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Dynamic Solution of General Coalition-Game

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The presented paper is a free continuation of the author's paper [3]. The general coalition-game model, suggested in [3], is further investigated, and the concept of the dynamic, i.e. not strong, solution of such game is presented here. Some of its properties are derived and the mutual connections between both, strong and dynamic, solutions are discussed. The solution of the game is considered separately for coalition structures and for configurations formed by coalition structures and their admissible imputations.

0. INTRODUCTION

The concept of the general coalition-game was suggested in [3] and some of its special modifications were mentioned there. In the same paper [3], the strongly stable solution of such game was defined. This solution is a generalization of the "strong" or "ideal" solutions defined for some well known types of coalition-games.

The strong solutions have some ideal properties following from the fact that they fulfil the demands of all players and coalitions. On the other hand, such solutions are not achievable for lots of games. For such games, many other solutions were defined which do not fulfil all theoretically possible demands, but which exist in all (or all important) games and describe the behaviour of rational players sufficiently well. These "weak" solutions are rather various, adapted for different modifications of the game model and for different purposes of applications. It means that it is practically impossible to find such a solution of the general coalition-game which would be a generalization of all of them. The solution of the general coalition-game suggested in this paper is an analogy of the solution of the coalition-game with side-payments presented by the author in [2]. The main reason for choosing exactly that solution is that the solution deals with coalition structures and with imputations (or configurations) separately. In this way, the finding of the optimal coalitions and the finding of their optimal imputations may be considered separately. It is advantageous for a wide class of applications. Moreover, the solution presented here is divided into a few relatively simple steps which enable certain variability and modifications of it, if necessary. Some more discussion about this topic is introduced in [2], and also in [3] and in Section 4 of this paper. It is shown in the following sections that the strongly stable solution of the general coalition-game is a special, most pretentious, modification of the dynamicaly stable solution given here.

1. GENERAL COALITION-GAME

The concept of the general coalition-game was introduced and discussed in [3]. The definition of it is briefly repeated and a few auxiliary notions are introduced in this section. In the whole paper we denote by R the set of all real numbers.

Let us suppose that there exists a non-empty and finite set I, and the class of all non-empty subsets of I, denoted by $\mathscr{I}$, i.e.

$$\mathscr{I} = 2^I - \{\emptyset\} .$$

Let us suppose, further, that there exists a mapping V from 2^I into the class of subsets of R^I, i.e.

$$V : 2^I \to 2^{R^I} ,$$

such that for all $K \in 2^I$

(1.1) $V(K)$ is closed ;

(1.2) if $\mathbf{x} = (x_i)_{i \in I} \in V(K)$, $\mathbf{y} = (y_i)_{i \in I} \in R^I$, $x_i \geqq y_i$ for all $i \in K$, then $\mathbf{y} \in V(K)$;

(1.3) $V(K) \neq \emptyset$, $V(K) = R^I \Leftrightarrow K = \emptyset$.

Then the pair

$$\Gamma = (I, V)$$

is called the *general coalition-game*. Elements of I and $\mathscr{I}$ are called *players* and *coalitions*, respectively. The mapping V is the *general characteristic function* of the game Γ.

Every partition of I into disjoint non-empty coalitions is called a *coalition structure*. The class of all coalition structures in the given game is denoted by $\mathbf{K}$. If $\mathscr{K}$ and $\mathscr{L}$ are two coalition structures then we say that *$\mathscr{K}$ is a subpartition of $\mathscr{L}$* iff for every coalition $K \in \mathscr{K}$ there exists a coalition $L \in \mathscr{L}$ such that $K \subset L$.

If $\mathbf{M} \subset \mathbf{K}$ is a class of coalition structures then we denote by $\bigcup \mathbf{M}$ the set of coalitions

$$\bigcup \mathbf{M} = \{K \in \mathscr{I} : K \in \mathscr{K} \textit{ for some } \mathscr{K} \in \mathbf{M}\} .$$

If $K \in \mathscr{I}$ is a coalition then the set

$$(1.4) \qquad V^*(K) = \{\mathbf{x} = (x_i)_{i\in I}: \textit{if } \mathbf{y} = (y_i)_{i\in I} \in R^I,\ y_i \geqq x_i \textit{ for all } i \in K \textit{ and } y_j > x_j \textit{ for some } j \in K, \textit{ then } \mathbf{y} \notin V(K)\} =$$
$$= \{\mathbf{x} = (x_i)_{i\in I}: \textit{for all } \mathbf{z} = (z_i)_{i\in I} \in V(K) \textit{ is either } x_i > z_i \textit{ for some } i \in K \textit{ or } x_i = z_i \textit{ for all } i \in K\},$$

is called the *superoptimum* of the coalition K.

If $\mathcal{M} \subset \mathcal{I}$ is a non-empty class of coalitions then we denote

$$V(\mathcal{M}) = \bigcap_{K\in\mathcal{M}} V(K), \quad V^*(\mathcal{M}) = \bigcap_{K\in\mathcal{M}} V^*(K),$$

and for the empty subclass $\emptyset$ of $\mathcal{I}$ is by (1.3) and (1.4),

$$V(\emptyset) = V^*(\emptyset) = R^I.$$

It is easy to see that for $\mathcal{M} \subset \mathcal{I}$, $\mathcal{N} \subset \mathcal{I}$ is

$$V(\mathcal{M} \cup \mathcal{N}) = V(\mathcal{M}) \cap V(\mathcal{N}), \quad V^*(\mathcal{M} \cup \mathcal{N}) = V^*(\mathcal{M}) \cap V^*(\mathcal{N}).$$

It was shown in [3] already that for every coalition $K \in \mathcal{I}$ and for every coalition structure $\mathcal{K} \in \mathbf{K}$ is

$$(1.5) \qquad V(K) \cup V^*(K) = R^I$$

and

$$(1.6) \qquad V(K) \cap V^*(K) \neq \emptyset, \quad V(\mathcal{K}) \cap V^*(\mathcal{K}) \neq \emptyset.$$

Every real valued vector $\mathbf{x} = (x_i)_{i\in I} \in R^I$ is called an *imputation*. If there exists a coalition structure $\mathcal{K} \in \mathbf{K}$ such that $\mathbf{x} \in V(\mathcal{K})$ then $\mathbf{x}$ is called an *admissible imputation* in $\mathcal{K}$.

Every pair $(\mathcal{K}, \mathbf{x})$, $\mathcal{K} \in \mathbf{K}$, $\mathbf{x} \in R^I$, is called a *configuration*. If, moreover, $\mathbf{x} \in V(\mathcal{K})$, then the configuration $(\mathcal{K}, \mathbf{x})$ is called *admissible* in the given general coalition-game.

It is obvious that for every coalition $K \in \mathcal{I}$ and every pair of imputations $\mathbf{x} \in R^I$, $\mathbf{y} \in R^I$ such that $x_i \leqq y_i$ for all $i \in K$, the relation $\mathbf{x} \in V^*(K)$ implies $\mathbf{y} \in V^*(K)$, and, on the other hand, the relation $\mathbf{y} \in V(K)$ implies $\mathbf{x} \in V(K)$. Moreover, if $\mathbf{x} \in V(K) - V^*(K)$ then there exists an imputation $\mathbf{z} \in V(K) \cap V^*(K)$ such that $x_i \leqq z_i$ for all $i \in I$. The last statement was proved in [3]. It is useful to introduce the following property of the general characteristic function V and of the superoptimum function V^*.

Lemma 1. If $K, L \in \mathcal{I}$ are coalitions such that $V(K) \cap V^*(L) \neq \emptyset$ then also $V^*(K) \cap V^*(L) \neq \emptyset$ and $V^*(K) \cap V(K) \cap V^*(L) \neq \emptyset$.

Proof. Let us consider an arbitrary imputation $\mathbf{x} \in V(K) \cap V^*(L)$. If $\mathbf{x} \in V^*(K)$ then the statement holds. If $\mathbf{x} \in V(K) - V^*(K)$ then there exists an imputation $\mathbf{y} \in V(K) \cap V^*(K)$ such that $y_i \geqq x_i$ for all $i \in I$. It means that $\mathbf{y} \in V^*(L)$, too.

2. EFFECTIVE COALITION STRUCTURES

The concept of effectivity, introduced in this section, is an important auxiliary concept by means of which the coalitions and coalition structures may be classified, and those ones having real chance to be realized may be chosen. The effectivity, defined below, represents a generalization of the analogous notion defined in [2], and it was partly used in [3], too. The main purpose of the effectivity concept is to exclude the coalition structures which can not appear during the bargaining process. In the second degree, the concept of effectivity helps to exclude the coalition structures which will be, in an extremally simple way, substituted by some other coalition structures, more advantageous for all players. For this purpose, two kinds of effectivity of coalition structures are introduced here. The effectivity from below, excluding the coalition structures in which at least one coalition can not fulfil the rational demands of its members and sub-coalitions, and the effectivity from above (which is less important) helping to eliminate the coalition structures which may appear but which will be immediately substituted by some universally more useful ones, only by means of simple union of coalitions. First of all, the following preliminary notion is defined.

Definition 1. A coalition $K \in \mathscr{I}$ is called *effective* iff there exists an imputation $\mathbf{x} \in V(K)$ which belongs to the superoptimum of all subcoalitions of K, i.e. iff

$$V(K) \cap \Big(\bigcap_{J \in \mathscr{I}, J \subset K} V^*(J) \Big) \neq \emptyset\,.$$

Remark 1. It follows from (1.6) immediately that all one-player coalitions are effective.

