Tomáš Roubíček

A generalized solution of a nonconvex minimization problem and its stability

Kybernetika, Vol. 22 (1986), No. 4, 289--298

Persistent URL: `http://dml.cz/dmlcz/124370`

A GENERALIZED SOLUTION OF A NONCONVEX MINIMIZATION PROBLEM AND ITS STABILITY

TOMÁŠ ROUBÍČEK[1]

It is well known that the set of the solutions of a minimization problem on an infinite-dimensional space X is not stable with respect to a perturbation of the minimized function. Here a generalized solution is defined as an element of a suitable completion of X. A necessary and sufficient condition for the completion of X to guarantee the stability of the set of the generalized solutions is given. It is shown that the generalized solution can be considered as a certain minimizing filter on X, which generalizes the notion of the minimizing sequence.

1. INTRODUCTION

Let us consider a real-valued function f on a space X and the problem:

$$\text{(1)} \qquad \begin{array}{c} \text{minimize } f(x) \\ \text{subject to } x \in X \,. \end{array}$$

Hereinafter, we denote its infimum by $\inf f = \inf f(X)$ and the set of the (classical) solutions by $\operatorname{Arginf} f = \{x \in X; f(x) = \inf f\}$. In case when X is a reflexive Banach space it is well known that either some type of convexity or the finite dimension of X results, roughly speaking, in a "good behaviour" of the mapping $f \mapsto \operatorname{Arginf} f$ with respect to perturbations of f. However, in a nonconvex infinite-dimensional case the solution of the minimization problem may fail to exist, although for a "near" minimized function it may exist. This pathology implies that the mapping $f \mapsto$ $\mapsto \operatorname{Arginf} f$ cannot be stable (i.e. upper semi-continuous) in any separated topology, and it also indicates that the classical solution is not a "natural" notion, at least for nonconvex functions.

Besides, optimization algorithms do not yield any solution of the minimization problem (except some special problems), but only a minimizing sequence. This was probably the reason that Golshtein [6] introduced the minimizing sequences as generalized solutions; see also [8]. Motivated by it, we try to introduce a "natural"

[1] This paper was prepared while the author was with the General Computing Centre of the Czechoslovak Academy of Sciences.

topology on the space of generalized solutions, with respect to which the set of the generalized solutions is stable. Of course, in the general-topology framework the notion of sequences is not a sufficiently powerful tool, thus in Section 4 the generalized solutions will be characterized as certain minimizing filters on X. For such purpose any linear structure on X will not be needed, and (except Sec. 5) X will be considered as a uniform space only. Also the space of the functions on X will be endowed with a corresponding uniformity (as a generalization of the well-known Hausdorff pseudometric on the hyperspace of epigraphs). In Section 2 the generalized solutions are constructed by extension of (1) onto a completion of the original space X with regard to some uniformity on X. In Section 3 the set of the generalized solutions is shown to be stable if this uniformity is sufficiently coarse in comparison with the original uniformity, Moreover, there exists an "optimal stable" uniformity. Considering the "optimal" uniformity, in Section 4 the set of the generalized solutions is characterized by a minimizing filter constructed by using the level sets of f. Finally, in some special cases the coincidence of the generalized and the classical solutions is shown, and some conditions of the first order for the generalized solution are stated.

For the "classical" approach to the stability of the mapping $f \mapsto \operatorname{Arginf} f$ we refer e.g. to Alt [1], Zollezzi [9], where further references can be found.

2. BASIC CONCEPTS AND THE GENERALIZED SOLUTION

First, we briefly recall some concepts from general topology; for details see e.g. [2], [3]. A uniformity $\mathscr{U}$ on a set X is a filter on $X \times X$ such that, for each $U \in \mathscr{U}$, $\Delta = \{(x, x); x \in X\} \subset U$, $U^{-1} = \{(x, y);\ (y, x) \in U\} \in \mathscr{U}$, and there exists $V \in \mathscr{U}$ such that $V^2 = V \circ V = \{(x, y);\ (x, z) \in V,\ (z, y) \in V \text{ for some } z \in X\} \subset U$. The pair $(X, \mathscr{U})$, shortly X, is called a uniform space. E.g., if (X, d) is a (pseudo) metric space, then the collection of all $\{(x, y); d(x, y) < \varepsilon\}$ with $\varepsilon > 0$ forms a filter base of a certain unique uniformity on X. We say that this uniformity is induced by d and any metric (especially normed linear) space will be considered as a uniform space endowed with this uniformity. If $\mathscr{U}$, $\mathscr{V}$ are two uniformities on X and the identity mapping on X is uniformly continuous from $(X, \mathscr{U})$ to $(X, \mathscr{V})$, shortly $(\mathscr{U}, \mathscr{V})$-uniformly continuous, we say that $\mathscr{V}$ is coarser than $\mathscr{U}$, or $\mathscr{U}$ is finer than $\mathscr{V}$ (in other words, $\mathscr{V} \subset \mathscr{U}$). For $U \in \mathscr{U}$, $S \subset X$, we put $U(S) = \{x \in X;\ (y, x) \in U$ for some $y \in S\}$. If for each $U \in \mathscr{U}$ there is a finite set S such that $U(S) = X$, the uniformity $\mathscr{U}$ is called precompact. For any uniformity on a set X there is the finest uniformity among all uniformities that are precompact and coarser than $\mathscr{U}$, which is called the precompact modification of $\mathscr{U}$ and will be denoted by $\mathscr{U}^*$. At the same time, $\mathscr{U}^*$ is the coarsest uniformity on X that makes uniformly continuous all the uniformly continuous function from $(X, \mathscr{U})$ to the interval $[0, 1]$, or equivalently to the usual two-point compactification $\bar{R}$ of the real line R (i.e. $\bar{R} = R \cup \{-\infty, +\infty\}$).

