Tran Quoc Chien

Duality in vector optimization. III. Vector partially quasiconcave programming and vector fractional programming

*Kybernetika*, Vol. 20 (1984), No. 6, 458--472

Persistent URL: http://dml.cz/dmlcz/124604

KYBERNETIKA — VOLUME 20 (1984), NUMBER 6

# DUALITY IN VECTOR OPTIMIZATION

## Part III. Vector Partially Quasiconcave Programming and Vector Fractional Programming

TRAN QUOC CHIEN

In the last part of the tripaper so-called $T_2$-duality concept for a certain class of vector optimization programs is introduced. Further, both the $T_1$-duality theory and the $T_2$-duality theory are applied for some classes of vector fractional programming. Finally, for completeness some $T_1$-duality theorems are formulated and proved in the last section.

### 6. $T_2$-DUALITY IN VECTOR PARTIALLY QUASICONCAVE PROGRAMMING

Let $D$ be a nonempty convex in $R^n$. A function $f : D \to R$ is called *r-quasiconcave* $(r \in R)$ on $D$ if the sets

$$\{x \in D \mid f(x) \geqq \alpha\}$$

are convex for all $\alpha \geqq r$.

Given $F = [f_1, \ldots, f_q]$, where $f_k$, $k = 1, \ldots, q$ are functions defined on $D$, we say that $F$ is *r-quasiconcave*, where $r = (r_1, \ldots, r_q) \in R^q$, if $f_k$ are $r_k$-quasiconcave on $D$ for all $k = 1, \ldots, q$.

**Remark 1.** We see that quasiconcave functions are $-\infty$-quasiconcave. Hence all results in this section remain valid also for quasiconcave programming.

For the further development one will need the following theorem

**Theorem 6.1.** (Separation Theorem.)

Let $\{A_k\}_{k=1}^q$ be a finite family of convex subsets of $R^n$. Then if $\bigcap_{k=1}^q A_k = \emptyset$ there exist a family of vectors of $R^n$, not simultaneously equal 0, $\{l^k\}_{k=1}^q$ and a family of real numbers $\{\lambda_k\}_{k=1}^q$ such that

(i) $A_k \subset \{x \in R^n \mid \langle l^k, x\rangle \leqq \lambda_k\} \quad \forall k = 1, \ldots, q$,

(ii) $\sum_{k=1}^q l^k = 0$ and $\sum_{k=1}^q \lambda_k \leqq 0$.

Proof. Consider the sets $A = A_1 \times A_2 \times \ldots \times A_q$ and $B = \{(\underbrace{x, x, \ldots, x}_{q\text{-times}}) \mid x \in R^n\}$.

We have $A \cap B = \emptyset$ for $\bigcap_{k=1}^{q} A_k = \emptyset$. Hence, by the known separation theorem of two convex sets, there exists a family of vectors of $R^n$, not simultaneously equal 0, $\{l^k\}_{k=1}^q$ such that

$$\sum_{k=1}^{q} \langle l^k, x^k \rangle \leqq \langle \sum_{k=1}^{q} l^k, x \rangle \quad \forall x^k \in A_k, k = 1, \ldots, q, \quad \forall x \in R^n .$$

From this inequality it follows that $\sum_{k=1}^{q} l^k = 0$ and

$$\lambda_k = \sup_{x \in A_k} \langle l^k, x \rangle < +\infty \quad \forall k = 1, \ldots, q .$$

So the families $\{l^k\}_{k=1}^q$ and $\{\lambda_k\}_{k=1}^q$ satisfy condition (i), (ii). □

Now let $F = [f_1, \ldots, f_q]$ be $r$-quasiconcave on a nonempty convex set $D \subset R^n$ where $r = (r_1, \ldots, r_q) \in R^q$. Consider the program

$$\text{(6.1)} \qquad \text{find} \quad \operatorname{Sup}^w \{F(x) \mid x \in D\}$$

which is called a *vector partially quasiconcave program.*

Similarly as in Section 4 of [2], instead of program (6.1), we shall deal with its modified program

$$\text{(6.2)} \qquad \text{find} \quad \operatorname{Sup}^w \bigcup_{x \in D} \mu_F(x) = S$$

where

$$\mu_F(x) = F(x) - R^q_+ \quad \forall x \in D .$$

Denote

$$\text{(6.3)} \quad \mathscr{L} = \{L = (l^1, \ldots, l^q) \mid l^k \in R^n, \ \forall k = 1, \ldots, q, \sum_{k=1}^{q} l^k = 0 \quad \text{and} \quad \exists l^j \neq 0\}$$

$$\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_q) \in R^q \mid \sum_{k=1}^{q} \lambda_k \leqq 0\}$$

and

$$\text{(6.4)} \quad D^* = \{(L, \lambda) \in \mathscr{L} \times \Lambda \mid \exists j \in \{1, \ldots, q\} \ ; \ D \cap \{x \in R^n \mid \langle l^j, x \rangle > \lambda_j\} \neq \emptyset\} .$$