Definition 2. Let $\mathscr{K} \in \mathbf{K}$ be a coalition structure. Then $\mathscr{K}$ is called *effective from below* iff each coalition $K \in \mathscr{K}$ is effective. The class of all coalition structures being effective from below will be denoted by $\mathbf{K}_{\text{ef}}$. Further, $\mathscr{K}$ is called *effective from above*, iff for every coalition structure $\mathscr{L}$ effective from below, $\mathscr{L} \in \mathbf{K}_{\text{ef}}$, such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$, there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that $\mathbf{x} \notin V(\mathscr{L}) - V^*(\mathscr{L})$. The class of all coalition structures being effective from above will be denoted by $\mathbf{K}^{\text{ef}}$. The coalition structure $\mathscr{K}$ is called *effective* iff it is effective from below and effective from above. The class of all effective coalition structures will be denoted by $\mathbf{K}^{\text{ef}}_{\text{ef}}$.

Remark 2. It follows from Remark 1 immediately that the coalition structure containing exactly all one-player coalitions is effective from below. Moreover, if some coalition $K \in \mathscr{I}$ is effective then there always exists a coalition structure effective from below and containing K.

Lemma 2. A coalition structure $\mathscr{K} \in \mathbf{K}$ is effective from below iff there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that for all $\mathscr{J} \in \mathbf{K}$ which are subpartitions of $\mathscr{K}$, is $\mathbf{x} \in V^*(\mathscr{J})$.

Proof. If $\mathscr{K} \in \mathbf{K}$ is effective from below then all coalitions $K \in \mathscr{K}$ are effective and, consequently, for every $K \in \mathscr{K}$ there exists an imputation

$$\mathbf{x}^K = (x_i^K)_{i \in I} \in V(K)$$

such that $\mathbf{x}^K \in V^*(J)$ for all $J \in \mathscr{I}$, $J \subset K$. As the coalitions in $\mathscr{K}$ are disjoint, it is possible to construct an imputation

$$\mathbf{x} = (x_i)_{i \in I} \in R^I, \quad x_i = x_i^K, \quad i \in K, \quad K \in \mathscr{K}.$$

Then

$$\mathbf{x} \in \bigcap_{K \in \mathscr{K}} V(K) = V(\mathscr{K}),$$

and $\mathbf{x} \in V^*(J)$ for all $J \in \mathscr{I}$, such that $J \subset K$ for some $K \in \mathscr{K}$. It means

$$\mathbf{x} \in \bigcap_{K \in \mathscr{K}} \Big(\bigcap_{J \in \mathscr{I}, J \subset K} V^*(J) \Big).$$

Consequently, $\mathbf{x} \in V^*(\mathscr{J})$ for all coalition structures $\mathscr{J} \in \mathbf{K}$ which are subpartitions of $\mathscr{K}$. The opposite implication is obvious.

Lemma 3. If $L \in \mathscr{I}$ is not effective then there exists a set of coalitions

$$\mathscr{I}^*(L) = \{J \colon J \in \mathscr{I},\ J \subset L,\ J \textit{ is effective}\} \tag{2.1}$$

such that

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}^*(L)} V^*(J).$$

Proof. Let us denote

$$\mathscr{I}(L) = \{J \colon J \in \mathscr{I},\ J \subset L\}.$$

If L is not effective then

$$V(L) \cap \Big(\bigcap_{J \in \mathscr{I}(L)} V^*(J) \Big) = \emptyset.$$

As $V(L) \cup V^*(L) = R^I$, according to (1.5), the inclusions

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}(L)} V^*(J)$$

and

$$(2.2) \qquad V^*(L) \supset \bigcap_{J \in (\mathscr{I}(L) - \{L\})} V^*(J)$$

are true. The same is true for all non-effective coalitions $J \in \mathscr{I}$. Consequently, it is true even for all non-effective coalitions from $\mathscr{I}(L) - \{L\}$. Hence, using (2.2), the inclusion

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}^*(L)} V^*(J)$$

is proved.

Remark 3. The sets $\mathscr{I}^*(L)$ used in the previous lemma are non-empty for all coalitions $L \in \mathscr{I}$, as they contain at least the effective one-player coalitions. It means that also

$$\bigcup_{J \in \mathscr{I}^*(L)} J = L$$

for all coalitions $L \in \mathscr{I}$.

Remark 4. It follows from Lemma 3 immediately that a coalition $K \in \mathscr{I}$ is effective iff

$$V(K) \cap \left(\bigcap_{J \in \mathscr{I}^*(K)} V^*(J) \right) \neq \emptyset$$

where the set of coalitions $\mathscr{I}^*(K)$ is defined by (2.1).

Remark 5. A coalition structure $K \in \mathscr{K}$ is effective from above iff there does not exist any $\mathscr{L} \in \mathbf{K}_{\text{ef}}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$ and $V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})$, as follows from Definition 2 immediately.

Lemma 4. If the coalition structure $\mathscr{K}$ is not effective from above, i.e. $\mathscr{K} \in \mathbf{K} - \mathbf{K}^{\text{ef}}$, then there exists an effective coalition structure $\mathscr{M} \in \mathbf{K}_{\text{ef}}^{\text{ef}}$ such that

$$V(\mathscr{K}) \subset V(\mathscr{M}) - V^*(\mathscr{M}) .$$

Proof. If $\mathscr{K} \in \mathbf{K} - \mathbf{K}^{\text{ef}}$ then there exists $\mathscr{L} \in \mathbf{K}_{\text{ef}}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$ and $V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})$, as follows from Definition 2. Let us denote by $\mathbf{K}(\mathscr{K})$ the class of exactly all coalition structures $\mathscr{L}$ having the properties introduced above. Let us choose $\mathscr{M} \in \mathbf{K}(\mathscr{K})$ such that for no $\mathscr{N} \in \mathbf{K}(\mathscr{K})$ the inclusion

$$V(\mathscr{N}) - V^*(\mathscr{N}) \supset V(\mathscr{M})$$

holds. It is obvious that such a coalition structure $\mathscr{M}$ exists in the finite class $\mathbf{K}(\mathscr{K})$, and that it is effective from above and effective from below.

Theorem 1. There exists at least one effective coalition structure in every general coalition-game, i.e.

$$\mathbf{K}_{\text{ef}}^{\text{ef}} = \mathbf{K}_{\text{ef}} \cap \mathbf{K}^{\text{ef}} \neq \emptyset .$$

Proof. The coalition structure containing exactly all one-player coalitions is effective from below, as follows from Remark 2. It means that the class $\mathbf{K}_{\text{ef}}$ is non-empty. Let us define the partial ordering relation on $\mathbf{K}_{\text{ef}}$ in such a way that for $\mathscr{K}, \mathscr{L} \in \mathbf{K}_{\text{ef}}$ is $\mathscr{L} \succ \mathscr{K}$ iff $\mathscr{K}$ is a subpartition of $\mathscr{L}$ and $V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})$. This relation is antireflexive, antisymetric and transitive. As the class $\mathbf{K}_{\text{ef}}$ is finite and non-empty, there exists at least one maximal element in $\mathbf{K}_{\text{ef}}$ according to the partial ordering relation given above. The class $\mathbf{K}_{\text{ef}}^{\text{ef}}$ is equal to the class of exactly all those maximal elements.

3. SAFE COALITION STRUCTURES

By means of the concept of effectivity from below, we are able to eliminate the coalition structures which can not arize in the given game. Moreover, by means of the effectivity from above, we may distinguish the coalition structures effective from below, the "life" of which is limited by the ability of their coalitions to increase their profit by union of their cooperation. In certain sense, the effectivity of coalition structures expresses their ability to withstand the objections of their subpartitions and, on the other hand, of the coalition structures the subpartition of which the given coalition structure is.

In this section, we are interested in the ability of coalition structures to resist the objections of the remaining coalitions and coalition structures. Coalition structure preserves its existence by choosing such an imputation which corresponds with the rational demands of as many coalitions as possible. The coalition structure may be considered to be safe against the objections (or demands) of some set of coalitions, if it chooses its admissible imputations which can not be dominated by any of the objecting coalitions and which preserves its effectivity from below. This idea is formulated, more exactly, in the following definition.

Definition 3. Let $\mathscr{K} \in \mathbf{K}$ be a coalition structure and $\mathscr{M} \subset \mathscr{I}$ be a set of coalitions. We say that $\mathscr{K}$ *is safe against* $\mathscr{M}$, and write $\mathscr{K}\ \sigma\ \mathscr{M}$, iff there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that $\mathbf{x} \in V^*(\mathscr{M})$. If $\mathscr{K}$ is not safe against $\mathscr{M}$, we write $\mathscr{K}$ non $\sigma\ \mathscr{M}$.