A uniformity $\mathscr{U}$ on X defines a unique topology on X by declaring the collection $\{U(\{x\}); U \in \mathscr{U}\}$ to be the filter of neighbourhoods of x for each $x \in X$. This topology is called to be induced by $\mathscr{U}$ and every uniform space is to be understood as the topological space with the topology just described. This topology is completely regular and, besides, every completely regular topology can be induced by a uniformity. Any uniformity on X inducing the topology of a given topological (or uniform) space X is called admissible. E.g. $\mathscr{U}^*$ is admissible for $(X, \mathscr{U})$.

In what follows, we suppose that $(X, \mathscr{U})$ is a Hausdorff uniform space (in applications, X will be mostly a metric space and $\mathscr{U}$ will be induced by its metric as described above). Let $\mathscr{V}$ be an admissible uniformity on X. Recall that there is a unique Hausdorff uniform space $(\bar{X}, \bar{\mathscr{V}})$, called the completion of $(X, \mathscr{V})$, such that $(\bar{X}, \bar{\mathscr{V}})$ is complete, X is dense in $\bar{X}$, and the trace of $\bar{\mathscr{V}}$ on $X \times X$ is just $\mathscr{V}$. Of course, $\bar{X}$ depends on $\mathscr{V}$. E.g., $\bar{X}$ is compact iff $\mathscr{V}$ is precompact.

Now, we extend the minimization problem (1) to $\bar{X}$. For an arbitrary function $f: X \to \bar{R}$, we define the extended function $\bar{f}: \bar{X} \to \bar{R}$ by

$$\bar{f}(x_0) = \liminf_{\substack{x \to x_0 \\ x \in X}} f(x) = \sup_{A \in \mathscr{N}(x_0)} \inf_{x \in A} f(x)\,, \tag{2}$$

where $\mathscr{N}(x_0) = \{B \cap X;\ B \text{ is a neighbourhood of } x_0 \text{ in } \bar{X}\}$, which is obviously a filter on X for any $x_0 \in \bar{X}$. Clearly, $\bar{f}$ is defined everywhere on $\bar{X}$ and is lower semicontinuous (briefly l.s.c.), and $\inf \bar{f}(\bar{X}) = \inf f$. The function $\bar{f}$ is called a l.s.c. regularization of f and, in the notation of Ioffe, Tihomirov [7], the pair $(\bar{X}, \bar{f})$ is called a regular extension of (X, f). Moreover, if f itself is l.s.c., then the restriction of $\bar{f}$ to X is just f.

Definition 1. An element $x \in \bar{X}$ will be called a generalized solution of the minimization problem (1) if $\bar{f}(x) = \inf f$. The set of all generalized solutions will be denoted by $\mathscr{A}rginf f$.

Of course, the generalized solution of (1) is just the classical solution of the extended problem:

$$\text{minimize } \bar{f}(x)$$
$$\text{subject to } x \in \bar{X}\,.$$

It should be emphasized that $\mathscr{A}rginf f$ depends on $\mathscr{V}$. Obviously, $\mathscr{A}rginf f$ is a nonempty compact set in $\bar{X}$ provided $\mathscr{V}$ is precompact. Now, we state a simple relation between the set of the classical solutions of (1) and the set of the generalized ones:

Theorem 1. $\operatorname{Arginf} f \subset X \cap \mathscr{A}rginf f$. Besides, if f is l.s.c., then $\operatorname{Arginf} f = X \cap \mathscr{A}rginf f$.

Proof. Let $x \in X$, $f(x) = \inf f$. Due to (2), we have $\bar{f}(x) \leqq f(x)$, hence $\bar{f} \geqq \inf f$ implies $\bar{f}(x) = \inf f$. Conversely, let $x \in X$, $\bar{f}(x) = \inf f$. If f is l.s.c., then $\bar{f}(x) = f(x)$. $\square$

In Section 1 we have mentioned the minimizing sequence, i.e. the sequence $\{x_n\}$, $x_n \in X$ and $f(x_n) \to \inf f$; see [6]. Now we have immediately:

Theorem 2. Let $\mathscr{V}$ be precompact. Then every minimizing sequence $\{x_n\}$ has got a cluster point in $\bar{X}$ and each such a cluster point belongs to $\mathscr{A}rginf\, f$.

Proof. As $\bar{X}$ is compact, hence countably compact as well, $\{x_n\}$ has a cluster point x in $\bar{X}$. Since $\{x_n\}$ is minimizing and $\bar{f}$ is l.s.c., we get $\bar{f}(x) = \inf f$. $\square$

3. STABILITY OF THE SET OF THE GENERALIZED SOLUTIONS

To investigate stability of the mapping $f \mapsto \mathscr{A}rginf\, f$, we need a topology on the set $\bar{R}^X$ of all functions $f: X \to \bar{R}$.

