

Christian Commault; Jacques Descusse; Jean-Michel Dion; Jorge A. Torres Muñoz  
Block decoupling invariants: Geometric and transfer matrix characterizations

*Kybernetika*, Vol. 27 (1991), No. 4, 345--361

Persistent URL: <http://dml.cz/dmlcz/124566>

## Terms of use:

© Institute of Information Theory and Automation AS CR, 1991

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library*  
<http://project.dml.cz>

## **BLOCK DECOUPLING INVARIANTS: GEOMETRIC AND TRANSFER MATRIX CHARACTERIZATIONS**

CHRISTIAN COMMAULT, JACQUES DESCUSSE, JEAN-MICHEL DION,  
J. A. TORRES\*

In this paper we consider the block decoupling problem for linear time invariant systems in the general case i.e. the system transfer matrix is not supposed to be surjective. The aim of this work is twofold, first to introduce new lists of integers called "Block Essential Structures" of the system and then to provide several equivalent characterizations of these invariants within both transfer matrix and geometric approaches.

It turns out these integers represent precisely the minimal infinite structure achievable for the blocks of the decoupled system through precompensation. When the system is decouplable by dynamic state feedback the minimal infinite structure achievable is the same as previously.

### **1. INTRODUCTION**

During the last twenty years a great deal of interest has been brought to the theory of decoupling for linear time invariant systems. Important steps in the development of the decoupling theory as well as references can be found in Falb and Wolovich [12], Morse and Wonham [16], Wonham [19], Hautus and Heymann [14]. In the recent years a great deal of interest has been devoted to the structural features of decoupling problems. Various lists of integers strongly related with the deep structure of the system have been introduced. In particular Descusse and Dion in [5] expressed the solvability condition of [12] in terms of infinite structure equalities. In [4] the essential orders characterizing the minimal infinite structure of the decoupled system are presented for the row by row decoupling problem.

In this paper we will focus our interest on the block decoupling problem. We are interested in obtaining the simplest possible decoupled systems. More precisely we

\* Torres's work is sponsored by CONACYT-MEXICO.

look for decoupled systems possessing the least possible infinite structure. We consider a linear system whose transfer matrix  $T(s)$  is partitioned in subsystems  $T_i(s)$  according to a desired output partition. We assume that the rank of  $T(s)$  is equal to the sum of the ranks of the subsystems  $T_i(s)$ , then block decoupling by precompensation is always possible [14]. The aim of the paper is mainly to introduce new lists of integers called "the  $i$ th block essential orders" which represent precisely the minimal infinite structure achievable for the  $i$ th block of the decoupled system through precompensation. We will define the "block essential orders" as the union of the  $i$ th block essential orders. We will prove that the block essential orders are feedback invariants and will provide several equivalent characterizations of these invariants within both transfer matrix and geometric approaches.

The present work generalizes that one already done by Commault and Coworkers [4] where only the row decoupling problem was considered.

This paper is organized as follows: Section two is devoted to notations and basic concepts. In the third section we introduce the block decoupling problem and some preliminaries are given. The new lists of invariants called block essential orders are introduced in section four. We give also several geometric and transfer matrix characterizations of these invariants. Its application to the block decoupling problem through precompensation, static or dynamic state feedback take place also in section four. We show that the block essential orders represent the least infinite structure achievable for the decoupled system. In the last section an illustrative example exhibits both geometric and transfer matrix points of view.

## 2. NOTATIONS AND PRELIMINARIES

In this paper we will deal with linear time invariant  $(C, A, B)$  systems described by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where  $x \in \mathcal{X} \sim \mathbb{R}^n$ ,  $u \in \mathcal{U} \sim \mathbb{R}^m$  and  $y \in \mathcal{Y} \sim \mathbb{R}^p$  with  $B$  monic and  $C$  epic.

Associated with  $(C, A, B)$  we will consider a  $k$ -partition of the output  $y$  into  $k$  nonempty subsets of components  $y_i$ , each of dimension  $p_i$  ( $0 < p_i$ ,  $\sum_{i=1}^k p_i = p$ ). This partition induces a corresponding partition of  $C$ , written as  $(C_1^T, C_2^T, \dots, C_k^T)^T$  with  $T$  denoting transposition. We will denote by  $C^i$  the matrix obtained from  $C$  by removing the  $i$ th row-block  $C_i$ .

Hereafter, we will use  $\mathcal{V}_i^*$  the maximal  $(A, B)$ -invariant subspace contained in  $\text{Ker } C_i$ , also written  $\sup I(A, B, \text{Ker } C_i)$ ,  $\mathcal{F}_i^*$  that maximal one contained in  $\text{Ker } C^i$ .  $\mathcal{R}_i^*$  will denote the maximal controllability subspace contained in  $\mathcal{F}_i^*$ . The classical tools of the geometric approach are derived from the following algorithms [19].

*(A, B)-invariant subspace algorithm:*      *Controllability subspace algorithm:*

$$\begin{aligned} \mathcal{V}_i^0 &= \mathcal{X} & \mathcal{R}_i^0 &= 0 \\ \mathcal{V}_i^{\mu+1} &= \text{Ker } C_i \cap A^{-1}(\mathcal{B} + \mathcal{V}_i^\mu) & \mathcal{R}_i^{\mu+1} &= \mathcal{T}_i^* \cap (A\mathcal{R}_i^\mu + \mathcal{B}) \end{aligned} \quad (2)$$

with  $\mathcal{B}$  the image of  $B$ .

Define

$$p'_{\mu+1} = \dim(\mathcal{B} \cap \mathcal{V}^\mu / \mathcal{B} \cap \mathcal{V}^*); \quad \mu \geq 0 \quad (3)$$

where  $\mathcal{V}^\mu$  and  $\mathcal{V}^*$  are related to  $\text{sup } I(A, B, \text{Ker } C)$ .

The infinite zero structure of  $(C, A, B)$  is the list of integers  $\{n_i\}$  defined by  $n_i = \text{card } \{p'_\mu \geq i\}$ . By the way,  $(C, A, B)$  has  $p'_1$  infinite zeros the orders of which are  $n_i$ . Recall that  $p'_1 = \text{rank } T(s)$ .