Remark 6. The preceding definition implies immediately that for $\mathscr{K} \in \mathbf{K}$ and $\mathscr{M} \subset \mathscr{I}$ is $\mathscr{K}$ safe against $\mathscr{M}$ iff

$$V(\mathscr{K}) \cap \big(\bigcap_{M \in \mathscr{M}} V^*(\mathscr{M}) \big) \neq \emptyset .$$

Lemma 5. If $\mathscr{K} \in \mathbf{K}$ is a coalition structure then

(1) for any set of coalitions $\mathscr{M} \subset \mathscr{I}$, exactly one of the relations $\mathscr{K}\ \sigma\ \mathscr{M}$ and $\mathscr{K}$ non $\sigma\ \mathscr{M}$ is true;

(2) if $\mathscr{K} \in \mathbf{K} - \mathbf{K}_{ef}$ then there exists a coalition $K \in \mathscr{K}$ and a set of coalitions $\mathscr{M} \subset \{J\colon J \in \mathscr{I},\ J \subset K,\ J \textit{ is effective}\}$, such that $\mathscr{K}$ non $\sigma\ \mathscr{M}$;

(3) if $\mathscr{K} \in \mathbf{K} - \mathbf{K}^{ef}$ then there exists $\mathscr{L} \in \mathbf{K}^{ef}_{ef}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$, $\mathscr{K}$ non $\sigma\ \mathscr{L}$ and $\mathscr{L}\ \sigma\ \mathscr{K}$;

(4) if $\mathscr{M} \subset \mathscr{N} \subset \mathscr{I}$ and $\mathscr{K}\ \sigma\ \mathscr{N}$ then $\mathscr{K}\ \sigma\ \mathscr{M}$;

(5) if $\mathscr{M} \subset \mathscr{I}$ then $\mathscr{K}\ \sigma\ \mathscr{M}$ implies $\mathscr{K}\ \sigma\ (\mathscr{M} \cup \mathscr{K})$;

(6) if $\emptyset$ is the empty subset of $\mathscr{I}$ then always $\mathscr{K}\ \sigma\ \emptyset$.

Proof. The first statement follows from Definition 3 immediately. If $\mathscr{K}$ is not effective from below then there exists a coalition $K \in \mathscr{K}$ which is not effective, i.e

$$V(K) \cap \Big(\bigcap_{J \in \mathscr{I}, J \subset K} V^*(J) \Big) = \emptyset\,,$$

and, according to Lemma 3, for every non-effective coalition $L \in \mathscr{I}$, $L \subset K$, it is

$$\bigcap_{J \in \mathscr{I}^*(L)} V^*(J) \subset V^*(L)\,,$$

where $\mathscr{I}^*(L) = \{J\colon J \in \mathscr{I},\ J \subset L,\ J \textit{ is effective}\}$. It means that

$$V(K) \cap \Big(\bigcap_{J \in \mathscr{I}^*(K)} V^*(J) \Big) = \emptyset\,,$$

too. Consequently, $V(\mathscr{K}) \cap V^*(\mathscr{M}) = \emptyset$ for some $\mathscr{M} \subset \mathscr{I}^*(K)$. If $\mathscr{K}$ is not effective from above then there exists an effective coalition structure $\mathscr{L}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$ and

$$V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})\,,$$

as follows from Lemma 4. It means that $V(\mathscr{K}) \cap V^*(\mathscr{L}) = \emptyset$ and, according to Remark 6, $\mathscr{K}$ non $\sigma\ \mathscr{L}$. On the other hand, $V(\mathscr{K}) \subset V(\mathscr{L})$ and $V(\mathscr{K}) \cap V^*(\mathscr{K}) \neq \emptyset$, as follows from (1.6). It means that $V(\mathscr{L}) \cap V^*(\mathscr{K}) \neq \emptyset$ and $\mathscr{L}\ \sigma\ \mathscr{K}$. If $\mathscr{M} \subset \mathscr{N} \subset \mathscr{I}$ and $\mathscr{K}\ \sigma\ \mathscr{N}$ then, according to Definition 3 or Remark 5, $\mathscr{K}\ \sigma\ \mathscr{M}$, too. If $\mathscr{K}\ \sigma\ \mathscr{M}$ for some $\mathscr{M} \subset \mathscr{I}$, then there exists an imputation $\mathbf{x} \in V(\mathscr{K}) \cap V^*(\mathscr{M})$. Condition (2.1), Lemma 1 and the fact that coalitions in $\mathscr{K}$ are disjoint imply that there exists an imputation $\mathbf{y} \in V(\mathscr{K}) \cap V^*(\mathscr{K}) \cap V^*(\mathscr{M})$, and, consequently, $\mathscr{K}\ \sigma\ (\mathscr{M} \cup \mathscr{K})$. Finally, if $\emptyset$ is the empty subset of $\mathscr{I}$ then $V^*(\emptyset) = R^I$, according to Section 1. Hence, $V(\mathscr{K}) \cap V^*(\emptyset) \neq \emptyset$ for all coalition structures $\mathscr{K}$, and $\mathscr{K}\ \sigma\ \emptyset$.

Lemma 6. If $\mathscr{J} \in \mathbf{K}$, $\mathscr{K} \in \mathbf{K}$ and if $V(\mathscr{J}) \subset V(\mathscr{K}) - V^*(\mathscr{K})$ then $\mathscr{J}$ non $\sigma\ \mathscr{K}$ and $\mathscr{K}\ \sigma\ \mathscr{J}$.

Proof. If $V(\mathscr{J}) \subset V(\mathscr{K}) - V^*(\mathscr{K})$ then $V(\mathscr{J}) \cap V^*(\mathscr{K}) = \emptyset$. Hence, $\mathscr{J}$ non σ $\mathscr{K}$.
Further, as $V(\mathscr{J}) \subset V(\mathscr{K})$, relation (1.6) and the fact that coalitions in any coalition structure are disjoint imply that

$$\emptyset \neq \bigcap_{J \in \mathscr{J}} (V(J) \cap V^*(J)) \subset \bigcap_{J \in \mathscr{J}} V(J) \subset \bigcap_{K \in \mathscr{K}} V(K) .$$

It is equivalent with

$$\emptyset \neq V^*(\mathscr{J}) \cap V(\mathscr{J}) \subset V(\mathscr{K}) ,$$

then

$$V(\mathscr{K}) \cap V^*(\mathscr{J}) \neq \emptyset , \quad \mathscr{K} \sigma \mathscr{J} .$$

Investigating the property of safety of coalition structures, it is sufficient to be interested in the effective coalitions only, as follows from the next statement.

Lemma 7. Let $\mathscr{K} \in \mathbf{K}$, $\mathscr{M} \subset \mathscr{J}$ and let $\mathscr{K}$ non σ $\mathscr{M}$. If there exists a coalition $L \in \mathscr{M}$ which is not effective then there exists a set of effective coalitions $\mathscr{J} \subset \mathscr{J}^*(L)$, where $\mathscr{J}^*(L)$ is defined by (2.1), such that $\mathscr{K}$ non σ $((\mathscr{M} - \{L\} \cup \mathscr{J})$.

Proof. If $L \in \mathscr{M}$ is not effective then, according to Lemma 3,

$$V^*(L) \supset \bigcap_{J \in \mathscr{J}^*(L)} V^*(J) .$$

It means that

$$\emptyset = V(\mathscr{K}) \cap V^*(\mathscr{M}) = V(\mathscr{K}) \cap V^*(L) \cap \Big(\bigcap_{M \in \mathscr{M} - \{L\}} V^*(M)\Big) \supset$$

$$\supset V(\mathscr{K}) \cap V^*(\mathscr{J}) \cap \Big(\bigcap_{M \in \mathscr{M} - \{L\}} V^*(M)\Big) = V(\mathscr{K}) \cap V^*((\mathscr{M} - \{L\}) \cup \mathscr{J})$$

for at least one set of coalitions $\mathscr{J} \subset \mathscr{J}^*(L)$.

Corollary. It follows from the previous lemma that if $\mathscr{K} \in \mathbf{K}$, $\mathscr{M} \subset \mathscr{J}$, if $\mathscr{K}$ non σ $\mathscr{M}$ and if there exists $\mathscr{L} \in \mathbf{K} - \mathbf{K}_{ef}$ such that $\mathscr{L} \subset \mathscr{M}$ then there exists a class of coalition structures $\{\mathscr{J}_1, \ldots, \mathscr{J}_r\} \subset \mathbf{K}_{ef}$ such that

$$\mathscr{K} \text{ non } \sigma \left((\mathscr{M} - \mathscr{L}) \cup (\mathscr{J}_1 \cup \ldots \cup \mathscr{J}_r)\right) .$$

4. STABILITY IN GENERAL COALITION-GAMES

The auxiliary concepts of effectivity and safety introduced in the preceding sections enable us to define the concept of stability in the considered general coalition-game. The stability will be considered for coalition structures and for configurations separately. In this model, the stability of the coalition structures is the primary one, and the stability of configurations (or imputations) is derived from it in a simple way.

The coalition structures which are considered to be stable must fulfil a few of natural conditions. Namely, their realization in the given game must be possible, i.e. they should be effective from below. Moreover, they must be safe against as many other coalitions as possible. Among the coalition structures fulfilling those conditions we prefer the ones which are stable from above. Analogously, the stable configurations must be formed by stable coalition structures and by the imputations which guarantee their stability.

In this paper, two kinds of stability are distinguished, the strong and the dynamic one. If the strong stability exists then the considered coalition structure or configuration fulfils the reasonable demands of all players and coalitions. It means that it is safe against all objections and can not be replaced by another one, more advantageous for some coalition. Such strong stability was already introduced and discussed in [3]. It is not difficult to verify that the strongly stable coalition structures do not exist in some games. Because of it, some weaker solutions, existing in all or almost all games, were presented in the literature. The main goal of this paper is to introduce one of such solutions for the general coalition-games. This solution will be called dynamically stable.