For any $(L, \lambda) \in D^*$ put

$$\mathscr{S}_k(l^k, \lambda_k) = \sup \{f_k(x) \mid x \in D, \langle l^k, x \rangle > \lambda_k\}$$

$$\Phi(L, \lambda) = [\mathscr{S}_1(l^1, \lambda_1), \ldots, \mathscr{S}_q(l^q, \lambda_q)]$$

and

$$\text{(6.5)} \qquad v_\Phi(L, \lambda) = \Phi(L, \lambda) + R^q_+ .$$

The program

$$\text{(6.6)} \qquad \text{find} \quad \operatorname{Inf}^w \bigcup_{(L,\lambda) \in D^*} v_\Phi(L, \lambda) = I$$

is called a $T_2$-*dual* for the program (6.1) (or (6.2)).

**Theorem 6.2.** (Weak $T_2$-Duality Principle.)
If $f_k$ are lower semicontinuous for all $k = 1, \ldots, q$, then we have

$$\forall x \in D \ \forall (L, \lambda) \in D^* : F(x) - \Phi(L, \lambda) \notin \operatorname{int} R^q_+ .$$

Proof. Let $x \in D$ and $(L, \lambda) \in D^*$. If $F(x) > \Phi(L, \lambda)$, then

$$x \in \{z \in R^n \mid \langle l^k, z \rangle \leqq \lambda_k\} \quad \forall k = 1, \ldots, q .$$

Let $j \in \{1, \ldots, q\}$ be such that (see (6.4))

$$D \cap \{z \in R^n \mid \langle l^j, z \rangle > \lambda_j\} \neq \emptyset .$$

Since $f_j(x) > \mathscr{S}_j(l^j, \lambda_j)$ and $f_j$ is lower semicontinuous we have $\langle l^j, x \rangle < \lambda_j$. Hence

$$\sum_{k=1}^{q} \langle l^k, x \rangle < \sum_{k=1}^{q} \lambda_k \leqq 0$$

which contradicts the property $\sum_{k=1}^{q} l^k = 0$ for any $L \in \mathscr{L}$. □

Further put

$$M = \operatorname{Min}^w \bigcup_{(L,\lambda) \in D^*} v_\Phi(L, \lambda) \tag{6.7}$$

and

$$R_r = (r_1, \infty) \times \ldots \times (r_q, \infty) .$$

**Theorem 6.3.** (Partially Strong $T_2$-Duality Principle.)
Suppose that the relative interior of the set $D$, ri $D$, is nonempty and $f_k$ are lower semicontinuous for all $k = 1, \ldots, q$. Then the following equalities

$$S \cap R_r = M \cap R_r = I \cap R_r$$

hold.

Proof. It suffices to prove the assertion for the case when int $D$ is nonempty since then it can be analogously carried out in the affine hull of $D$ if int $D = \emptyset$. Suppose thus that int $D \neq \emptyset$. We shall prove it in the following steps
(i) $S \cap R_r \subset M$: Let $y = (y_1, \ldots, y_q) \in S \cap R_r$. Put

$$A_k = \{x \in D \mid f_k(x) > y_k\} , \quad k = 1, \ldots, q .$$

Obviously $\bigcap_{k=1}^{q} A_k = \emptyset$ for the optimality of $y$. By Theorem 6.1 there is $(L, \lambda) \in \mathscr{L} \times \Lambda$ such that

$$A_k \subset \{x \in R^n \mid \langle l^k, x \rangle \leqq \lambda_k\} , \quad k = 1, \ldots, q .$$

If $D \subset \{x \in R^n \mid \langle l^k, x \rangle \leqq \lambda_k\}$, $\forall k = 1, \ldots, q$, then for any interior point $x \in D$ we have $\sum_{k=1}^{q} \langle l^k, x \rangle < \sum_{k=1}^{q} \lambda_k \leqq 0$ which contradicts $\sum_{k=1}^{q} l^k = 0$. Consequently $(L, \lambda) \in D^*$.

It is easy to see that $\Phi(L, \lambda) \leqq y$ or $y \in v_\Phi(L, \lambda)$. From the weak $T_2$-duality principle it follows $y \in M$.

(ii) $I \cap R_r \subset S$: Let $y \in I \cap R_r$. Then for any $z \in R^q$, $r \leqq z < y$ there is an $x \in D$ such that $z \leqq F(x)$ (otherwise by the same consideration as above we have $z \in \bigcup_{(L,\lambda)\in D^*} v_\Phi(L, \lambda)$ which contradict $y \in I$). It means $y \in \overline{\bigcup_{x\in D} \mu_F(x)}$, which implies $y \in S$ in view of the weak $T_2$-duality principle.

