpha)!2^{n-k_\alpha}} \left[1 - \frac{1}{2} e^{-2c(\theta-\theta_0)}\right]^{k_\alpha - 1} \left[e^{-2c(\theta-\theta_0)}\right]^{n-k_\alpha+1}.$$

The first term in both derivatives does not depend on θ. The second one converges to 1,

$$\lim_{\theta\to\infty} \left[1 - \frac{1}{2} e^{-2c(\theta - k_n^{DE})}\right]^{(n-1)/2} = \lim_{\theta\to\infty} \left[1 - \frac{1}{2} e^{-2c(\theta-\theta_0)}\right]^{k_\alpha - 1} = 1.$$

And since $n - k_\alpha + 1 < (n+1)/2$ the last term in both expressions makes that $\beta'_{\phi_m}(\theta)$ goes to zero more quickly than $\beta'_{\phi_s}(\theta)$; then, there exists a θ_1 such that

$$\beta'_{\phi_m}(\theta) < \beta'_{\phi_s}(\theta), \quad \forall\, \theta > \theta_1.$$

Since also it is

$$\lim_{\theta\to\infty} \beta_{\phi_m}(\theta) = \lim_{\theta\to\infty} \beta_{\phi_s}(\theta) = 1$$

it must be

$$\beta_{\phi_s}(\theta) < \beta_{\phi_m}(\theta), \quad \forall\, \theta > \theta_1$$

and because both power functions are increasing in θ, there will exist an interval $(\beta_2, 1)$ in which

$$\left(\beta_{\phi_s}^{DE}\right)^{-1}(\gamma) > \left(\beta_{\phi_m}^{DE}\right)^{-1}(\gamma), \quad \forall\, \gamma \in (\beta_2, 1).$$

Taking $\beta = \max\{\beta_1, \beta_2\}$, there will exist an interval $(\beta, 1)$ in which

$$\left(\beta_{\phi_s}^{U}\right)^{-1}(\gamma) < \left(\beta_{\phi_m}^{U}\right)^{-1}(\gamma), \quad \forall\, \gamma \in (\beta, 1)$$

and

$$\left(\beta_{\phi_s}^{DE}\right)^{-1}(\gamma) > \left(\beta_{\phi_m}^{DE}\right)^{-1}(\gamma), \quad \forall\, \gamma \in (\beta, 1)$$

i.e., in which

$$\Delta_{\phi_m}(\gamma) < \Delta_{\phi_s}(\gamma)$$

and the proof follows. □

Remark 2. A complementary study, in a future paper, of the behaviour of the power and level of these tests would be interesting in order to confirm the result of Proposition 4, but now considering "neighborhoods" of a distribution F_0 specified in terms, for instance, of ϵ-contamination

$$\mathcal{P}_\epsilon = \{F | F = (1-\epsilon)F_0 + \epsilon H\,,\; H \in \mathcal{M}\}$$

where $\mathcal{M}$ is the space of probability measures. Specially with F_0 equal to the normal distribution.

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REFERENCES

[1] R. E. Barlow and F. Proschan: Statistical Theory of Reliability and Life Testing. Holt, New York 1975.
[2] T. Bednarski: Minimax testing between Prohorov neighbourhoods. Statist. and Decisions *2* (1984), 281–292.
[3] Y. Benjamini: Is the t test really conservative when the parent distribution is long-tailed? J. Amer. Statist. Assoc. *78* (1983), 383, 645–654.
[4] A. Buja: Simultaneously least favorable experiments I: Upper standard functionals and sufficiency. Z. Wahrsch. verw. Geb. *65* (1984), 367–384.
[5] A. Buja: Simultaneously least favorable experiments II: Upper standard loss functions and their applications. Z. Wahrsch. verw. Geb. *69* (1985), 387–420.
[6] A. Buja: On the Huber-Strassen theorem. Probab. Theory Related Fields *73* (1986), 367–384.
[7] C. A. Field and E.M. Ronchetti: Small Sample Asymptotics. Institute of Mathematical Statistics, Lecture Notes-Monograph Series, Vol. 13, Hayward, California 1990.
[8] A. García-Pérez: Contraste de hipótesis con modelo variable. Estadística Española *29* (1987), 115, 63–86.
[9] A. García-Pérez: On robustness for hypotheses testing. In: Proc. 47th Session of the International Statistical Institute, Book 1, Paris, France 1989, pp. 363–364.
[10] A. García-Pérez: A note on robustness for hypotheses testing. In: Proc. 2nd Bernoulli Society World Congress and 53rd IMS Annual Meeting, Uppsala, Sweden 1990,ápp. 105.
[11] A. García-Pérez: A maximin test for approximate hypotheses using a tail ordering. In: Proc. 48th Session of the International Statistical Institute, Book 2, Cairo, Egypt 1991, pp. 527.
[12] A. García-Pérez: On robustness for hypotheses testing. International Statistical Review *61* (1993), 3, 369–385.
[13] P. Hall: The Bootstrap and Edgewoth Expansion. Springer-Verlag, New York 1992.
[14] P. J. Huber: A robust version of the probability ratio test. Ann. Math. Statist. *36* (1965), 1753–1758.
[15] P. J. Huber and V. Strassen: Minimax tests and the Neyman-Pearson lemma for capacities. Ann. Statist. *1* (1973), 251–263; *2* (1974), 223–224.
[16] D. Lambert: Influence functions for testing. J. Amer. Statist. Assoc. *76* (1981), 375, 649–657.
[17] D. Lambert: Qualitative robustness of tests. J. Amer. Statist. Assoc. *77* (1982), 378, 352–357.
[18] D. Lambert: Robust two-sample permutation tests. Ann. Statist. *13* (1985), 606–625.
[19] M. J. Lawrence: Inequalities of s-ordered distributions. Ann. Statist. *3* (1975), 413–428.
[20] N. N. Lebedev: Special functions and their applications. Dover, New York 1972.
[21] E. L. Lehmann: Ordered families of distributions. Ann. Math. Statist. *26* (1955), 399–419.
[22] Wei-Yin Loh: Bounds on AREs for restricted classes of distributions defined via tail-orderings. Ann. Statist. *12* (1984), 685–701.
[23] N. Reid: Saddlepoint methods and statistical inference. Statistical Science *3* (1988), 2, 213–238.
[24] H. Rieder: Least favorable pairs for special capacities. Ann. Statist. *5* (1977), 909–921.
[25] L. P. Rivest: Bartlett's, Cochran's, and Hartley's tests on variances are liberal when the underlying distribution is long-tailed. J. Amer. Statist. Assoc. *81* (1986), 393, 124–128.

[26] P. J. Rousseeuw and E. M. Ronchetti: The influence curve for tests. Research Report 21. Fachgruppe für Statistik, ETH, Zurich 1979.
[27] P. J. Rousseeuw and E. M. Ronchetti: Influence curves for general statistics. J. Comput. Appl. Math. 7 (1981), 161–166.
[28] W. R. Van Zwet: Convex Transformations of Random Variables. Math. Centrum, Amsterdam 1964.

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