Let  $\mathbf{R}(s)$  be the field of rational functions. A rational function  $f(s) = n(s)/d(s)$  is said to be proper (resp. strictly proper) if  $\text{deg}(d(s)) \geq \text{deg}(n(s))$  (resp.  $\text{deg}(d(s)) > \text{deg}(n(s))$ ) where  $\text{deg}(n(s))$  denotes the polynomial degree of  $n(s)$ .

Denote by  $\mathbf{R}_p(s)$  the ring of proper rational functions and  $\mathbf{R}_p^{p \times m}(s)$  the set of proper rational  $p \times m$  transfer matrices.

The units (invertible elements) of the ring  $\mathbf{R}_p^{m \times m}(s)$  are called bicausal matrices and are characterized by the property that  $B(s)$  is a bicausal matrix if and only if:

$$\det(\lim_{s \rightarrow \infty} B(s)) \neq 0$$

**Definition 1.** A full column rank proper rational matrix  $V(s)$  is said to be a right bicausal matrix if there exists a proper rational matrix  $W(s)$  such that  $[V(s) \ W(s)]$  is a bicausal matrix.

The transfer matrix of system (1) is a  $p \times m$  strictly proper rational matrix with null left and right static kernels, since  $B$  is monic and  $C$  is epic.

$$T(s) = C(sI - A)^{-1} B$$

For a  $p \times m$  rational matrix  $G(s)$ , there exists Smith-McMillan factorizations at infinity which were studied in [17] and in [8]:

$$G(s) = B_1(s) \Lambda(s) B_2(s)$$

where  $B_1(s), B_2(s)$  are bicausal matrices and  $\Lambda(s)$  is uniquely defined by:

$$\Lambda(s) = \begin{bmatrix} s^{-n_1} & & & 0 \\ & \ddots & & \\ & & s^{-n_r} & \\ 0 & & & 0 \end{bmatrix}$$

where  $r$  is the rank of  $G(s)$ . If  $n_i$  is negative,  $-n_i$  is the order of a pole at infinity, whereas if  $n_i$  is positive,  $n_i$  is the order of a zero at infinity.

In the whole paper the infinite structure of  $G(s)$  will be denoted by:

$$\Sigma_{\infty}(G(s)) = \{n_1, \dots, n_r\} \quad (4)$$

we will also denote:  $-\Sigma_{\infty}(G(s)) = \{-n_1, \dots, -n_r\}$ .

In case, when dealing with a realization  $(C, A, B)$  of a strictly proper transfer matrix  $T(s)$ , the infinite structure will be denoted either by  $\Sigma_{\infty}(C, A, B)$  or by  $\Sigma_{\infty}(T(s))$ .

Another useful notation, in the spirit of compactness, will be  $d_{\infty}(T(s))$  to denote the sum of infinite pole and zero orders of  $T(s)$ , this means:

$$d_{\infty}(T(s)) = \sum_{i=1}^k |n_i| \quad (5)$$

The equivalence between Morse's second list  $I_2$  and the ordered list of the degrees of a minimal polynomial basis for the right kernel of the transfer matrix is well known [13], [18]. In particular the dimension of the largest controllability subspace  $\mathcal{R}^*$  contained in  $\text{Ker } C$  of a minimal realization  $(A, B, C)$  of  $T(s)$  is equal to the sum of the degrees of a minimal polynomial basis for the right kernel of  $T(s)$ . In fact we will use the dual formulation of this result namely:

$$\dim \left( \frac{\mathcal{X}}{\mathcal{S}^* + \mathcal{V}^*} \right) = \sigma(T(s)) \quad (6)$$

where  $\sigma(T(s))$  denotes the sum of the degrees of a minimal polynomial basis of the left kernel of  $T(s)$  and  $\mathcal{S}^*$  is the minimal  $(C, A)$ -invariant subspace containing the image of  $B$ .

### 3. THE BLOCK DECOUPLING PROBLEM

#### Problem formulation

We will consider now the block decoupling problem. In order to avoid trivialities we will require the compensated system to be just as "output controllable" as the original system is.

We will say that the proper precompensator  $C(s)$  is *admissible* if

$$\text{rank } T(s) C(s) = \text{rank } T(s)$$

This admissibility condition is equivalent to the preservation of the  $C^{\infty}$  controlled output trajectories, see [1].

Let  $T(s)$  be a  $p \times m$  proper rational matrix, partitioned in row-blocks relatively to a given list of positive integers  $(p_1, \dots, p_k)$ , such that  $\sum_{i=1}^k p_i = p$ , in the following way:

$$T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_k(s) \end{bmatrix}; \quad \text{with } T_i(s) \in \mathbf{R}_p^{p_i \times m}(s) \quad \text{for } i = 1, \dots, k$$

The system with transfer matrix  $T(s)$  is said to be block decoupled relatively to the partition  $\{p_i\}$  if there exist positive integers  $m_1, \dots, m_k$  satisfying  $\sum_{i=1}^k m_i = m$ , such that  $T(s)$  has the block diagonal form:

$$T(s) = \begin{bmatrix} T_{11}(s) & 0 & & \\ 0 & & & \\ & & & \\ & & & T_{kk}(s) \end{bmatrix} = \text{diag} (T_{11}(s), \dots, T_{kk}(s))$$

with  $T_{ii}(s) \in \mathbf{R}_p^{p_i \times m_i}(s)$  for  $i = 1, \dots, k$

This means that each above defined input block influences only one output block. If one wants this influence to be effective the  $T_{ii}(s)$  must be non null for each  $i$ , in this case the system is called non degenerate.

The decoupling problem can be formulated as follows: Is it possible to exhibit an admissible proper precompensator  $C(s)$  such that  $T(s)C(s)$  is block decoupled and non degenerate?

It is well known that the above defined decoupling problem is solvable by precompensation when the row blocks  $T_i(s)$  are independent ( $r = \sum_{i=1}^k r_i$ ,  $r = \text{rank } T(s)$  and  $r_i = \text{rank } T_i(s)$ ), Hautus and Heyman [14].

In this paper we focus our attention on the *simplest achievable decoupled systems*. More precisely we will give the minimal McMillan degree and the minimal infinite structure achievable for the blocks of the decoupled system.

#### 4. THE BLOCK ESSENTIAL STRUCTURES

In the first part of this section we are concerned with some lists of integers strongly related with the infinite structure of the system under study. Such lists, called the  $i$ th block essential structures, are feedback invariant and represent, as we will see later, the minimal infinite structure achievable for the  $i$ th block of the decoupled system.