First, for any uniformity $\mathscr{Y}$ on a given set we define a uniformity $\mathscr{Y}^{\mathrm{H}}$ on the hyperspace of all its subsets by means of the base $\{V^{\mathrm{H}}; V \in \mathscr{Y}\}$, where $V^{\mathrm{H}} = \{(M, N); M \subset V(N) \,\&\, N \subset V(M)\}$; see [2, Chap. II]. Let us remark that if $\mathscr{Y}$ is induced by a metric d, then $\mathscr{Y}^{\mathrm{H}}$ is nothing else than the uniformity induced by the well-known Hausdorff pseudometric d^{H} on the hyper-space, which is defined by (see [2, Chap. IX]):

$$d^{\mathrm{H}}(M, N) = \begin{cases} \max\left(\sup\limits_{x \in M} \inf\limits_{y \in N} d(x, y),\ \sup\limits_{y \in N} \inf\limits_{x \in M} d(x, y)\right) & \text{for } M, N \text{ non-empty}, \\ 0 & \text{for } M, N \text{ empty}, \\ +\infty & \text{elsewhere.} \end{cases}$$

Now, on $\bar{R}^X$ we define a uniformity $\mathscr{U}^{\mathrm{F}}$ as the preimage of the uniformity $(\mathscr{U} \times \mathscr{W})^{\mathrm{H}}$ under the mapping $f \mapsto \operatorname{epi} f$, where $\operatorname{epi} f = \{(x, a) \in X \times \bar{R};\ f(x) \leqq a\}$, $\mathscr{W}$ is the unique admissible uniformity on $\bar{R}$, and, of course, $\mathscr{U}$ is the uniformity on X considered in the previous section. Analogously, we define the uniformity $\bar{\mathscr{V}}^{\mathrm{F}}$ on the set $\bar{R}^{\bar{X}}$ of all functions $\bar{f}: \bar{X} \to \bar{R}$. The uniformity $\mathscr{U}^{\mathrm{F}}$ on $\bar{R}^X$ naturally corresponds to the given uniformity $\mathscr{U}$ on X, and therefore it seems reasonable to require the stability (it means here upper semicontinuity) of the mapping $f \mapsto \mathscr{A}rginf\, f$ in the topology induced by $\mathscr{U}^{\mathrm{F}}$.

Definition 2. The uniformity $\mathscr{V}$ will be called stable (with respect to $\mathscr{U}$) if the set-valued mapping $f \mapsto \mathscr{A}rginf\, f$ is upper semicontinuous with respect to the topologies induced by $\mathscr{U}^{\mathrm{F}}$ and $\bar{\mathscr{V}}$; i.e. for any f and any neighbourhood A of $\mathscr{A}rginf\, f$ in $\bar{X}$ there is $U \in \mathscr{U}^{\mathrm{F}}$ such that $g \in U(\{f\})$ implies $\mathscr{A}rginf\, g \subset A$.

Theorem 3. $\mathscr{V}$ is stable (with respect to $\mathscr{U}$) if and only if $\mathscr{V}$ is coarser than the precompact modification of $\mathscr{U}$, i.e. $\mathscr{V} \subset \mathscr{U}^*$.

The proof of the "if" part follows immediately from Lemma 1 and 2, the "only if" part follows from Lemma 3 and 4.

Lemma 1. If $\mathscr{V} \subset \mathscr{U}$, then the mapping $f \mapsto \bar{f}$ of $\bar{R}^X$ into $\bar{R}^{\bar{X}}$ is $(\mathscr{U}^{\mathrm{F}}, \bar{\mathscr{V}}^{\mathrm{F}})$-uniformly continuous.

Proof. The assertion follows from the facts that $\operatorname{epi}\bar{f} = \operatorname{cl}_{\bar{X}\times\bar{R}} \operatorname{epi} f$, where $\operatorname{cl}_{\bar{X}\times\bar{R}}$ denotes the closure in $\bar{X} \times \bar{R}$ (see after a slight modification the proof of Corollary 2.1 in [4, Chap. I]); the mapping $M \mapsto \operatorname{cl}_{\bar{X}\times\bar{R}} M$ is $((\mathscr{V} \times \mathscr{W})^{\mathrm{H}}, (\bar{\mathscr{V}} \times \mathscr{W})^{\mathrm{H}}$-uniformly continuous; and the uniformity $(\mathscr{U} \times \mathscr{W})^{\mathrm{H}}$ is finer than $(\mathscr{V} \times \mathscr{W})^{\mathrm{H}}$. □

Lemma 2. Let $\mathscr{V}$ be precompact and A a neighbourhood of $\operatorname{Arginf}\bar{f}$ in $\bar{X}$ ($\bar{f} \in \bar{R}^{\bar{X}}$ is l.s.c.). Then there is $V \in \bar{\mathscr{V}}^{\mathrm{F}}$ such that $\bar{g} \in \bar{R}^{\bar{X}}$, $\bar{g} \in V(\{\bar{f}\})$ imply $\operatorname{Arginf}\bar{g} \subset A$.