The dynamically stable solution need not fulfil the demands of all players and coalitions. It means that it is not, generally, safe against all objections. During the bargaining process, the coalition structure or configuration having the properties of the dynamic stability, but not of the strong one, repetitively appears. After the objections of some players or coalitions, against which it is not safe, this coalition structure or configuration is substituted by another one. The new coalition structure or configuration is not safe against all objections, too, and it is substituted by another one. Each of these dynamically stable coalition structures or configurations may appear, after a few steps of the described process, again. They are equivalent in the sense that it is impossible to decide in which one of them the bargaining process really stops.

The solutions presented in this paper are analogous to those ones given in [2] for coalition-games with side-payments. It was written in [3], already, that the solutions of different types of coalition-games presented in literature may be roughly divided into two groups. The first one contains the strong solutions fulfilling the demands of all coalitions. The strong stability, investigated in [3] and mentioned also in the following sections of this paper, is a generalization of some solutions from this first group. This fact was proved for the core of the coalition-game with side-payments in [3]. It can also be proved for strong solutions of some other coalition-games which are special cases of the general coalition-game. The second group of solutions contains the weak ones, not fulfilling the demands of all coalitions. These solutions are rather various and they can not be generalized by one weak solution of the general coalition-game. However, the dynamically stable solution suggested in the following section reflects at least some useful properties of the most important weak solutions known from the literature.

The solution suggested in [2] was chosen to be generalized for the general coalition-games according to its ability to separate the stability of coalition structures from the stability of imputations and configurations. It is useful in some non-traditional applications of the coalition-games theory, namely for clustering and groups forming models. This separation cannot be absolute, of course. The stability of configurations is derived from the stability of coalition structures, and, on the other hand, the stability of coalition structures is defined by means of some existentional properties of imputations. It means that both concepts, stability of coalition structures and stability of configurations, are closely connected. However, this particular separation of the concepts enables wider variability of their applications.

The general ideas, discussed above, are more exactly formulated and investigated in the following sections.

5. STABILITY OF COALITION STRUCTURES

Before introducing the definitions of stability (strong or dynamic) of coalition structures, we define a useful auxiliary notion.

Definition 4. The mapping Δ from the class of the coalition structures into the family of subclasses of the class of the coalition structures effective from below, Δ: $\mathbf{K} \to 2^{\mathbf{K}_{ef}}$, such that for all $\mathcal{K} \in \mathbf{K}$ is

$$(5.1) \qquad \Delta(\mathcal{K}) = \{\mathcal{J} \in \mathbf{K}_{ef}: \textit{there exist } \mathbf{M} \subset \mathbf{K}_{ef} \textit{ such that } \mathcal{K}\ \sigma\,(\bigcup \mathbf{M}) \textit{ and } \mathcal{K} \text{ non } \sigma\,((\bigcup \mathbf{M}) \cup \mathcal{J})\}$$

is called the *domination structure* in the given general coalition-game.

Remark 7. It follows from Definition 3 immediately that for any $\mathcal{K} \in \mathbf{K}_{ef}$ is

$$(5.2) \qquad \Delta(\mathcal{K}) = \{\mathcal{J} \in \mathbf{K}_{ef}: \textit{there exists } \mathbf{M} \subset \mathbf{K}_{ef} \textit{ such that } V(\mathcal{K}) \cap (\bigcap_{\mathcal{M} \in \mathbf{M}} V^*(\mathcal{M})) \neq \emptyset \textit{ and } V(\mathcal{K}) \cap (\bigcap_{\mathcal{M} \in \mathbf{M}} V^*(\mathcal{M})) \cap V^*(\mathcal{J}) = \emptyset\}\,.$$

The previous remark implies that the concept of safety of coalition structures may be avoided in the definition of the domination structure Δ. In fact, it is an auxiliary concept without any deeper influence on the definition of stability. However, the concept of the safety of coalition structures enables us to simplify the formulation and formal description of some steps of the following explanation. It is also useful for the easy interpretation and understanding of some steps of the solution model suggested here.

Remark 8. It follows from Lemma 5 that for any $\mathscr{K} \in \mathbf{K}$ is $\mathscr{K} \notin \Delta(\mathscr{K})$.

Lemma 8. If a coalition structure $\mathscr{K}$ is not effective, $\mathscr{K} \in \mathbf{K} - \mathbf{K}_{ef}^{ef}$, then $\Delta(\mathscr{K}) \neq \emptyset$ and also $\Delta(\mathscr{K}) \cap \mathbf{K}_{ef}^{ef} \neq \emptyset$.

Proof. Let $\mathscr{K} \in \mathbf{K} - \mathbf{K}_{ef}$. Then there exists the class of coalition structures

$$\mathbf{J} = \{\mathscr{J}: \mathscr{J} \in \mathbf{K},\ \mathscr{J} \textit{ is a subpartition of } \mathscr{K}\}$$

such that

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{J} \in \mathbf{J}} V^*(\mathscr{J})) = \emptyset\,.$$

Let us construct another class

$$\mathbf{J}^* = \{\mathscr{J}: \mathscr{J} \in \mathbf{K}_{ef},\ \mathscr{J} \textit{ is a subpartition of } \mathscr{K}\}\,.$$

It is obvious tha $\mathbf{J}^* \subset \mathbf{J}$ and $\mathbf{J}^* \subset \mathbf{K}_{ef}$. It follows from Lemma 3 immediately that

$$\bigcap_{\mathscr{J} \in \mathbf{J}} V^*(\mathscr{J}) = \bigcap_{\mathscr{J} \in \mathbf{J}} \bigcap_{J \in \mathscr{J}} V^*(J) \supset \bigcap_{\mathscr{J} \in \mathbf{J}^*} \bigcap_{J \in \mathscr{J}} V^*(J) = \bigcap_{\mathscr{J} \in \mathbf{J}^*} V^*(\mathscr{J})\,,$$

and, consequently,

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{J} \in \mathbf{J}^*} V^*(\mathscr{J})) = \emptyset\,.$$

It means that there exists a class of coalition structures $\mathbf{L} \subset \mathbf{J}^*$ and a coalition structure $\mathscr{J} \in \mathbf{J}^* - \mathbf{L} \subset \mathbf{K}_{ef}$, such that

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{L} \in \mathbf{L}} V^*(\mathscr{L})) \neq \emptyset$$

and

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{L} \in \mathbf{L}} V^*(\mathscr{L})) \cap V^*(\mathscr{J}) = \emptyset\,.$$

Hence, $\mathscr{J} \in \Delta(\mathscr{K})$. Let, on the other hand, $\mathscr{K} \in \mathbf{K}_{ef} - \mathbf{K}^{ef}$. It follows from Lemma 5 that $\mathscr{K}\ \sigma\ \emptyset$ where $\emptyset$ is the empty class of coalitions. Moreover, it follows from Lemma 5 that there exists a coalition structure $\mathscr{L} \in \mathbf{K}_{ef}^{ef}$ such that $\mathscr{K}$ non $\sigma\ \mathscr{L}$. It means that $\mathscr{L} \in \Delta(\mathscr{K})$.

Lemma 9. If $\mathscr{K} \in \mathbf{K}$ is a coalition structure then $\Delta(\mathscr{K}) = \emptyset$ if and only if $\mathscr{K}\ \sigma\ (\bigcup \mathbf{K}_{ef})$.

Proof. Let $\mathscr{K}\ \sigma\ (\bigcup \mathbf{K}_{ef})$. Then

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{L} \in \mathbf{K}_{ef}} V^*(\mathscr{L})) \neq \emptyset$$

and there exists no $\mathscr{M} \in \mathbf{K}_{ef}$ such that $\mathscr{M} \in \Delta(\mathscr{K})$, as follows from (5.2). If, on the other hand, $\Delta(\mathscr{K}) = \emptyset$ then, according to Definition 4, $\mathscr{K}\ \sigma\ (\bigcup \boldsymbol{M})$ for all $\boldsymbol{M} \subset \mathbf{K}_{ef}$, and also for $\boldsymbol{M} = \mathbf{K}_{ef}$.

Remark 9. It follows from Lemma 8 and Lemma 9 immediately that $\Delta(\mathscr{K}) = \emptyset$ iff $\mathscr{K} \in \mathbf{K}_{ef}$ and $\mathscr{K}\ \sigma\ (\bigcup \mathbf{K}_{ef})$.

In the following two definitions, we introduce the concepts of stability, strong and dynamic, of coalition structures. The notion of strong stability was already defined and briefly investigated in [3], and both types of stability were interpreted and discussed in Section 4 of this paper.

Definition 5. A coalition structure $\mathscr{K} \in \mathbf{K}$ is called *strongly stable* iff the value of the dominantion structure $\Delta(\mathscr{K})$ is empty. The class of strongly stable coalition structures in the given general coalition-game is denoted by $\mathbf{S}^*$.

Remark 10. It follows from the previous definition and from Lemma 8 immediately that $\mathbf{S}^* \subset \mathbf{K}_{ef}^{ef}$.

Theorem 2. A coalition structure $\mathscr{K}$ is strongly stable if and only if it is safe against the set of all effective coalitions.

Proof. It follows from Lemma 9 immediately that $\mathscr{K} \in \mathbf{S}^*$ iff $\mathscr{K}\ \sigma\ (\bigcup \mathbf{K}_{ef})$. Remark 2 implies that

$$\bigcup \mathbf{K}_{ef} = \{M \in \mathscr{I} : \exists\ \mathscr{M} \in \mathbf{K}_{ef},\ M \in \mathscr{M}\} = \{M \in \mathscr{I} :\ M \text{ is effective}\}\,,$$

and the statement is proved.