We are also interested in the minimal McMillan degree achievable for the blocks of the decoupled system. We will recall the geometric definition and some transfer matrix properties of the block decoupling invariants  $n_{ie}$ , which were introduced in [11].

##### 4.1. Geometric and Transfer Matrix Characterizations

We will give first a geometric definition of the block essential structure of a system  $(C, A, B)$ . Then we will give two suitable possible interpretations of the block essential structure in the transfer matrix framework.

**Definition 2.** Let  $(C, A, B)$  be the system (1) decomposed according to the output partition  $(p_1, \dots, p_k)$ . Let  $F_i$  denote any matrix such that:  $(A + BF_i) \mathcal{R}_i^* \subset \mathcal{R}_i^*$ , where  $\mathcal{R}_i^*$  is the maximal controllability subspace contained in  $\text{Ker } C^i$  and  $C^i$  is

the matrix obtained from  $C$  by removing its  $i$ th row-block. Consider  $\mathcal{B}_i := \mathcal{B} \cap \mathcal{R}_i^*$ ,  $B_i$  such that  $\text{Im}(B_i) = \mathcal{B}_i$  and  $A_i := A + BF_i$ .

We define the  $i$ th block essential structure of  $(C, A, B)$  relatively to  $\{p_i\}$ , denoted  $\Sigma_{ei}(C, A, B)$ , as the structure at infinity of the system  $(C_i, A_i, B_i)$ , i.e.:

$$\Sigma_{ei}(C, A, B) := \Sigma_{\infty}(C_i, A_i, B_i) \quad \text{for } i = 1, \dots, k$$

The block essential structure of  $(C, A, B)$  relatively to  $\{p_i\}$ , denoted  $\Sigma_e(C, A, B)$  is defined by:

$$\Sigma_e(C, A, B) := \bigcup_{i=1}^k \Sigma_{ei}(C, A, B)$$

In what follows we will consider the system  $(C, A, B)$  of transfer matrix  $T(s) = C(sI - A)^{-1}B$ , partitioned according to  $(p_1, \dots, p_k)$ , that means:

$$T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_k(s) \end{bmatrix}; \quad \text{with } T_i(s) = C_i(sI - A)^{-1}B$$

where  $C_i$  denotes the  $i$ th row-block of matrix  $C$ .

In the whole paper it will be assumed that:

$$\text{rank } T(s) = \sum_{i=1}^k \text{rank } T_i(s) \quad (7)$$

The notation  $T^i(s)$  will be used to denote the matrix obtained from  $T(s)$  by removing its  $i$ th row-block, in other words:

$$T^i(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_{i-1}(s) \\ T_{i+1}(s) \\ \vdots \\ T_k(s) \end{bmatrix}$$

In order to give a first transfer matrix interpretation of the block essential structure of  $(C, A, B)$ , let us introduce the following:

**Definition 3.** Let  $\mathfrak{G}(s)$  be a  $k$ -dimensional rational subspace and  $M(s)$  a polynomial matrix which columns are a minimal polynomial basis for  $\mathfrak{G}(s)$ . Denote  $h_i$  the  $i$ th column degrees of  $M(s)$ .  $N(s) := M(s) \text{diag}(s^{-h_1}, \dots, s^{-h_k})$  will be called a minimal polynomial basis in  $s^{-1}$  for  $\mathfrak{G}(s)$ .

Remark that a minimal basis in  $s^{-1}$  is a right bicausal matrix.

**Theorem 1.** The  $i$ th block essential structure  $\Sigma_{ei}(C, A, B)$  of the system  $(C, A, B)$ , which transfer matrix is  $T(s)$ , relatively to  $\{p_i\}$  satisfies:

$$\Sigma_{ei}(C, A, B) = \Sigma_{\infty}(T_i(s) V_i(s)) \quad \text{for } i = 1, \dots, k$$

where  $T_i(s)$  is the  $i$ th row-block of  $T(s)$  and  $V_i(s)$  is any right bicausal matrix which is a basis for  $\text{Ker } T^i(s)$ , with  $i = 1, \dots, k$ .

Notice that there always exists such a basis for  $\text{Ker } T^i(s)$ , e.g. consider a minimal polynomial basis in  $s^{-1}$  for  $\text{Ker } T^i(s)$ .

**Proof.** Prove first that  $\Sigma_\infty(T_i(s) V_i(s))$  is independent of the choice of  $V_i(s)$ . Let  $V_i(s)$  and  $W_i(s)$  be two right bicausal matrices which are basis matrices for  $\text{Ker } T^i(s)$ .  $V_i(s) = W_i(s) Q_i(s)$ , with  $Q_i(s)$  a nonsingular rational matrix; since that  $V_i(s)$  and  $W_i(s)$  are full column rank at infinity, then  $Q_i(s)$  is a bicausal matrix. It follows that:  $\Sigma_\infty(T_i(s) V_i(s)) = \Sigma_\infty(T_i(s) W_i(s))$ .

By definition:

$$\Sigma_{ei}(C, A, B) := \Sigma_\infty(C_i, A + BF_i, BG_i)$$

where  $F_i$  is any matrix such that  $(A + BF_i) \mathcal{R}_i^* \subset \mathcal{R}_i^*$  and  $G_i$  is such that  $\text{Im } BG_i = \mathcal{B} \cap \mathcal{R}_i^*$ .

The transfer matrix of the system  $(C_i, A + BF_i, BG_i)$ , denoted  $T_{F_i}(s)$  can be written as follows:

$$T_{F_i}(s) = T_i(s) V_i(s)$$

with:

$$V_i(s) = (I - F_i(sI - A)^{-1} B)^{-1} G_i \quad i = 1, \dots, k$$

$V_i(s)$  is clearly a right bicausal matrix, we will show now that  $V_i(s)$  is a basis for the null space of  $T^i(s)$ . First, observe that  $\mathcal{R}_i^*$  is contained in  $\text{Ker } C^i$ , for all  $i \neq j$ , this means that  $T^j(s) V_i(s) = 0$  for all  $i \neq j$ . We will show now that  $\text{rank } (V_i(s)) = \dim (\text{Ker } T^i(s))$ , which will end the proof.

For this we have:

$$\text{rank } (T^i(s)) = r - r_i = \dim \left( \frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{R}_i^*} \right) = m - \dim (\mathcal{B} \cap \mathcal{R}_i^*)$$

then:  $\dim (\mathcal{B} \cap \mathcal{R}_i^*) = m - r + r_i$ .