Proof. Since $\bar{R}$ is equivalent (as a uniform, ordered space) to the interval $I = [0, 1]$, we may replace $\bar{R}$ by I in this proof. As $\operatorname{Arginf}\bar{f}$ is compact, there is an open symmetric entourage $U \in \bar{\mathscr{V}}$ such that $U^2(\operatorname{Arginf}\bar{f}) \subset A$. The set $B = \bar{X} \setminus U(\operatorname{Arginf}\bar{f})$ is compact and $\bar{f}$ is l.s.c., hence $\inf\bar{f}(B) > \inf\bar{f}$. Thus we have $\varepsilon = (\inf\bar{f}(B) - \inf\bar{f})/2 > 0$. Then $W = \{((x, a), (y, b)) \in (\bar{X} \times I) \times (\bar{X} \times I);\ (x, y) \in U,\ |a - b| < \varepsilon\}$ is an entourage from the uniformity of $\bar{X} \times I$. Let $\bar{g} \in I^{\bar{X}}$ and $\operatorname{epi}\bar{g} \in W^{\mathrm{H}}(\operatorname{epi}\bar{f})$. Then $\operatorname{epi}\bar{g} \subset W(\operatorname{epi}\bar{f})$ implies that for any $x \in \bar{X} \setminus U^2(\operatorname{Arginf}\bar{f})$ there is $y \in \bar{X}$ such that $(x, y) \in U$ and $\bar{f}(y) < \bar{g}(x) + \varepsilon$. Therefore $y \in B$, hence $\bar{f}(y) \geqq \inf\bar{f} + 2\varepsilon$, and also $\bar{g}(x) > \inf\bar{f} + \varepsilon$. Furthermore, we take some $z \in \operatorname{Arginf}\bar{f}$. From $\operatorname{epi}\bar{f} \subset W(\operatorname{epi}\bar{g})$ we obtain some $y \in U(\{z\})$ with $\bar{g}(y) < \bar{f}(z) + \varepsilon = \inf\bar{f} + \varepsilon$, which shows that x cannot belong to $\operatorname{Arginf}\bar{g}$. Therefore we get the estimate $\operatorname{Arginf}\bar{g} \subset U^2(\operatorname{Arginf}\bar{f}) \subset A$ and, taking for V the preimage of W^{H} under the mapping $\bar{f} \mapsto \operatorname{epi}\bar{f}$, we come to the required assertion. □

Lemma 3. If $\mathscr{V}$ is stable and precompact, then $\mathscr{V} \subset \mathscr{U}^*$.

Proof. Suppose the contradiction, i.e. $\mathscr{V} \not\subset \mathscr{U}^*$. Hence the identity on X is not $(\mathscr{U}^*, \mathscr{V})$-uniformly continuous. Since $\mathscr{V}$ is precompact, the identity is not even proximally continuous; see [3, Thm. 3.2.77]. Thus we can choose $M \subset X$ and $V \in \mathscr{V}$ such that, for any $U \in \mathscr{U}^*$, $U(M) \not\subset V^2(M)$. In fact, we have $W(M) \not\subset V^2(M)$ for any $W \in \mathscr{U}$ because for each $W \in \mathscr{U}$ there is $U \in \mathscr{U}^*$ such that $U(M) \subset W(M)$, namely $U = \{(x, y) \in X \times X;\ |g(x) - g(y)| \leqq \frac{1}{2}\}$ where $g: X \to [0, 1]$ is a $\mathscr{U}$-uniformly continuous function such that $g(x) = 0$ for $x \in M$ and $g(x) = 1$ for $x \in X \setminus W(M)$. For $S \subset X$, we denote by δ_S the indicator function of S, i.e. $\delta_S(x) = 0$ for $x \in S$ and $\delta_S(x) = +\infty$ for $x \in X \setminus S$. Clearly, for any neighbourhood G of δ_M in $(\bar{R}^X, \mathscr{U}^{\mathrm{F}})$ there exists $W \in \mathscr{U}$ such that $\delta_{W(M)} \in G$. However, $\mathscr{Arginf}\,\delta_{W(M)} = \operatorname{cl}_{\bar{X}} W(M) \not\subset \operatorname{cl}_{\bar{X}} V(M)$ for any $W \in \mathscr{U}$, and $\operatorname{cl}_{\bar{X}} V(M)$ is a neighbourhood of $\operatorname{cl}_{\bar{X}} M = \mathscr{Arginf}\,\delta_M$ in $\bar{X}$, which shows that $\mathscr{V}$ is not stable. □

Lemma 4. If $\mathscr{V}$ is stable, then $\mathscr{V}$ is precompact.