Corollary. It follows from the preceding theorem immediately that

$$\mathscr{K} \in \mathbf{S}^* \Leftrightarrow \mathscr{K}\ \sigma\ (\bigcup \mathbf{K}_{ef})\,.$$

The definition of the strong stability of coalition structures given in [3] was formally different from Definition 5 written above. However, Theorem 6 from [3], the previous Theorem 2 and Remark 7 of this work imply that both definitions are equivalent. As the strong stability of coalition structures was already investigated in [3], the dynamic stability and the mutual connections between both types of stability will be the main object of the presented paper.

Definition 6. A coalition structure $\mathscr{K}$ is called *dynamically stable* iff for every finite sequence of coalition structures

$$\{\mathscr{K}_1, \mathscr{K}_2, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{ef}\,,$$

such that

$$\mathscr{K}_1 = \mathscr{K}, \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}), \quad r = 2, \ldots, n,$$

there exists a finite sequence of coalition structures

$$\{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{ef}$$

such that

$$\mathscr{L}_1 = \mathscr{K}_n, \quad \mathscr{L}_r \in \Delta(\mathscr{L}_{r-1}), \quad r = 2, \ldots, m, \quad \mathscr{K} \in \Delta(\mathscr{L}_m).$$

The class of all dynamically stable coalition structures is denoted by $\mathbf{S}$.

Remark 11. It follows from the previous definitions immediately that $\mathbf{S}^* \subset \mathbf{S} \subset \mathbf{K}_{ef}$.

Lemma 10. If $\mathscr{K}, \mathscr{L} \in \mathbf{K}_{ef}$ and $\mathscr{L} \in \Delta(\mathscr{K})$, $\mathscr{K} \in \mathbf{S}$, then also $\mathscr{L} \in \mathbf{S}$.

Proof. If $\mathscr{K} \in \mathbf{S}$ and $\mathscr{L} \in \Delta(\mathscr{K})$ then there exists for every finite sequence

$$\{\mathscr{K}, \mathscr{L}, \mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{ef}$$

such that

$$\mathscr{K}_1 \in \Delta(\mathscr{L}), \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}), \quad r = 2, \ldots, n,$$

there exists a finite sequence

$$\{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{ef}$$

such that

$$\mathscr{L}_1 \in \Delta(\mathscr{K}_n), \quad \mathscr{L}_r \in \Delta(\mathscr{L}_{r-1}), \quad r = 2, \ldots, m, \quad \mathscr{K} \in \Delta(\mathscr{L}_m),$$

and, consequently, $\mathscr{L} \in \mathbf{S}$.

Lemma 11. If $\mathscr{K}, \mathscr{L} \in \mathbf{K}$, $\mathscr{K} \in \mathbf{S}^*$ and if there exists a finite sequence of coalition structures $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{ef}$ such that

$$\mathscr{K}_1 \in \Delta(\mathscr{L}), \quad \mathscr{K} \in \Delta(\mathscr{K}_n), \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}), \quad r = 2, \ldots, n,$$

then the coalition structure $\mathscr{L}$ is not dynamically stable, i.e. $\mathscr{L} \notin \mathbf{S}$.

Proof. As $\Delta(\mathscr{K}) = \emptyset$, there is no finite sequence of coalition structures $\{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{ef}$ such that $\mathscr{L}_1 \in \Delta(\mathscr{K})$, $\mathscr{L} \in \Delta(\mathscr{L}_m)$ and $\mathscr{L}_r \in \Delta(\mathscr{L}_{r-1})$ for $r = 2, \ldots, m$. Consequently, $\mathscr{L} \notin \mathbf{S}$.

Lemma 12. If $\mathscr{K} \in \mathbf{K} - \mathbf{K}^{ef}$ and $\mathscr{L} \in \mathbf{K}_{ef}$ is the coalition structure for which $\mathscr{K}$ is a subpartition of $\mathscr{L}$ and $V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})$ then $\mathscr{K} \notin \mathbf{S}^*$ and if $\mathscr{K} \in \mathbf{S}$ then also $\mathscr{L} \in \mathbf{S}$.

Proof. If $\mathcal{K} \in \mathbf{K} - \mathbf{K}^{\text{cf}}$ then $\Delta(\mathcal{K}) \neq \emptyset$, according to Lemma 8. Consequently, $\mathcal{K} \notin \mathbf{S}^*$. If $\mathcal{L}$ fulfils the assumptions of this lemma, then, according to Lemma 6, $\mathcal{K}$ non $\sigma\,\mathcal{L}$. As $\mathcal{K}\ \sigma\ \emptyset$, where $\emptyset$ is the empty class of coalitions, it follows from Definition 4 that $\mathcal{L} \in \Delta(\mathcal{K})$. Lemma 10 implies that for $\mathcal{K} \in \mathbf{S}$ is also $\mathcal{L} \in \mathbf{S}$.

In the intuitive considerations introduced in Section 4, as well as in the Introduction, we have supposed that the bargaining process hardly ever stops in the coalition structure not effective from above, but that it is substituted by some effective unions of its coalitions. The previous Lemma 12 implies that this intuitively supposed step my be done without loss of the dynamic stability of the resultant coalition structure.

It means that looking for the rational bargaining result of the general coalition-game for the class of coalition structures, we may limit ourselves to two classes of them. Namely, to the class of strongly stable coalition structures (which are always effective, as follows from Remark 10), and to the class of effective and dynamically stable coalition structures. In symbols, we are interested in classes

$$\mathbf{S}^* \quad \text{and} \quad \mathbf{S} \cap \mathbf{K}^{\text{ef}}_{\text{cf}} .$$

The main properties and mutual connections between these two classes of coalition structures are discussed and investigated in the remaining part of this section.

When discussing the motivation of introducing the weak (in our terminology dynamically stable) solutions of the general coalition-games, it was claimed that the strong solution does not exist in some games. In the following example, we show that this statement is really true.

Example 1. Let us consider a general coalition-game $\Gamma = (I, \mathbf{V})$ such that $I = \{1, 2, 3\}$ and

$$\begin{aligned}
\mathbf{V}(\{i\}) &= \{\mathbf{x} = (x_k)_{k=1,2,3} : x_i \leqq 0\}\,, \quad i = 1, 2, 3\,,\\
\mathbf{V}(\{i, j\}) &= \{\mathbf{x} = (x_k)_{k=1,2,3} : x_i + x_j \leqq 1\}\,, \quad i, j = 1, 2, 3,\ i \neq j\,,\\
\mathbf{V}(I) &= \{\mathbf{x} = (x_k)_{k=1,2,3} : x_1 + x_2 + x_3 \leqq 1\}\,.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{V}^*(\{i\}) &= \{\mathbf{y} = (y_k)_{k=1,2,3} : y_i \geqq 0\}\,, \quad i = 1, 2, 3\,,\\
\mathbf{V}^*(\{i, j\}) &= \{\mathbf{y} = (y_k)_{k=1,2,3} : y_i + yj \geqq 1\}\,, \quad i, j = 1, 2, 3,\ i \neq j\,,\\
\mathbf{V}^*(I) &= \{\mathbf{y} = (y_k)_{k=1,2,3} : y_1 + y_2 + y_3 \geqq 1\}\,.
\end{aligned}$$

Let us denote the coalition structures

$$\mathcal{K}_0 = \{\{1\}, \{2\}, \{3\}\}\,, \quad \mathcal{K}_1 = \{\{1\}, \{2, 3\}\}\,, \quad \mathcal{K}_2 = \{\{2\}, \{1, 3\}\}\,,$$
$$\mathcal{K}_3 = \{\{3\}, \{1, 2\}\}\,, \quad \mathcal{K}_4 = \{I\}\,.$$

 It can be easily verified that

$$\bigcap_{i=1}^{3} V^*(\mathscr{K}_i) = \bigcap_{i=1}^{3} \{\mathbf{x} = (x_k)_{k=1,2,3}: x_i \geqq 0,\ x_j + x_m \geqq 1,$$
$$j, m = 1, 2, 3,\ j \neq i \neq m \neq j\} =$$
$$= \{\mathbf{x} = (x_k)_{k=1,2,3}: x_i \geqq 0,\ x_j + x_m \geqq 1,$$
$$i, j, m = 1, 2, 3,\ m \neq i \neq j \neq m\},$$

and

$$V(\mathscr{K}_4) \cap (\bigcap_{i=1}^{3} V^*(\mathscr{K}_i)) = \emptyset .$$

It means that $\mathscr{K}_4 \notin \mathbf{K}_{ef}$. It is easy to verify, in an analogous way, that

$$\mathbf{K}_{ef} = \{\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\} .$$

Further

$$V(\mathscr{K}_0) = \{\mathbf{x} = (x_k)_{k=1,2,3}: x_i \leqq 0,\ i = 1, 2, 3\}$$

and for any $i \in I$ is

$$V(\mathscr{K}_i) = \{\mathbf{x} = (x_k)_{k=1,2,3}: x_i \leqq 0,\ x_j + x_m \leqq 1,$$
$$j, m = 1, 2, 3,\ j \neq i \neq m \neq i\} .$$

Hence,

$$V(\mathscr{K}_0) \subset V(\mathscr{K}_i) - V^*(\mathscr{K}_i)$$

for all $i = 1, 2, 3$ and, consequently, $\mathscr{K}_0 \notin \mathbf{K}^{ef}$. It is obvious, then, that

$$\mathbf{K}^{ef} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_4\} \quad \text{and} \quad \mathbf{K}^{ef}_{ef} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\} .$$