From the definition of  $G_i$  and  $V_i(s)$ :

$$\dim (\mathcal{B} \cap \mathcal{R}_i^*) = \text{rank } G_i = \text{rank } V_i(s)$$

On the other hand, because of the independence between the blocks  $T_i(s)$ ,  $\dim (\text{Ker } T^i(s)) = m - r + r_i$ . Then:

$$\text{rank } V_i(s) = \dim (\text{Ker } T^i(s)) \quad \text{for } i = 1, \dots, k$$

which ends the proof. □

Give now a technical result which will be used to provide a transfer matrix characterization of  $\Sigma_{ei}(C, A, B)$ .

**Lemma 1.** Let  $T(s)$  be a  $p \times m$  proper rational matrix partitioned in row-blocks relatively to  $(p_1, \dots, p_k)$ , such that  $\sum_{i=1}^k p_i = p$ . Let  $r_i$  denote the rank of  $T_i(s)$ , for

$i = 1, \dots, k$ . If  $T(s)$  is such that:

$$\text{rank } T(s) = \sum_{i=1}^k r_i$$

then  $T(s)$  can be decomposed as:

$$T(s) = B(s) \tilde{I} \tilde{T}(s)$$

where  $\tilde{T}(s)$  is an  $r \times m$  full row rank proper rational matrix partitioned in row-blocks relatively to  $\{r_i\}$ .  $B(s) = \text{diag}(B_1(s), \dots, B_k(s))$ ,  $B_i(s)$  is a  $p_i \times p_i$  bicausal matrix and  $\tilde{I} = \text{diag}(\tilde{I}_1, \dots, \tilde{I}_k)$ ,  $\tilde{I}_i = \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}$ ;  $I_{r_i}$  is the  $r_i \times r_i$  identity matrix.

The proof is quite similar to that of Lemma 1 of [11]. It can be performed by replacing the Smith-McMillan factorization of  $T_i(s)$  by a Smith-McMillan factorization at infinity of  $T_i(s)$ .

Now, we are able to give another interesting characterization of the block essential structure  $\Sigma_{ei}(C, A, B)$ , in the rational transfer matrix framework. In fact this is a non trivial generalization of [4] concerning the essential orders for the row by row case.

**Theorem 2.** Let  $(C, A, B)$  be the system (1) of transfer matrix  $T(s)$ , decomposed according to the output partition  $(p_1, \dots, p_k)$ . Assume that  $\text{rank } T(s) = \sum_{i=1}^k r_i$ , where  $r_i = \text{rank } T_i(s)$  and let  $T(s)$  be decomposed as in Lemma 1, i.e.:  $T(s) = B(s) \tilde{I} \tilde{T}(s)$ .

Let  $\tilde{T}(s)$  be factorized as follows:

$$\tilde{T}(s) = \begin{bmatrix} R(s) & 0 \end{bmatrix} B'(s)$$

where  $R(s)$  is an  $r \times r$  strictly proper rational matrix and  $B'(s)$  is an  $m \times m$  bicausal matrix. Consider  $R^{-1}(s)$  decomposed in the following way:

$$R^{-1}(s) = [\bar{R}_1(s), \dots, \bar{R}_k(s)]; \quad \bar{R}_i(s) \in \mathbf{R}^{r \times r_i}(s)$$

Then the  $i$ th block essential structure of  $(C, A, B)$  relatively to  $\{p_i\}$ ,  $\Sigma_{ei}(C, A, B)$ , satisfies:

$$\Sigma_{ei}(C, A, B) = -\Sigma_{\infty}(\bar{R}_i(s)) \quad \text{for } i = 1, \dots, k$$

**Proof.** Using Theorem 1, it is equivalent to prove:

$$\Sigma_{\infty}(T_i(s) V_i(s)) = -\Sigma_{\infty}(\bar{R}_i(s)) \quad \text{for } i = 1, \dots, k$$

where  $V_i(s)$  is any right bicausal matrix which is a basis for  $\text{Ker } T^i(s)$ . In the proof we choose  $V_i(s)$  a polynomial basis in  $s^{-1}$  for  $\text{Ker } T^i(s)$ . So, we begin by rewriting the  $i$ th row-block of  $T(s)$  in Lemma 1 as follows:

$$T_i(s) = B_i(s) \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix} \tilde{T}_i(s)$$

where  $B_i(s)$  is a  $p_i \times p_i$  bicausal matrix.  $\tilde{T}_i(s)$  is an  $r_i \times m$  full row-rank proper

rational matrix and where  $I_{r_i}$  is the  $r_i \times r_i$  identity matrix. A possible solution is to consider the Smith-McMillan factorization at infinity of  $T_i(s)$ .

In addition, since that subspaces  $\text{Ker } T^i(s)$  and  $\text{Ker } \tilde{T}^i(s)$  are identical, and since  $B_i(s)$  is bicausal we have:

$$\Sigma_\infty(T_i(s) V_i(s)) = \Sigma_\infty(\tilde{T}_i(s) V_i(s)) \quad (8)$$

Therefore it is sufficient to perform the proof for:  $\tilde{T}(s) = \begin{bmatrix} \tilde{T}_1(s) \\ \vdots \\ \tilde{T}_k(s) \end{bmatrix}$  which is a full row-rank rational matrix, because  $\text{rank } T(s) = r_1 + r_2 + \dots + r_k$ .

Let  $R_i(s)$  denote the  $i$ th row-block of the matrix  $R(s)$  partitioned respectively to  $\{r_i\}$ . Recall that  $R^i(s)$  is the  $(r - r_i) \times r$  matrix obtained by removing the  $i$ th row-block  $R_i(s)$  from  $R(s)$ . From  $V_i(s)$  we will first construct a polynomial basis in  $s^{-1}$ ,  $U_i(s)$ , for  $\text{Ker } R^i(s)$ . Then we will prove that  $\Sigma_\infty(T_i(s) V_i(s)) = \Sigma_\infty(R_i(s) U_i(s))$ . From this, the result will follow, basically because the infinite structures  $-\Sigma_\infty(R^{-1}(s))$ ,  $\Sigma_\infty(R(s))$  and  $\Sigma_\infty(T(s))$  are identical.