Proof. Suppose $\mathscr{V}$ not to be precompact. Thus there is a symmetric entourage $V \in \mathscr{V}$ such that $X \setminus V(S) \neq \emptyset$ for any finite set $S \subset X$. It is easy to choose an infinite sequence $\{x_n\}$ in X such that $m \neq n$ implies $(x_n, x_m) \notin V$. Define functions $f, f_n: X \to \bar{R}$ by

$$f(x) = \begin{cases} 1/k & \text{if } x = x_k \text{ for some } k, \\ +\infty & \text{elsewhere}, \end{cases}$$

$$f_n(x) = \begin{cases} 0 & \text{for } x = x_n, \\ f(x) & \text{elsewhere}. \end{cases}$$

Since the set of all x_n is $\mathscr{V}_1$-uniformly discrete in $\bar{X}$, we have $\mathscr{A}rginf f = \emptyset$. On the other hand, $\mathscr{A}rginf f_n = \{x_n\}$ and every neighbourhood of f in $(\bar{R}^X, \mathscr{U}^F)$ contains some f_n. Consequently, $\mathscr{V}$ is not stable. □

Remark. Theorem 3 shows that the precompact modification of $\mathscr{U}$ may be considered as the optimal uniformity which yields the stable generalized solutions. Denoting by $\bar{X}^*$ the "optimal" completion of X, i.e. the completion with respect to $\mathscr{U}^*$, then for every stable uniformity $\mathscr{V}$ the corresponding space $\bar{X}$ is a continuous image of $\bar{X}^*$ under a mapping that leaves X pointwise fixed; see [5, Chap. 15]. The space $\bar{X}^*$ is sometimes called the Samuel compactification of the uniform space $(X, \mathscr{U})$.

4. THE GENERALIZED SOLUTIONS AND MINIMIZING FILTERS ON X

Now it will be shown that each generalized solution of (1) may be considered as a minimizing filter on X with suitable properties. We recall some more or less usual definitions. A filter $\mathscr{F}$ on X is said to be f-minimizing if $\lim\inf_{\mathscr{F}} f = \inf f$ (i.e. $\inf f(A) = \inf f$ for any $A \in \mathscr{F}$); $\mathscr{U}$-Cauchy if $\forall V \in \mathscr{U}\ \exists R \in \mathscr{F}: R \times R \subset V$; $\mathscr{U}$-round if $\forall R \in \mathscr{F}\ \exists S \in \mathscr{F} \exists V \in \mathscr{U}: V(S) \subset R$; and $\mathscr{U}$-compressed if $\forall A, B \subset X$: $(\exists V \in \mathscr{U}: (A \times B) \cap V = \emptyset) \Rightarrow \exists R \in \mathscr{F}: (A \cup B) \cap R = \emptyset$. We consider the set of all filters on X ordered by the usual inclusion $\subset$, hence we can speak about minimal or maximal filters with given properties. Recall that in Section 2 we have denoted by $\mathscr{N}(x)$, $x \in \bar{X}$, the trace on X of the beighbourhood filter of x in $\bar{X}$.

Theorem 4. The mapping $x \mapsto \mathscr{N}(x)$ defines a one-to-one correspondence between the points $x \in \mathscr{A}rginf f$ and

a) f-minimizing minimal $\mathscr{V}$-Cauchy filters on X;

b) f-minimizing $\mathscr{V}$-round $\mathscr{V}$-Cauchy filters on X;

c) f-minimizing $\mathscr{V}$-round $\mathscr{V}$-compressed filters on X, provided $\mathscr{V}$ is precompact;

d) f-minimizing maximal $\mathscr{V}$-round filters on X, provided $\mathscr{V}$ is precompact.

Proof. Clearly, $x \in \mathscr{A}rginf f$ iff $\mathscr{N}(x)$ is f-minimizing. Because of the construction of the completion of $(X, \mathscr{V})$ used in [2, Chap. II], we can see that every $x \in \bar{X}$ corresponds by $x \mapsto \mathscr{N}(x)$ to exactly one minimal $\mathscr{V}$-Cauchy filter on X. Similar reasoning based on the construction used in [3, Thm. 6.3.23] gives the one-to-one correspondence with the $\mathscr{V}$-round $\mathscr{V}$-Cauchy filters. If $\mathscr{V}$ is precompact, then the $\mathscr{V}$-Cauchy filters coincide with the $\mathscr{V}$-compressed filters (see [3, Thm. 5.2.18]), and the $\mathscr{V}$-round $\mathscr{V}$-compressed filters coincide with the maximal $\mathscr{V}$-round filters (see [3, Thm. 6.3.12]). □

Remark. If $\mathscr{V} = \mathscr{U}^*$ (= the "optimal" stable uniformity), we can state further characterization using the assertions b), c), d) of Theorem 4, because the $\mathscr{U}^*$-round filters coincide with $\mathscr{U}$-round filters. Note that then d uses the original uniformity) $\mathscr{U}$ only. On the other hand, maximal filters with given properties can be "constructed" generally only by means of the axiom of choice.

To give fuller characterization of $\mathscr{A}rginf f$, we employ the common notion of the level sets of f, defined as $\text{lev}_\varepsilon f = \{x \in X;\ f(x) \leqq \inf f + \varepsilon\}$. The collection $\{V(\text{lev}_\varepsilon f);\ \varepsilon > 0,\ V \in \mathscr{U}\}$ is obviously a filter on X, we shall denote it by $\mathscr{A}(f)$.

Theorem 5. If $\mathscr{V} \supset \mathscr{U}^*$, then $\bigcap \{\mathscr{N}(x);\ x \in \mathscr{A}rginf f\} \supset \mathscr{A}(f)$. If $\mathscr{V} \subset \mathscr{U}^*$, then $\bigcap \{\mathscr{N}(x);\ x \in \mathscr{A}rginf f\} \subset \mathscr{A}(f)$.