It can be easily verified that

$$\mathscr{K}_0 \text{ non } \sigma\, \mathscr{K}_i \quad \text{for} \quad i = 1, 2, 3 ,$$

and

$$\mathscr{K}_i\, \sigma\, \mathscr{K}_j , \quad \mathscr{K}_i \text{ non } \sigma\, (\mathscr{K}_j \cup \mathscr{K}_k)$$

for all $i, j, k = 1, 2, 3,\ i \neq j \neq k \neq i$. Moreover,

$$\mathscr{K}_4\, \sigma\, \mathscr{K}_j , \quad \mathscr{K}_4 \text{ non } \sigma\, (\mathscr{K}_j, \mathscr{K}_k) , \quad j, k = 1, 2, 3 , \quad j \neq k .$$

Consequently,

$$\Delta(\mathscr{K}_0) = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\} ,$$
$$\Delta(\mathscr{K}_i) = \{\mathscr{K}_j, \mathscr{K}_k\} , \quad i, j, k = 1, 2, 3 , \quad i \neq j \neq k \neq i ,$$
$$\Delta(\mathscr{K}_4) = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\} .$$

Hence,

$$\mathbf{S}^* = \emptyset\,, \quad \mathbf{S} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\} = \mathbf{K}_{\text{ef}}^{\text{ef}}\,.$$

On the other hand, the second one of the considered classes of coalition structures, the class $\mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}}$, is always non-empty, as follows from the next theorem.

Theorem 3. There exists at least one effective and dynamically stable coalition structure in every general coalition-game, i.e. $\mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}} \neq \emptyset$.

Proof. First, we prove that the class $\mathbf{S}$ is non-empty. Let us define a binary relation $\succ$ on the class $\mathbf{K}$ such that for $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ is $\mathscr{L} \succ \mathscr{K}$ iff there exists a finite sequence

$$\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{\text{ef}}\,,$$

$$\mathscr{K}_1 = \mathscr{K}\,, \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1})\,, \quad r = 2, \ldots, m\,, \quad \mathscr{L} \in \Delta(\mathscr{K}_n)\,,$$

and there is no finite sequence $\{\mathscr{L}_1, \ldots, \mathscr{L}_m\}$, such that

$$\mathscr{L}_1 = \mathscr{L}\,,\ \mathscr{L}_r \in \Delta(\mathscr{L}_{r-1})\,,\ r = 2, \ldots, m\,,\ \mathscr{K} \in \Delta(\mathscr{L}_m)\,.$$

Then $\succ$ is a partial ordering relation on the finite class $\mathbf{K}$. It evidently follows from the definition of the relation $\succ$ that it is antireflexive, antisymetric and transitive. There always exists at least one coalition structure $\mathscr{L} \in \mathbf{K}$ such that the relation $\mathscr{M} \succ \mathscr{L}$ is not true for any $\mathscr{M} \in \mathbf{K}$. It follows from the definition of the relation $\succ$ that the class of such maximal elements according to $\succ$ is equal to the class of dynamically stable coalition structures. It means that $\mathbf{S} \neq \emptyset$. Moreover, $\mathbf{S} \subset \mathbf{K}_{\text{ef}}$, as follows from Remark 11, so that also $\mathbf{S} \cap \mathbf{K}_{\text{ef}} \neq \emptyset$. If there exists some $\mathscr{K} \in \mathbf{S} - \mathbf{K}^{\text{ef}} = \mathbf{S} - \mathbf{K}_{\text{ef}}^{\text{ef}}$ then Lemma 12 and Lemma 4 imply that there exists $\mathscr{M} \in \mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}}$.

Considering the mutual relations among the important classes of coalition structures, it is useful to answer the question whether the class $\mathbf{S}$ is really different from $\mathbf{K}_{\text{ef}}^{\text{ef}}$. We know that $\mathbf{S} \subset \mathbf{K}_{\text{ef}}$, nevertheless, it was not shown, yet, that $\mathbf{S}$ and $\mathbf{K}_{\text{ef}}^{\text{ef}}$ are generally different classes. The next example proves that it is so.

Example 2. Let us consider a general coalition-game $\Gamma = (I, V)$, with $I = \{1, 2, 3, 4\}$ and with

$$V(\{i\}) = \{\mathbf{x} = (x_k)_{k \in I}\colon x_i \leqq 0\}\,, \quad i = 1, 2, 3, 4\,,$$

$$V(\{i, j\}) = \{\mathbf{x} = (x_k)_{k \in I}\colon x_i + x_j \leqq 0\}\,, \quad i, j \in I,\ i \neq j\,,$$

$$V(\{i, j, m\}) = \{\mathbf{x} = (x_k)_{k \in I}\colon x_i + x_j + x_m \leqq 1\}\,, \quad i, j, m \in I,\ i \neq j \neq m \neq i\,,$$

$$V(I) = \{\mathbf{x} = (x_k)_{k \in I}\colon x_1 + x_2 + x_3 + x_4 \leqq 1\}\,.$$

Let us denote the coalition structures

$$\mathscr{K}_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\} \qquad \mathscr{K}_9 = \{\{3\}, \{1, 2\}, \{4\}\}$$

$$\mathscr{K}_2 = \{\{1\}, \{2, 3\}, \{4\}\} \qquad \mathscr{K}_{10} = \{\{3\}, \{1, 2, 4\}\}$$
$$\mathscr{K}_3 = \{\{1\}, \{2\}, \{3,4\}\} \qquad \mathscr{K}_{11} = \{\{4\}, \{1, 2, 3\}\}$$
$$\mathscr{K}_4 = \{\{1\}, \{3\}, \{2, 4\}\} \qquad \mathscr{K}_{12} = \{\{1, 2\}, \{3, 4\}\}$$
$$\mathscr{K}_5 = \{\{1\}, \{2, 3, 4\}\} \qquad \mathscr{K}_{13} = \{\{1, 3\}, \{2, 4\}\}$$
$$\mathscr{K}_6 = \{\{2\}, \{1, 3\}, \{4\}\} \qquad \mathscr{K}_{14} = \{\{1, 4\}, \{2, 3\}\}$$
$$\mathscr{K}_7 = \{\{2\}, \{3\}, \{1, 4\}\} \qquad \mathscr{K}_{15} = \{I\}$$
$$\mathscr{K}_8 = \{\{2\}, \{1, 3, 4\}\} .$$

It can be easily verified, by methods used in Example 1, that

$$\mathbf{K}_{\text{ef}} = \mathbf{K} - \{\mathscr{K}_{15}\} ,$$
$$\mathbf{K}_{\text{ef}}^{\text{ef}} = \{\mathscr{K}_5, \mathscr{K}_8, \mathscr{K}_{10}, \mathscr{K}_{11}, \mathscr{K}_{12}, \mathscr{K}_{13}, \mathscr{K}_{14}\}$$

and that

$$\Delta(\mathscr{K}_5) = \{\mathscr{K}_8, \mathscr{K}_{10}, \mathscr{K}_{11}\} ,$$
$$\Delta(\mathscr{K}_8) = \{\mathscr{K}_5, \mathscr{K}_{10}, \mathscr{K}_{11}\} ,$$
$$\Delta(\mathscr{K}_{10}) = \{\mathscr{K}_5, \mathscr{K}_8, \mathscr{K}_{11}\} ,$$
$$\Delta(\mathscr{K}_{11}) = \{\mathscr{K}_5, \mathscr{K}_8, \mathscr{K}_{10}\} ,$$
$$\Delta(\mathscr{K}_{12}) = \Delta(\mathscr{K}_{13}) = \Delta(\mathscr{K}_{14}) = \{\mathscr{K}_5, \mathscr{K}_8, \mathscr{K}_{10}, \mathscr{K}_{11}\} .$$

Consequently,

$$\mathbf{S}^* = \emptyset \quad \text{and} \quad \mathbf{S} = \{\mathscr{K}_5, \mathscr{K}_8, \mathscr{K}_{10}, \mathscr{K}_{11}\} \neq \mathbf{K}_{\text{ef}} \neq \mathbf{K}_{\text{ef}}^{\text{ef}} \neq \mathbf{S} .$$

It was already mentioned in this paper that the game solution model, introduced here, is in certain degree analogical to the model given in [2] for coalition-games with side-payments. It was proved in [2] that if there exists at least one strongly stable coalition structure in the considered type of game, then all dynamically stable coalition structures are strongly stable. This strong result is true for the very special case of the coalition-games with side-payments and for some games very closely related to them, but it is not true for all general coalition-games. It is shown in the following example.