Define  $W_i(s) = B'(s) V_i(s)$ , which forms a basis of  $\text{Ker } [R^i(s) \ 0]$ . We can write:

$$\begin{aligned} \tilde{T}(s) [W_1(s), \dots, W_k(s)] &= \begin{bmatrix} R_1(s) & 0 \\ \vdots & \\ R_k(s) & 0 \end{bmatrix} [W_1(s), \dots, W_k(s)] = \\ &= \text{diag} [D_1(s), \dots, D_k(s)] \end{aligned} \quad (9)$$

where  $W_i(s)$  is an  $m \times (m - r + r_i)$  proper rational matrix. Notice that, since  $V_i(s)$  is a right bicausal matrix and  $B'(s)$  is a bicausal matrix,  $W_i(s)$  is also a right bicausal matrix.

From  $W_i(s)$  we can obtain a basis, denoted  $U_i(s)$ , for  $\text{Ker } R^i(s)$ . This can be made by choosing the first  $r$  rows of  $W_i(s)$ , i.e.,  $W'_i(s) := [I_r \ 0] W_i(s)$ , where  $I_r$  is the  $r \times r$  identity matrix. Since  $\dim(\text{Ker } R^i(s))$  is  $r_i$  and since  $W_i(s)$  possesses  $m - r + r_i$  columns independent at infinity we can take  $r_i$  independent columns at infinity from  $W'_i(s)$  to constitute basis  $U_i(s)$ . Therefore  $W'_i(s)$  can be factorized as follows:

$$W'_i(s) = [U_i(s), Z_i(s)] P_i$$

where  $P_i$  is an  $(m - r + r_i)$  permutation matrix. Because  $U_i(s)$  is a basis of span  $(W'_i(s))$ , we have:

$$Z_i(s) = U_i(s) \Omega_i(s)$$

we can write:  $Z_i(s) = [U_i(s), Y_i(s)] \begin{bmatrix} \Omega_i(s) \\ 0 \end{bmatrix}$  where  $Y_i(s)$  is chosen in such a way that  $[U_i(s), Y_i(s)]$  is a bicausal matrix. Since  $Z_i(s)$  is proper then  $\Omega_i(s)$  is also proper.

We can then construct a bicausal matrix  $B'_i(s)$  such that:

$$[U_i(s), Z_i(s)] = [U_i(s), 0] \begin{bmatrix} I_{r_i} & \Omega_i(s) \\ 0 & I_{m-r} \end{bmatrix} = [U_i(s), 0] B'_i(s)$$

it follows that:

$$[I_r \ 0] W_i(s) = W'_i(s) = [U_i(s) \ 0] B'_i(s) \quad (10)$$

where  $B_i''(s) = B_i'(s) P_i$  is bicausal. With this in mind, write the following expression:

$$\begin{bmatrix} R_1(s) \\ \vdots \\ R_k(s) \end{bmatrix} [U_1(s), \dots, U_k(s)] = \text{diag} [E_1(s), \dots, E_k(s)] \quad (11)$$

where  $E_i(s)$  is an  $r_i \times r_i$  proper rational matrix. By construction  $U_i(s)$  constitutes a right bicausal matrix.

Consider  $R^{-1}(s)$  partitioned in column-blocks relatively to  $\{r_i\}$  as follows:

$$R^{-1}(s) = [\bar{R}_1(s), \dots, \bar{R}_k(s)]$$

so, we have:

$$[U_1(s), \dots, U_k(s)] = [\bar{R}_1(s), \dots, \bar{R}_k(s)] \text{diag} [E_1(s), \dots, E_k(s)]$$

Since  $r = \sum_{i=1}^k r_i$ , and  $[U_1(s), \dots, U_k(s)]$  is non singular, then  $E_i(s)$  are non singular matrices. This allows us to rewrite the last expression in the following way:

$$[U_1(s), \dots, U_k(s)] \text{diag} [E_i^{-1}(s), \dots, E_k^{-1}(s)] = [\bar{R}_1(s), \dots, \bar{R}_k(s)]$$

then, for each block we have:

$$[U_i(s) X_i(s)] \begin{bmatrix} E_i^{-1}(s) \\ 0 \end{bmatrix} = \bar{R}_i(s); \quad \text{for } i = 1, \dots, k$$

where  $X_i(s)$  is any  $r \times (r - r_i)$  rational matrix. Since  $U_i(s)$  is a right bicausal matrix, we choose  $X_i(s)$  in order to make  $[U_i(s) X_i(s)]$  a bicausal matrix. Therefore the infinite structure of  $[U_i(s) X_i(s)] \begin{bmatrix} E_i^{-1}(s) \\ 0 \end{bmatrix}$  and  $E_i^{-1}(s)$  are equal.

From the latter, we have:

$$\Sigma_\infty(E_i(s)) = -\Sigma_\infty(\bar{R}_i(s)); \quad \text{for } i = 1, \dots, k$$

putting together expressions (9), (10) and (11), we obtain:

$$D_i(s) B_i''^{-1}(s) = [E_i(s) \ 0]; \quad \text{for } i = 1, \dots, k$$

Finally, since  $B_i''(s)$  is bicausal we get:

$$\Sigma_\infty(D_i(s)) = -\Sigma_\infty(\bar{R}_i(s)) \quad \text{for } i = 1, \dots, k$$

which by (8) and (9) ends the proof.  $\square$

*Remark.* In the proof of the above theorem the choice  $V_i(s)$  minimal basis in  $s^{-1}$  for  $\text{Ker } T_i(s)$  is not actually necessary. Any right bicausal matrix being a basis for  $\text{Ker } T^i(s)$  would do the job.

**Corollary 1.** With the above notations, the  $i$ th block essential structure  $\Sigma_{ei}(C, A, B)$  is feedback invariant.

*Proof.* Immediate from Theorem 2 and the fact that any state feedback control action can always be represented by a bicausal precompensator [14].  $\square$

In order to study the minimal McMillan degree achievable for the blocks of the decoupled system, let us recall the following “geometric” definition [11].