Proof. Suppose $\mathscr{U}^* \subset \mathscr{V}$ and $A \in \mathscr{A}(f)$. Then $A \supset V(\text{lev}_\varepsilon f)$ for some $\varepsilon > 0$ and $V \in \mathscr{U}^*$. Thus also $V \in \mathscr{V}$ and we can take a symmetric entourage $U \in \bar{\mathscr{V}}$ such that $U^2 \cap (X \times X) \subset V$ and put $B = U(\text{lev}_{\varepsilon/2} \bar{f})$. Clearly, $B \supset U(\mathscr{A}rginf f)$. Let $x_0 \in B \cap X$. Then there is $y \in U(x_0)$ such that $\bar{f}(y) \leqq \inf f + \varepsilon/2$. In view of (2), there is $z \in U(y) \cap X$ with $f(z) \leqq \inf f + \varepsilon$. Thus $x_0 \in U^2(z) \subset V(z) \subset A$. In others words, $B \cap X \subset A$ and, since B is a neighbourhood of $\mathscr{A}rginf f$ in $\bar{X}$, we have $A \in \mathscr{N}(x)$ for each $x \in \mathscr{A}rginf f$.

To prove the second part of the theorem, we suppose $\mathscr{V} \subset \mathscr{U}^*$ and $A \in \mathscr{N}(x)$ for each $x \in \mathscr{A}rginf f$. Hence for each $x \in \mathscr{A}rginf f$ there exists an open neighbourhood B_x of x such that $B_x \cap X \subset A$. Put $B = \bigcup\{B_x;\ x \in \mathscr{A}rginf f\}$. Then B is open in $\bar{X}$, $B \supset \mathscr{A}rginf f$, and $B \cap X \subset A$. Since $\mathscr{V}$ is precompact, hence $\bar{X}$ is compact, and (similarly as in the proof of Lemma 2) there are $U \in \bar{\mathscr{V}}$ and $\varepsilon > 0$ such that $U^2(\mathscr{A}rginf f) \subset B$ and $\inf \bar{f}(\bar{X} \setminus U(\mathscr{A}rginf f)) \geqq \inf f + \varepsilon$. Clearly, $U^2(\mathscr{A}rginf f) \cap X \subset A$, $V = U \cap (X \times X) \in \mathscr{V} \subset \mathscr{U}$, and $V(\text{lev}_\varepsilon f) \subset A$ (since $\text{lev}_\varepsilon f \subset U(\mathscr{A}rginf f) \cap X$), which proves $A \in \mathscr{A}(f)$. □

Remark. In the case when (X, d) is a metric space and $\mathscr{U}$ is induced by d, the filter $\mathscr{A}(f)$ has the base $\{M_\varepsilon;\ \varepsilon > 0\}$ where $M_\varepsilon = \{x \in X;\ f(y) \leqq \inf f + \varepsilon$ for some $y \in X$ with $d(x, y) \leqq \varepsilon\}$. The sets M_ε can be considered as the sets of "ε-approximate solutions" of the minimization problem (1). Thus for $\mathscr{V} = \mathscr{U}^*$ we have got certain effective characterization of $\mathscr{A}rginf f$ in terms of the original uniformity $\mathscr{U}$, whereas the particular elements of $\mathscr{A}rginf f$ can be described only by means of either the axiom of choice or the uniformity $\mathscr{V}$.

5. SOME CONNECTIONS TO THE CLASSICAL CONCEPTS

In this section, X will be a Banach space with the norm $\|\cdot\|$, and, as above, $f: X \to \bar{R}$ will be the function to be minimized. We denote the effective domain of f by $\text{dom} f = \{x \in X;\ f(x) < +\infty\}$, and recall some usual definitions: f is proper if $\text{dom} f \neq \emptyset$ and $f(x) > -\infty$ for any $x \in X$; f is coercive if $\text{lev}_\varepsilon f$ is bounded for some $\varepsilon > 0$; and f is locally uniformly convex if $\forall x \in \text{dom} f\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \text{dom} f: f(x) + f(y) - 2f((x + y)/2) \leqq \delta \Rightarrow \|x - y\| \leqq \varepsilon$.

Theorem 6. Let X be a reflexive Banach space, $\mathscr{V}$ an admissible (with regard to the norm topology) uniformity on X, $f\colon X \to \bar{R}$ l.s.c., coercive and proper. If either f is locally uniformly convex or X has a finite dimension, then $\mathscr{Arginf} f = \operatorname{Arginf} f$.

Proof. The convex case: Since f is also strictly convex, there is exactly one classical solution (see e.g. [4, Chap. II]), denote it by x_0. By Theorem 1 we have $x_0 \in \mathscr{Arginf} f$. Let us consider $x \in \bar{X}$, $x \neq x_0$. Then there are disjoint neighbourhoods A, B of x_0, x in $\bar{X}$, respectively. Clearly, $A \cap X$ is a neighbourhood of x_0 in X and, in view of the uniform convexity of f, $f(y) > f(x_0) + \delta$ for some $\delta > 0$ and any $y \in X \setminus A$. Especially, $\inf f(B \cap X) \geqq f(x_0) + \delta$, and therefore $\bar{f}(x) \geqq f(x_0) + \delta$, which shows that $x \notin \mathscr{Arginf} f$.