Example 3. Let us consider a general coalition-game $\Gamma = (I, V)$, where $I = \{1, 2, 3\}$ and

$$V(\{i\}) = \{\mathbf{x} = (x_k)_{k\in I}: x_i \leqq 1\} , \quad i = 1, 2, 3 ,$$
$$V(\{1, 2\}) = \{\mathbf{x} = (x_k)_{k\in I}: x_1 \leqq 4,\ x_2 \leqq 2\} ,$$
$$V(\{1, 3\}) = \{\mathbf{x} = (x_k)_{k\in I}: x_1 \leqq 2,\ x_3 \leqq 4\} ,$$
$$V(\{2, 3\}) = \{\mathbf{x} = (x_k)_{k\in I}: x_2 \leqq 4,\ x_3 \leqq 2\} ,$$
$$V(I) = \{\mathbf{x} = (x_k)_{k\in I}: x_k \leqq 3,\ k = 1, 2, 3\} .$$

Let us denote, analogously to Example 1, the coalition structures

$$\mathscr{K}_0 = \{\{1\},\{2\},\{3\}\},$$
$$\mathscr{K}_i = \{\{i\},\{j,k\}\}, \quad i = 1,2,3, \quad j,k = 1,2,3, \quad i \neq j \neq k \neq i,$$
$$\mathscr{K}_4 = \{I\}.$$

It is easy to verify that

$$\mathbf{K}_{\text{ef}} = \mathbf{K} \quad \text{and} \quad \mathbf{K}_{\text{ef}}^{\text{ef}} = \mathbf{K}^{\text{ef}} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_4\}.$$

Further,

$$\mathscr{K}_4 \,\sigma\, \mathscr{K}_i \quad \text{for all} \quad i = 0,1,2,3 \quad \text{and} \quad \mathscr{K}_4 \,\sigma\, (\cup \mathbf{K}_{\text{ef}})$$

and, consequently,

$$\Delta(\mathscr{K}_4) = \emptyset.$$

On the other hand,

$$\mathscr{K}_i \,\sigma\, \mathscr{K}_0, \quad \mathscr{K}_i \,\sigma\, \mathscr{K}_4 \quad \text{for all} \quad i = 1,2,3,$$
$$\mathscr{K}_1 \,\sigma\, \mathscr{K}_3, \quad \mathscr{K}_2 \,\sigma\, \mathscr{K}_1, \quad \mathscr{K}_3 \,\sigma\, \mathscr{K}_2,$$
$$\mathscr{K}_1 \text{ non } \sigma\, \mathscr{K}_2, \quad \mathscr{K}_2 \text{ non } \sigma\, \mathscr{K}_3, \quad \mathscr{K}_3 \text{ non } \sigma\, \mathscr{K}_1,$$
$$\mathscr{K}_0 \text{ non } \sigma\, \mathscr{K}_i, \quad i = 1,2,3,4.$$

Then

$$\Delta(\mathscr{K}_1) = \mathscr{K}_2, \quad \Delta(\mathscr{K}_2) = \mathscr{K}_3, \quad \Delta(\mathscr{K}_3) = \mathscr{K}_1.$$

It means that

$$\mathbf{S}^* = \{\mathscr{K}_4\}, \quad \mathbf{S} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_4\} = \mathbf{K}^{\text{ef}} = \mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}}.$$

The coalition structure $\mathscr{K}_4$ is strongly stable, the coalition structures $\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3$ are dynamically but not strongly stable, $\mathscr{K}_0$ is neither effective nor dynamically stable.

One of the mutual connections between the classes of strongly stable and the others coalition structures is formulated in the following theorem and its corollary. Some further may be derived for some special types of general coalition-games, e.g. for superadditive coalition-games of for coalition-games with restricted or non-restricted side-payments and for some other modifications of the general model.

Theorem 4. If a coalition structure $\mathscr{L}$ is not strongly stable then there always exists a dynamically stable effective coalition structure $\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}}$ and a finite sequence of coalition structures $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{\text{ef}}$ such that $\mathscr{K}_1 \in \Delta(\mathscr{L})$, $\mathscr{K}_r \in \Delta(\mathscr{K}_{r-1})$, $r = 2, \ldots, n$, $\mathscr{K} \in \Delta(\mathscr{K}_n)$.

Proof. Let us consider a coalition structure $\mathscr{L} \in \mathbf{K}_{\text{ef}} - \mathbf{S}$, and let us suppose that the statement of theorem is not valid. It means, we suppose that there does not exist any sequence

$$\{\mathscr{K}_1, \ldots, \mathscr{K}_n, \mathscr{K}\} \subset \mathbf{K}_{\mathrm{ef}}$$

such that

$$\mathscr{K} \in \mathbf{S}\,, \quad \mathscr{K}_1 \in \Delta(\mathscr{L})\,, \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1})\,, \quad r = 2, \ldots, n\,, \quad \mathscr{K} \in \Delta(\mathscr{L})\,.$$

As $\mathscr{L} \notin \mathbf{S}^*$, there exists at least one coalition structure $\mathscr{L}_1$ which is effective from below and such that $\mathscr{L}_1 \in \Delta(\mathscr{L})$. The assumption that the statement of this theorem is not true implies that $\mathscr{L}_1 \notin \mathbf{S} \subset \mathbf{S}^*$, and there must exist some $\mathscr{L}_2 \in \mathbf{K}_{\mathrm{ef}}$ such that $\mathscr{L}_2 \in \Delta(\mathscr{L}_1)$. It is possible to continue in this way, and to construct a sequence

$$\{\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{\mathrm{ef}}$$

such that

$$\mathscr{L}_0 = \mathscr{L}\,, \quad \mathscr{L}_r \in \Delta(\mathscr{L}_{r-1})\,, \quad r = 1, \ldots, m\,; \quad \mathscr{L}_r \notin \mathbf{S}\,, \quad r = 0, 1, \ldots, m\,.$$

Moreover, we may construct that sequence so that

$$\mathscr{L}_r \neq \mathscr{L}_s \neq \mathscr{L} \quad \text{for} \quad r, s = 1, 2, \ldots, m,\ r \neq s\,.$$

As the class $\mathbf{K}_{\mathrm{ef}}$ is finite, it is obvious that after a finite number of steps we construct such $\mathscr{L}_m \in \mathbf{K}_{\mathrm{ef}}$ that for every $\mathscr{J} \in \Delta(\mathscr{L}_m)$ there exists $\mathscr{L}_i$, $0 \leqq i < m$, such that $\mathscr{J} = \mathscr{L}_i$. But, it means that $\mathscr{L}_m$ is necessarily dynamically stable, $\mathscr{L}_m \in \mathbf{S}$. If $\mathscr{L}_m \in \mathbf{S} - \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ then Lemma 8 implies that there exists $\mathscr{K} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ such that $\mathscr{K} \in \Delta(\mathscr{L}_m)$ and, according to Lemma 10, $\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$. If $\mathscr{L}_m \in \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$, we put $\mathscr{K} = \mathscr{L}_m$.

Corollary. The previous theorem implies that if the equality $\mathbf{S} = \mathbf{S}^*$ is valid in a general coalition-game then for every coalition structure $\mathscr{L} \in \mathbf{K} - \mathbf{S}$ there exists a coalition structure $\mathscr{K} \in \mathbf{S}^*$ and a sequence $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{\mathrm{ef}}$ such that $\mathscr{K}_1 \in \Delta(\mathscr{L})$, $\mathscr{K}_r \in \Delta(\mathscr{K}_{r-1})$, $r = 2, \ldots, n$, $\mathscr{K} \in \Delta(\mathscr{K}_n)$.

6. STABILITY OF IMPUTATIONS

In the last section of this paper, the notions of the strong and dynamic stability of configurations are introduced. According to Section 1 of this paper, a configuration is a pair formed by a coalition structure and an imputation, and we call it admissible if the imputation is admissible in the considered coalition structure. The motivation of the stability concepts, as well as the reason, why the stability of configurations is separated from the stability of coalition structures, were discussed in Section 4. The main properties of (strongly and dynamically) stable configurations are closely connected with the analogous properties of stable coalition structures, and they are derived from them.

Definition 7. A configuration $(\mathscr{K}, \mathbf{x})$, $\mathscr{K} \in \mathbf{K}$, $\mathbf{x} \in R^I$, is called *strongly stable* iff it is admissible and $\mathbf{x}$ is an element of the superoptimum of all coalitions, i.e.

$$\mathbf{x} \in V(\mathscr{K}) \cap \big(\bigcap_{L \in \mathscr{I}} V^*(L)\big).$$

Remark 12. It follows from the previous definition immediately that a configuration $(\mathscr{K}, \mathbf{x})$ is strongly stable iff

$$\mathbf{x} \in V(\mathscr{K}) \cap \big(\bigcap_{\mathscr{L} \in \mathbf{K}} V^*(\mathscr{L})\big).$$

Remark 13. It follows from the previous definition and from Definition 5 that if a configuration $(\mathscr{K}, \mathbf{x})$ is strongly stable then the coalition structure $\mathscr{K}$ is also strongly stable.

The following theorem is an analogy of the similar result introduced for strongly stable coalition structures in Theorem 2.

Theorem 5. If $(\mathscr{K}, \mathbf{x})$ is an admissible configuration, i.e. $\mathbf{x} \in V(\mathscr{K})$, then it is strongly stable if and only if $\mathbf{x}$ belongs to the superoptimum $V^*(J)$ of all effective coalitions J; this condition is equivalent with $\mathbf{x} \in V^*(\mathscr{J})$ for all $\mathscr{J} \in \mathbf{K}_{\mathrm{ef}}$.

Proof. It was already proved in Lemma 3 that for any non-effective coalition $L \in \mathscr{I}$ the inclusion

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}^*(L)} V^*(J),$$

where

$$\mathscr{I}^*(L) = \{J \in \mathscr{I} : J \subset L,\ J \text{ is effective}\},$$

is true. It means that if $\mathbf{x} \in V^*(J)$ for all effective coalitions then $\mathbf{x} \in V^*(L)$ for all coalitions in $\mathscr{I}$. A coalition structure $\mathscr{J} \in \mathbf{K}$ is effective from below iff it consists of effective coalitions only. On the other hand, every effective coalition belongs to at least one coalition structure which is effective from below, as follows from Remark 2. It means that

$$\bigcap_{J \in \mathscr{I}, J \text{ is effective}} V^*(J) = \bigcap_{J \in \mathbf{K}_{\mathrm{ef}}} V^*(J).$$

Theorem 6. If $\mathscr{K}$ is a strongly stable coalition structure then there always exists an imputation $\mathbf{x} \in R^I$ such that the configuration $(\mathscr{K}, \mathbf{x})$ is strongly stable.