**Definition 4.** Consider the linear system  $(C, A, B)$ , with  $B$  monic and  $C$  epic, and an output partition  $(p_1, \dots, p_k)$ , such that  $\sum_{i=1}^k p_i = p$ . Let  $\mathcal{V}^*$  (resp.  $\mathcal{T}_i^*$ ) denote the largest  $(A, B)$ -invariant subspace in  $\text{Ker } C$  (resp.  $\text{Ker } C^i$ ). The *block decoupling invariants* of  $T(s)$ , denoted  $n_{ie}$ , are defined as follows:

$$n_{ie} = \dim(\mathcal{T}_i^*/\mathcal{V}^*) \quad \text{for } i = 1, \dots, k.$$

The above definition gives rise to the following transfer matrix characterizations of the block decoupling invariants  $n_{ie}$ .

**Theorem 3.** Let  $T(s)$  be a  $p \times m$  strictly proper rational matrix of null left and right static kernels decomposed in row-blocks according to a given partition  $(p_1, \dots, p_k)$  and  $(C, A, B)$  a realization of  $T(s)$ . Assume that  $\text{rank } T(s) = \sum_{i=1}^k r_i$ , where  $r_i$  denotes the rank of  $T_i(s)$ . Then the block decoupling invariants  $n_{ie}$  of  $T(s)$  satisfy:

$$(i) \quad n_{ie} = \sum_{j=1}^{r_i} \mu_{ij} + \sigma(T_i(s))$$

where  $\{\mu_{i1}, \dots, \mu_{ir_i}\}$  is the  $i$ th block essential structure  $\Sigma_{ei}(C, A, B)$ .

$$(ii) \quad n_{ie} = d_\infty(T(s)) - d_\infty(T^i(s)) + \sigma(T_i(s))$$

where  $T_i(s)$  denotes the  $i$ th row-block of  $T(s)$ ,  $T^i(s)$  denotes the matrix obtained from  $T(s)$  by removing the  $i$ th row-block and  $\sigma(T_i(s))$  denotes the sum of the degrees of a minimal polynomial basis for the left kernel of  $T_i(s)$ .

Roughly speaking the idea of the proof of (ii) is as follows. Let  $\mathcal{X}$  be the state space of  $(C, A, B)$ , making invariant  $\mathcal{V}^*$ , the state space dimension of the reduced system is  $\dim(\mathcal{X}/\mathcal{V}^*) = d_\infty(T(s)) + \sigma(T(s))$ . This comes from the well known equivalences between Morse’s list  $I_3$  and the infinite structure of  $(C, A, B)$  and between Morse’s list  $I_4$  and  $\sigma(T(s))$  (see [15], [18] and [8]). Analogously, making invariant  $\mathcal{T}_i^*$ , we obtain  $\dim(\mathcal{X}/\mathcal{T}_i^*) = d_\infty(T^i(s)) + \sigma(T^i(s))$ . The proof is finished noting that  $\sigma(T_i(s)) = \sigma(T(s)) - \sigma(T^i(s))$ , because of the row block independence hypothesis, i.e.,  $\text{rank } T(s) = r_1 + r_2 + \dots + r_k$ .

In [11], it is shown that  $d_\infty(T(s)) - d_\infty(T^i(s)) = d_\infty(\bar{R}_i(s))$ , which implies (i), using Theorem 2 of this paper.

## 4.2. Application to the Block Decoupling Problem

We are now ready to give our results above the minimal infinite structure and the minimal McMillan degree achievable for the blocks of a decoupled system.

**Theorem 4.** Let  $T(s)$  be a  $p \times m$  strictly proper rational matrix of null left and right static kernels, decomposed in row-blocks according to a given partition

$(p_1, \dots, p_k)$  and  $(C, A, B)$  a realization of  $T(s)$ . If the system  $T(s)$  can be decoupled relatively to  $(p_1, \dots, p_k)$  by an admissible decoupling precompensator, then we have the following:

- (i) The minimal McMillan degree achievable for the  $i$ th block of the decoupled system is given by the block decoupling invariant  $n_{ie}$ .
- (ii) The minimal infinite structure of the  $i$ th block of the decoupled system is given by the  $i$ th block essential structure  $\Sigma_{ei}(T(s))$ .

Notice that when  $\text{rank } T(s) = \sum_{i=1}^k r_i$ , where  $r_i$  denotes the rank of  $T_i(s)$ , block decoupling via precompensation is always possible, [14].

Proof. (i) was proved in [11].

To prove (ii), consider  $T(s)$  decomposed as in Lemma 1:

$$T(s) = B(s) \tilde{I} \tilde{T}(s)$$

and let  $T(s)$  be factorized as:

$$\tilde{T}(s) = [R(s) \ 0] B'(s)$$

where  $R(s)$  is an  $r \times r$  full rank strictly proper rational matrix and  $B'(s)$  is an  $m \times m$  bicausal matrix.

Now partition  $R^{-1}(s)$  as follows:

$$R^{-1}(s) = [\bar{R}_1(s), \dots, \bar{R}_k(s)]; \quad \bar{R}_i(s) \in R^{r \times r_i}(s)$$

Consider now a Smith-McMillan factorization of  $\bar{R}_i(s)$  written as:

$$\bar{R}_i(s) = B_{i1}(s) \begin{bmatrix} \Delta_i(s) \\ 0 \end{bmatrix} B_{i2}(s) \tag{12}$$

$$\Delta_i(s) = \text{diag}(\mu_{i1}, \dots, \mu_{ir_i}).$$

Let us consider the following compensator  $C(s)$ :

$$C(s) := B'^{-1}(s) \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} \tag{13}$$

where  $Y(s) := R^{-1}(s) \text{diag}(B_{12}^{-1}(s) \Delta_1^{-1}(s), \dots, B_{k2}^{-1}(s) \Delta_k^{-1}(s))$  and  $X(s)$  any  $(m - r) \times r$  proper rational matrix. Clearly  $C(s)$  is an admissible precompensator.

The decoupled system is:

$$T(s) C(s) = B(s) \tilde{I} \text{diag}(B_{12}^{-1}(s) \Delta_1^{-1}(s), \dots, B_{k2}^{-1}(s) \Delta_k^{-1}(s))$$

where the infinite zero structure of the  $i$ th block is equal to the infinite pole structure of  $\bar{R}_i(s)$  which is, from Theorem 2,  $\Sigma_{ei}(T(s)) = \{\mu_{i1}, \dots, \mu_{ir_i}\}$ .

In [11] it is shown that the least value for the sum of the infinite zero orders for the  $i$ th block of the decoupled system is  $n_{ie} - \sigma(T_i(s))$ . Then, from (i) of Theorem 3 the result follows. □

**Comments.** It is proved in [2], that when the number of inputs is sufficiently large the system is decouplable by dynamic state feedback.