The finite-dimensional case: In consequence of the coercivity of f, there is a closed ball B in X such that $\inf f(X \setminus B) > \inf f$. Let $x \in \mathscr{Arginf} f$. Since $\mathscr{N}(x)$ is f-minimizing (cf. Thm. 4), $A \cap B \neq \emptyset$ for any $A \in \mathscr{N}(x)$ and $\mathscr{M} = \{A \cap B;\ A \in \mathscr{N}(x)\}$ is a filter on B. Obviously, $\mathscr{M}$ is $\mathscr{V}|_B$-Cauchy ($\mathscr{V}|_B$ denotes the trace of $\mathscr{V}$ on $B \times B$). Since B is compact, $\mathscr{V}|_B = \mathscr{U}|_B$ and $\mathscr{M}$ converges to some element of B. Thus $x \in B \subset X$, and the rest follows from Theorem 1. $\square$

Remark. Clearly, the proof of Theorem 6 needs the local uniform convexity of f at the minimizing point x_0 only; however, the corresponding assumption on f would be rather ineffective because the solution x_0 is usually unknown.

It can be proved that in the cases discussed in Theorem 6 the mapping $f \mapsto \operatorname{Arginf} f$ is stable. Consequently, there is no need to extend the set of the classical solutions, which is in harmony with the assertion of Theorem 6. On the other hand, the following example shows a situation in which an (even strictly convex) l.s.c., coercive, and proper function on a Hilbert space yields a generalized solution different from the classical one. Note that in this case the set of the classical solutions is not stable.

Let $X = l_2$ (the Hilbert space of all squared summable sequences), and let $\mathscr{V}$ be precompact and admissible (according to the norm topology). Denoting $x = (\xi_1, \xi_2, \ldots) \in l_2$, we consider $f\colon X \to \bar{R}$ defined by

$$f(x) = \begin{cases} \sum_{i=1}^{\infty} 2^{-i} \xi_i^2 & \text{for } \|x\| \leqq 1\,, \\ +\infty & \text{elsewhere}\,. \end{cases}$$

Obviously, $\operatorname{Arginf} f = \{0\}$. However, there is a minimizing sequence $\{e_n\}$ with $\|e_n\| = 1$, namely

$$e_n = (\underbrace{0, \ldots, 0}_{n\text{-times}}, 1, 0, \ldots)\,.$$

Due to Theorem 2 there is a cluster point x of $\{e_n\}$ in $\bar{X}$ and $x \in \mathscr{Arginf} f$. Since $\|e_n\| \geqq 1$ and $\mathscr{V}$ is admissible, we see that $x \neq 0$. In other words, $\mathscr{Arginf} f \neq \operatorname{Arginf} f$.

Finally, we generalize the classical necessary and sufficient condition of the first order. For this reason we suppose f to be Gâteaux differentiable (with the derivative

$f': X \to X^*$) and put $D_f(x) = \|f'(x)\|_*$, where $\|\cdot\|_*$ is the usual norm on the dual space X^*. We define $\bar{D}_f: \bar{X} \to \bar{R}$ by the same manner as it was done in (2) for $\bar{f}$, i.e.

$$\bar{D}_f(x_0) = \liminf_{\substack{x \to x_0 \\ x \in X}} D_f(x)$$

Analogously, we can extend the norm $\|\cdot\|$, defining $|||\cdot|||: \bar{X} \to \bar{R}$ by

$$|||x_0||| = \liminf_{\substack{x \to x_0 \\ x \in X}} \|x\| .$$

We remark that $|||\cdot|||$ has all the properties of the extended norm (see Def. 2.5 in [8]) but the fact that the addition $x + y$ cannot be well defined if both x and y belong to $\bar{X} \setminus X$.

Theorem 7. Let f be Gâteaux differentiable, $\mathscr{V} \subset \mathscr{U}$, $\mathscr{U}$ the norm uniformity, and $x \in \mathscr{A}rginf f$. Then $\bar{D}_f(x) = 0$.

Proof. Let $A \in \mathscr{N}(x)$. In view of $\mathscr{V} \subset \mathscr{U}$ and Theorem 4b, the filter $\mathscr{N}(x)$ is $\mathscr{U}$-round, thus we can take $B \in \mathscr{N}(x)$ and $\delta > 0$ such that $y \in A$ whenever $\|y - y_0\| \leqq \delta$ for some $y_0 \in B$. For any ε, $0 < \varepsilon \leqq \delta^2$, we can choose some $z \in B \cap \operatorname{lev}_\varepsilon f$ because $\mathscr{N}(x)$ is f-minimizing. Thanks to the well-known Ekeland ε-variational theorem, see e.g. [4, Chap. I, Cor. 6.1], there exists $y \in X$ with $\|y - z\| \leqq \sqrt{\varepsilon}$ and $\|f'(y)\|_* \leqq \sqrt{\varepsilon}$. Clearly, $y \in A$ and, consisting $\varepsilon \to 0$, we come to $\inf D_f(A) = 0$, hence $\bar{D}_f(x) = 0$. □

Theorem 8. Let f be Gâteaux differentiable and convex, $\mathscr{V} \supset \mathscr{U}^*$, $\mathscr{U}$ the norm uniformity, $x \in \bar{X}$, $\bar{D}_f(x) = 0$, and let one of the following conditions be **fulfilled:**

i) f is coercive and $\mathscr{V}$ is precompact,

ii) $|||x||| < +\infty$.