Proof. The statement of this theorem follows immediately from Definition 5 and Definition 7.

Theorem 7. If $(\mathscr{K}, \mathbf{x})$ is a strongly configuration, $\mathscr{L} \in \mathbf{K}$ is a coalition structure and if $\mathbf{x} \in V(\mathscr{L})$, then the configuration $(\mathscr{L}, \mathbf{x})$ is also strongly stable.

Proof. If $(\mathscr{K}, \mathbf{x})$ is strongly stable then $\mathbf{x} \in V(\mathscr{K})$ and $\mathbf{x} \in V^*(\mathscr{J})$ for all $\mathscr{J} \in \mathbf{K}$. If $\mathbf{x} \in V(\mathscr{L})$, too, then the second condition keeps valid, and, consequently, $(\mathscr{L}, \mathbf{x})$ is a strongly stable configuration.

It is evident, especially after the previous theorem, that the strong stability of configurations is, almost exclusively, a property of imputations. It keeps valid if the coalition structure is substituted by another one preserving the admissibility of the imputation. It means that it is possible to consider the strong stability of imputations as another form of the strong stability of configurations. It was done so in [3], where the strong stability of imputations is defined analogously to Definition 7, and some of its properties are derived. It follows from Theorem 6 that these properties may be, without any difficulties, transformed into analogous properties of the strongly stable configurations.

The situation is essentially different if we consider the dynamic stability of configurations. It is really a property of pairs where both members, coalition structure and imputation, play an important role, and where they are connected by relatively strong conditions.

Definition 8. A configuration $(\mathscr{K}, \mathbf{x})$, $\mathscr{K} \in \mathbf{K}$, $\mathbf{x} \in R^I$, is *dynamically stable* iff

(6.1) $$\mathbf{x} \in V(\mathscr{K})\,;$$

(6.2) $$\mathscr{K} \in \mathbf{S}\,;$$

(6.3) $$\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\,, \quad \mathbf{x} \notin V^*(\mathscr{M}) \Rightarrow \mathscr{M} \in \Delta(\mathscr{K})\,.$$

The preceding definition means that a configuration is dynamically stable iff it is formed by a dynamically stable coalition structure and by one of the imputations which characterize the dynamic stability. It means that the property of "dynamic stability" of imputations is dependent on the coalition structure in which the imputations are realized. The existence of dynamically stable configurations is guaraneed by Theorem 8 and by Theorem 3. Before presenting Theorem 8, it is useful to introduce the following lemma.

Lemma 13. If a configuration $(\mathscr{K}, \mathbf{x})$ is strongly stable then it is also dynamically stable.

Proof. If $(\mathscr{K}, \mathbf{x})$ is strongly stable, then $\mathbf{x} \in V(\mathscr{K})$ and $\mathbf{x} \in V^*(\mathscr{M})$ for all $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}$. Moreover, Remark 3 and Remark 11 imply that $\mathscr{K} \in \mathbf{S}$. It means that all conditions of Definition 8 are fulfilled and $(\mathscr{K}, \mathbf{x})$ is strongly stable.

Theorem 8. If a coalition structure $\mathscr{K}$ is dynamically stable, then there exists an imputation $\mathbf{x} \in R^I$ such that the configuration $(\mathscr{K}, \mathbf{x})$ is dynamically stable.

Proof. Let us choose a dynamically stable coalition structure $\mathcal{K} \in \mathbf{S}$, and for every class of coalition structures $\mathbf{M} \subset \mathbf{K}_{ef}$ denote

$$\mathbb{Y}_{\mathbf{M}}(\mathcal{K}) = V(\mathcal{K}) \cap (\bigcap_{\mathcal{M} \in \mathbf{M}} V^*(\mathcal{M})) . \tag{6.4}$$

It follows from (1.6) that the set $\mathbb{Y}_{\mathbf{M}}(\mathcal{K})$ is non-empty for at least one class of coalition structures $\mathbf{M} = \{\mathcal{K}\}$, where Remark 11 implies that $\mathbf{M} = \{\mathcal{K}\} \subset \mathbf{K}_{ef}$. Moreover, if $\mathbf{M} \subset \mathbf{N} \subset \mathbf{K}_{ef}$ then $\mathbb{Y}_{\mathbf{M}}(\mathcal{K}) \supset \mathbb{Y}_{\mathbf{N}}(\mathcal{K})$. If $\mathcal{J} \in \mathbf{K}_{ef}$ and $\mathbb{Y}_{\mathbf{M}}(\mathcal{K}) \neq \emptyset$ then $\mathbb{Y}_{\mathbf{M} \cup \{\mathcal{J}\}}(\mathcal{K}) = \emptyset$ only if $\mathcal{J} \in \Delta(\mathcal{K})$. Then also $\mathcal{J} \in \mathbf{S}$, as follows from Lemma 10. If $\mathbb{Y}_{\mathbf{M}}(\mathcal{K}) \neq \emptyset$ for all $\mathbf{M} \subset \mathbf{K}_{ef}$, then $\Delta(\mathcal{K}) = \emptyset$, and for all imputations

$$\mathbf{x} \in \mathbb{Y}_{\mathbf{K}_{ef}}(\mathcal{K})$$

the configurations $(\mathcal{K}, \mathbf{x})$ are strongly stable. It means, according to Lemma 13, that $(\mathcal{K}, \mathbf{x})$ is dynamically stable. If

$$\mathbb{Y}_{\mathbf{K}_{ef}}(\mathcal{K}) = \emptyset$$

then there exists $\mathbf{M} \subset \mathbf{K}_{ef}$ such that $\mathcal{K} \in \mathbf{M}$,

$$\mathbb{Y}_{\mathbf{M}}(\mathcal{K}) \neq \emptyset \quad \text{and} \quad \mathbb{Y}_{\mathbf{M} \cup \{\mathcal{J}\}}(\mathcal{K}) = \emptyset \quad \text{for all} \quad \mathcal{J} \in \mathbf{K}_{ef} - \mathbf{M} .$$

It means that all $\mathcal{J} \in \mathbf{K}_{ef} - \mathbf{M}$ belong to the set $\Delta(\mathcal{K})$, and all imputations $\mathbf{x} \in \mathbb{Y}_{\mathbf{M}}(\mathcal{K}) \neq \emptyset$ fulfil the condition (6.3). The coalition structure $\mathcal{K}$ fulfils (6.2), and (6.1) follows from (6.4). It means that $(\mathcal{K}, \mathbf{x})$ is dynamically stable.

Corollary. Theorem 8 and Theorem 3 imply that there exists at least one dynamically stable configuration in every general coalition-game.

Theorem 9. If a coalition structure $\mathcal{K}$ is strongly stable, $\mathcal{K} \in \mathbf{S}^*$, and if $\mathbf{x} \in R^I$ is an imputation such that $(\mathcal{K}, \mathbf{x})$ is dynamically stable, then the configuration $(\mathcal{K}, \mathbf{x})$ is strongly stable.

Proof. If $\mathcal{K} \in \mathbf{S}^*$ then $\Delta(\mathcal{K}) = \emptyset$, as follows from Definition 5, and $\mathbf{x} \in V^*(\mathcal{M})$ for all $\mathcal{M} \in \mathbf{K}_{ef}$, because of the dynamic stability of $(\mathcal{K}, \mathbf{x})$. That dynamic stability implies also that $\mathbf{x} \in V(\mathcal{K})$, and the conditions of Definition 7 are fulfilled.

The preceding theorem means that a strongly stable coalition structure cannot form a configuration which would be only dynamically but not strongly stable. A consequence of that fact is formulated in the following, last theorem of this paper.

Theorem 10. Let $\mathcal{K} \in \mathbf{K}$ be a coalition structure and let $\mathbf{x}, \mathbf{y} \in V(\mathcal{K})$ be admissible imputations. If the configuration $(\mathcal{K}, \mathbf{x})$ is strongly stable and the configuration $(\mathcal{K}, \mathbf{y})$ is not strongly stable, then $(\mathcal{K}, \mathbf{y})$ is not dynamically stable.

Proof. If $(\mathscr{K}, \mathbf{x})$ is strongly stable then Remark 13 implies that $\mathscr{K}$ is strongly stable coalition structure. If $(\mathscr{K}, \mathbf{y})$ were dynamically stable, then it should be, according to Theorem 9, also strongly stable. As we assume that $(\mathscr{K}, \mathbf{y})$ is not strongly stable, the statement of theorem must be true.

7. CONCLUSION

The concept of the dynamic stability of general coalition-games, introduced and investigated in this paper, represents only one of the possible "weak" solutions of such games. It is obvious that also some other solutions, suggested for special cases of such games, may be generalized and reformulated for the general coalition-game.

The results, derived in this paper and in paper [3], describe some of the main general properties of the investigated type of games. It is not difficult to see that further results may be derived for particular more special types of such games. It was done so in the literature for the known types of games. On the other hand, some further special cases, e.g. the general coalition-games with the superadditivity assumption, or coalition-games with restricted side-payments (cf. [3], Section 5) represent an unknown field, from this point of view, and it will be useful to pay some attention to them.

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