In this case the minimal infinite structure achievable when decoupling is possible is  $\Sigma_e(C, A, B)$ .

In the same vein it is possible to take into account stability requirements, the minimal McMillan degree achievable for the blocks of the decoupled system (with stability) is given in [3]. Some results generalizing those of this paper incorporating stability requirements are given without proofs in [10].

## 5. AN ILLUSTRATING EXAMPLE

Consider the linear system  $(C, A, B)$ :

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} x$$

which transfer matrix is:

$$T(s) = \begin{bmatrix} s^{-1} & 0 & 0 & s^{-2} \\ 0 & s^{-1} & 0 & 0 \\ s^{-1} & s^{-1} & s^{-2} & s^{-2} \end{bmatrix}$$

We will show later that this system is not row by row decouplable, but it is possible to decouple  $T(s)$  in blocks according to the output partition  $(2, 1)$ .

In this way, we will first compute the block essential structure of  $(C, A, B)$  relatively to  $(2, 1)$  to illustrate, in a second time, the transfer matrix characterizations of  $\Sigma_{ei}(C, A, B)$  given in this work. We will finish the example by computing a block decoupling precompensator which will exhibit the minimal infinite structure achievable for the block decoupled system.

Begin by decomposing the matrix  $C$  of  $(C, A, B)$  according to the partition  $(2, 1)$  as follows:

$$C^1 = [1 \ 1 \ 1 \ 0 \ 0]; \quad C^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The maximal controllability subspaces  $\mathcal{R}_i^*$  contained in  $\text{Ker } C^i$ , for  $i = 1, 2$ , are

respectively:

$$\mathcal{R}_1^* = \text{span} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathcal{R}_2^* = \text{span} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By definition the  $i$ th block essential structure of  $(C, A, B)$  is equal to the infinite structure of the subsystem  $(C_i, A + BF_i, B_i)$ , where:

$$F_i \text{ is such that: } (A + BF_i) \mathcal{R}_i^* \subset \mathcal{R}_i^*$$

$$G_i \text{ is such that: } \text{Im } BG_i = \mathcal{B} \cap \mathcal{R}_i^*$$

and  $B_i$  such that  $\text{Im}(B_i) = \mathcal{B}_i$ , for  $i = 1, 2$ . In this case we have that:

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfy the required properties. Then computing the infinite structures of systems  $(C_i, A + BF_i, B_i)$  we get the following  $i$ th block essential structure of  $(C, A, B)$  relatively to (2,1):

$$\Sigma_{e1}(C, A, B) = \Sigma_{\infty}(C_1, A + BF_1, BG_1) = \{1, 2\}$$

$$\Sigma_{e2}(C, A, B) = \Sigma_{\infty}(C_2, A + BF_2, BG_2) = \{2\}$$

which means that  $(C_1, A + BF_1, BG_1)$  has two infinite zeros of orders  $\{1, 2\}$  and  $(C_2, A + BF_2, BG_2)$  has one infinite zero of order  $\{2\}$ .

Let us give now the first matrix characterization of  $\Sigma_{ei}(C, A, B)$ . For this, compute as in the proof of Theorem 1, the following matrices:

$$V_i(s) = (I - F_i(sI - A)^{-1}B)^{-1}G_i \quad \text{for } i = 1, 2,$$

giving in this case:

$$V_1(s) = \begin{bmatrix} 1 & 0 & -s^{-1} \\ -1 & -s^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad V_2(s) = \begin{bmatrix} 0 & -s^{-1} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a simple calculation shows that:

$$T_1(s) V_1(s) = \begin{bmatrix} s^{-1} & 0 & 0 \\ -s^{-1} & -s^{-2} & 0 \end{bmatrix}; \quad T_2(s) V_1(s) = 0$$

$$T_2(s) V_2(s) = [s^{-2} \ 0]; \quad T_1(s) V_2(s) = 0$$

where clearly  $V_i(s)$  is a right bicausal basis for  $\text{Ker } T^i(s)$  and then:

$$\Sigma_{e1}(C, A, B) = \Sigma_{\infty}(T_1(s) V_1(s)) = \{1, 2\}$$

$$\Sigma_{e2}(C, A, B) = \Sigma_{\infty}(T_2(s) V_2(s)) = \{2\}$$

and stated in Theorem 1.

For illustrating Theorem 2, let us now factorize  $T(s)$  as follows:

$$T(s) = [R(s) \ 0] B(s)$$

where  $B(s)$  is a bicausal matrix and  $R(s)$  is equal to:

$$R(s) = \begin{bmatrix} s^{-1} & 0 & 0 \\ 0 & s^{-1} & 0 \\ s^{-1} & s^{-1} & s^{-2} \end{bmatrix}; \quad \text{with} \quad R^{-1}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ -s^2 & -s^2 & s^2 \end{bmatrix}$$

and let  $R^{-1}(s) = [\bar{R}_1(s), \bar{R}_2(s)]$  be partitioned in column-blocks according to  $(2, 1)$ .  $\bar{R}_1(s)$  has two infinite poles of orders  $\{1, 2\}$  and  $\bar{R}_2(s)$  has one infinite pole of order  $\{2\}$ , this confirms us that:

$$\Sigma_{e1}(C, A, B) = \Sigma_{\infty}(\bar{R}_1(s)) = \{1, 2\} \quad \Sigma_{e2}(C, A, B) = \Sigma_{\infty}(\bar{R}_2(s)) = \{2\}$$

as stated in Theorem 2.

### Compensator Construction

Below we will construct a block decoupling precompensator  $C(s)$  which will exhibit the minimal infinite structure achievable for the decoupled system. In fact  $T(s)$  is not row by row decouplable, since condition  $m \geq 2p - k$  does not hold [9], where  $m = 4$  is the number of inputs,  $p = 3$  is the number of outputs and  $k = 1$  is the column rank at infinity of  $R^{-1}(s)$ . However it is block decouplable according to partition  $(2, 1)$ , since condition  $m \geq 2p - k^*$  is satisfied, where  $k^* = 2$  is the maximal column rank at infinity that can be obtained from the column blocks of  $R^{-1}(s)$  [2].