Then $x \in \mathscr{A}rginf f$.

Proof. i) Consider the case when f is coercive and suppose, for a moment, that $\mathscr{N}(x)$ does not contain any bounded set. Take some $\varepsilon > 0$ for which $\operatorname{lev}_\varepsilon f$ is bounded, choose $y_0 \in \operatorname{lev}_{\varepsilon/2} f$, and put $B_K = \{y \in X;\ \|y - y_0\| \geqq K\}$. Clearly, $\{A \cap B_K;\ A \in \mathscr{N}(x)\ K < +\infty\}$ is a base of a $\mathscr{V}$-round filter because $\mathscr{V} \supset \mathscr{U}^*$, and we see that B_K must belong to $\mathscr{N}(x)$ since in the opposite case the filter in question would be larger than $\mathscr{N}(x)$ and therefore $\mathscr{N}(x)$ would not be maximal $\mathscr{V}$-round filter, which is not possible (note that $\mathscr{V}$ is precompact and cf. the reasoning in the proof of Thm. 4d). Thanks to the coercivity of f, $\inf f(B_K) \geqq \inf f + \varepsilon$ for K sufficiently large. For any $y \in B_K$, we consider the convex funtcion $f_y(a) = f(y_0 + a \,.\, s)$ with $s = (y - y_0)/\|y - y_0\|$. We have obviously the estimates $f_y(0) \leqq \inf f + \varepsilon/2$ and $f_y(K) \geqq \inf f + \varepsilon$ which imply $f_y'(a) \geqq \varepsilon/(2K)$ for any $a \geqq K$. Especially, the directional derivative of f at y with respect to the direction s is not less than $\varepsilon/(2K)$, hence $D_f(y) \geqq \varepsilon/(2K)$ for any $y \in B_K$. Therefore, $\bar{D}_f(x) \geqq \varepsilon/(2K) > 0$, which is a contradiction showing that $\mathscr{N}(x)$ contains a bounded set.

Now, set $\delta = \bar{f}(x) - \inf f$. If $\delta > 0$, then there are $A \in \mathcal{N}(x)$ and $0 < L < +\infty$ such that $\|y\| \leqq L$ for each $y \in A$ and $\inf f(A) \geqq \inf f + 2\delta/3$. Take $z \in \operatorname{lev}_{\delta/3} f$. Then, for any $y \in A$, $f(y) - f(z) \geqq \delta/3$ and $\|y - z\| \leqq L + \|z\|$. These estimates together with the convexity of f imply $D_f(y) \geqq \delta/(3L + 3\|z\|)$. Hence $\bar{D}_f(x) > 0$, which is a contradiction. Thus $\delta = 0$ and $x \in \mathcal{A}rginf f$.

ii) Now, if $|||x||| < +\infty$, then $\mathcal{N}(x)$ contains some bounded set (note that the function $|||\cdot|||$ is continuous on $\bar{X}$ because $\mathcal{V} \supset \mathcal{U}^*$). By the same arguments as in the precedent case, we get $x \in \mathcal{A}rginf f$. □

ACKNOWLEDGEMENTS

The author would like to thank to J. Hejcman (of the Mathematical Institute of the Czechoslovak Academy of Sciences) for helpful advice and thorough revision of the paper. Many thanks are also due to J. Jarušek and J. V. Outrata (both of the Institute of Information Theory and Automation, Czechoslovak Academy of Sciences) for useful remarks in course of preparing this paper.

(Received August 27, 1985.)

REFERENCES

[1] W. Alt: Lipschitzian perturbation of infinite optimization problems. In: Mathematical Programming with Data Perturbations II (A. V. Fiacco, ed.), Marcel Dekker, Inc., New York — Basel 1983, pp. 7—22.

[2] N. Bourbaki: General Topology. Hermann, Paris 1966.

[3] Á. Császár: General Topology. Akadémiai Kiadó, Budapest 1978.

[4] I. Ekeland and R. Teman: Convex Analysis and Variational Problems. North-Holland, Amsterdam 1976.

[5] L. Gillman and M. Jerison: Rings of Continuous Functions. Second edition. Springer-Verlag, Berlin — Heidelberg — New York 1976.

[6] E. G. Golshtein: Theory of Convex Programming. AMS, Transl. of Math. Monographs, Vol. 36, Providence, R. I. 1972.

[7] A. D. Ioffe and V. M. Tihomirov: Extension of variational problems. Trudy Moskov. Mat. Obšč. (Trans. Moscow Math. Soc.) *18* (1968), 207—273.

[8] E. Polak and Y. Y. Wardi: A study of minimizing sequences. SIAM J. Control Optim. *22* (1984), 599—609.

[9] T. Zolezzi: On stability analysis in mathematical programming. Math. Programming Study *21* (1984), 227—242.

Ing. Tomáš Roubíček, Ústav teorie informace a automatizace ČSAV, (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.