To illustrate the compensator construction of Theorem 3, let us consider the following Smith-McMillan factorizations at infinity of  $\bar{R}_1(s)$  and of  $\bar{R}_2(s)$ .

$$\bar{R}_1(s) = B_1(s) \begin{bmatrix} A_1(s) \\ 0 \end{bmatrix} B'_1(s) = \begin{bmatrix} s^{-1} & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\bar{R}_2(s) = B_2(s) \begin{bmatrix} A_2(s) \\ 0 \end{bmatrix} B'_2(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, from (13) given in the proof of Theorem 4 we have that a block decoupling

precompensator is given by:

$$C(s) = B^{-1}(s) \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix}$$

$$Y(s) = R^{-1}(s) \text{diag}(B_1^{-1}(s) A_1^{-1}(s), B_2^{-1}(s) A_2^{-1}(s))$$

and  $X(s)$  is any  $(m - r) \times r$  rational matrix. In this case we obtain:

$$Y(s) = \begin{bmatrix} s^{-1} & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \quad X(s) = [0 \ 0 \ 1]$$

yielding a compensator  $C(s)$ :

$$C(s) = B^{-1}(s) \begin{bmatrix} s^{-1} & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The decoupled system is finally:

$$T(s) C(s) = \begin{bmatrix} s^{-2} & -s^{-1} & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & s^{-2} \end{bmatrix} = \text{diag}(D_1(s), D_2(s))$$

As a matter of fact we have:

$$\Sigma_{\infty}(D_1(s)) = \{1, 2\}; \quad \Sigma_{\infty}(D_2(s)) = \{2\}$$

or  $D_1(s)$  has two infinite zeros of orders  $\{1, 2\}$  and  $D_2(s)$  has an infinite zero of order  $\{2\}$ , which verifies:

$$\Sigma_{ei}(T(s)) = \Sigma_{\infty}(D_i(s)) \quad \text{for } i = 1, 2.$$

Then  $D_1(s)$  and  $D_2(s)$  possess the minimal infinite structure achievable for the decoupled system.

To finish let us remark that the system  $(C, A, B)$  is block decouplable by non regular static state feedback. Actually we can verify that the following static state feedback:

$$F = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is equivalent to the precompensator  $C(s)$  given above. In fact:

$$(I - F(sI - A)^{-1} B)^{-1} G = \begin{bmatrix} 1 & 0 & -s^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} G = C(s)$$

Then the block decoupling by non regular static state feedback  $(F, G)$  provides us with the minimal infinite structure.

(Received November 5, 1990.)

## REFERENCES

- [1] R. Brockett and M. D. Mesarovic: The reproducibility of multivariable systems. *J. Math. Anal. Appl.* 2 (1965), 548—563.
- [2] C. Commault, J. M. Dion and J. Torres: Invariant spaces at infinity of linear systems: Application to block decoupling. *IEEE Trans. Automat. Control AC-35* (1990), 5, 618—623.
- [3] C. Commault, J. M. Dion and J. Torres: Minimal structure in the block decoupling problem with stability. *Automatica* 27 (1991), 2, 331—338.
- [4] C. Commault, J. Descusse, J. M. Dion, J. F. Lafay and M. Malabre: New decoupling invariants: the essential orders. *Internat. J. Control* 44 (1986), 3, 689—700.
- [5] J. Descusse and J. M. Dion: On the structure at infinity of linear square decouplable systems. *IEEE Trans. Automat. Control AC-27* (1982), 971—974.
- [6] J. Descusse, J. F. Lafay and M. Malabre: Solution to Morgan's problem. *IEEE Trans. Automat. Control AC-33* (1988), 8, 732—739.
- [7] J. Descusse: Block non interacting control with non regular static state feedback. *IFSC-Workshop, Prague-Czechoslovakia, Sept. 1989.*
- [8] J. M. Dion and C. Commault: Smith Mc Millan factorizations at infinity or rational matrix functions and their control interpretation. *Systems Control Lett. 1* (1982), 5, 312—320.
- [9] J. M. Dion and C. Commault: The minimal delay decoupling problem: feedback implementation with stability. *SIAM J. Control* 26 (1988), 1, 66—82.
- [10] J. M. Dion, C. Commault and J. Torres: Stable block decoupling invariants: Geometric and transfer matrix characterizations. In: *Proceedings of the Conference New Trends in Systems Theory, Genova, Birkhäuser, Boston 1989.*
- [11] J. M. Dion, J. A. Torres and C. Commault: New feedback invariants and the block decoupling problem. *Internat. J. Control* 51 (1990), 1, 219—235.
- [12] P. L. Falb and W. A. Wolovich: Decoupling in the design and synthesis of multivariable control systems. *IEEE Trans. Automat. Control AC-12* (1967), 6, 651—669.
- [13] G. D. Forney: Minimal bases of rational vector spaces with applications to multivariable linear systems. *SIAM J. Control* 13 (1973), 3, 493—520.
- [14] M. L. J. Hautus and M. Heymann: Linear feedback decoupling — transfer function analysis. *IEEE Trans. Automat. Control AC-28* (1983), 8, 823—832.
- [15] A. S. Morse: Structural invariants of linear multivariable systems. *SIAM J. Control Optim.* 11 (1973), 446—465.
- [16] A. S. Morse and W. M. Wonham: Status of non interacting control. *IEEE Trans. Automat. Control AC-16* (1973), 568—581.
- [17] G. Vergheze: *Infinite Frequency Behaviour in Generalized Dynamical Systems.* Ph. D. Dissertation, Dept. Elec. Eng. Stanford Univ., Stanford, Calif. 1978.
- [18] U.E. Warren and A. E. Eckberg Jr.: On the dimension of controllability subspaces: a characterization via polynomial matrices and Kronecker invariants. *SIAM J. Control* 13 (1975), 2, 434—445.
- [19] W. M. Wonham: *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York—Berlin—Heidelberg 1979.

*Dr. Christian Commault, Dr. Jean-Michel Dion, Laboratoire d'Automatique de Grenoble (C.N.R.S. URA 228), E.N.S.I.E.G. — I.N.P.G., B.P. 46 — 38402 Saint Martin d'Heres. France.*  
*Dr. Jacques Descusse, Laboratoire d'Automatique de Nantes (C.N.R.S. URA 823), E.N.S.M., 44072 Nantes Cedex 03. France.*  
*Dr. J. A. Torres, CINVESTAV del IPN, Ap. Postal 14-740, Mexico 07000 DF. Mexico.*