The inclusion $M \cap R_r \subset I$ is obvious. The proof is complete. □

**Corollary.** (Partially Direct $T_2$-Duality Theorem.)

If $x \in D$ is an optimal solution of the primal (6.2) such that $F(x) \geqq r$, then there is an optimal solution $(L, \lambda)$ of its $T_2$-dual (6.6) such that

$$F(x) \in v_\Phi(L, \lambda) .$$

**Theorem 6.4.** (Partially Converse $T_2$-Duality Theorem.)

Let $(L, \lambda) \in D^*$ be an optimal solution of the $T_2$-dual (6.6) such that $\Phi(L, \lambda) > r$. If $D$ is closed, $f_k$ are upper semicontinuous for all $k = 1, \ldots, q$ and there exists $j \in \{1, \ldots, q\}$ such that the set $A = \{x \in D \mid f_j(x) \geqq \mathcal{S}_j(l^j, \lambda_j)\}$ is nonempty and bounded, then there is an optimal solution of the primal (6.2) such that

$$F(x) \in v_\Phi(L, \lambda) .$$

Proof. Choose a sequence $\{y^m\}_{m=1}^{\infty} \subset R^q$ such that $\Phi(L, \lambda) > y^m \geqq r$ for all $m$ and $y_k^m \uparrow \mathcal{S}_k(l^k, \lambda_k)$ for all $k = 1, \ldots, q$. Put

$$A_k^m = \{x \in D \mid f_k(x) \geqq y_k^m\} .$$

From the optimality of $(L, \lambda)$ it follows that the sets $A^m = \bigcap_{k=1}^{q} A_k^m$ are nonempty, closed and convex for all $m$. In view of Lemma 5.9. in [2] there is an $m_0$ such that $A_j^m$ are bounded for all $m \geqq m_0$. Consequently $A^m$ are bounded for $m \geqq m_0$. Hence $\bigcap_{m=1}^{\infty} A^m \neq \emptyset$ and any $x \in \bigcap_{m=1}^{\infty} A^m$ is a required optimal solution with $F(x) \in$ $\in v_\Phi(L, \lambda)$.

The proof is complete. □

## 7. INTRODUCTION TO FRACTIONAL PROGRAMMING

In the literature problems of the type

$$\max \{f(x)/g(x) \mid x \in S\} \tag{7.1}$$

are called *fractional programs*. Optimization problems of this kind occur if rates of economic or technical terms (for instance cost/time) define the objective function. Many works (at least 400 according to Schaible [3]) have already appeared in this field. One may find a relatively complete survey on fractional programming in Schaible [3]. We shall now develop a duality theory for Vector Fractional Programming (V. F. P.), which is still less investigated. For the scalar fractional programming there are several approaches to define duals, see [3]–[7], and the most known of them is the transformation method. On the basis of this method one can transform program (7.1), where $-f, g$ are convex and $S$ is convex, to a concave program and then apply the known duality theory for convex programming. As regards V. F. P. these approaches are not applicable, since it is not generally possible to reduce simultaneously all components of objective functions to concave functions. That is why one should find a new method to develop a duality theory for V. F. P. Here, on the basis of the results achieved in the foregoing parts, we shall give a duality theory for some classes of V. P. F.

Now let $S \subset R^n$ be a nonempty set and $N_k$, $D_k$, $k = 1, \ldots, q$, be functions on $S$ such that $D_k$, $k = 1, \ldots, q$, do not vanish on $S$. Put

$$Q(x) = \{N_1(x)/D_1(x), \ldots, N_q(x)/D_q(x)\} .$$

The program

(7.2) $$\text{find} \quad Sup^w\{Q(x) \mid x \in S\}$$

is called a *Vector Fractional Program* (V. F. P.)

If $N_k$, $D_k$, $k = 1, \ldots, q$, are affine and $S$ is a polytope then the program (7.2) is called a *Vector Linear Fractional Program* (V. L. F. P.)

If $N_k$, $D_k$, $k = 1, \ldots, q$, are quadratic, and $S$ is a polytope it is called a *Vector Quadratic Fractional Program* (V. Q. F. P.)

If $N_k$, $k = 1, \ldots,$ are quadratic, $D_k$, $k = 1, \ldots, q$, are affine and $S$ is a polytope it is called a *Vector Quadratic-Affine Fractional Program* (V. Q. A. F. P.).

If $-N_k$, $D_k$, $k = 1, \ldots, q$, are convex and $S$ is a convex set it is called a *Vector Concave Convex Fractional Program* (V. C. C. F. P.)

*Quasiconcavity condition* (Q. C. C.):

(7.3) $$D_k \text{ are positive on } S \text{ for all } k = 1, \ldots, q$$

and

(7.4) $$N_k \text{ is nonnegative or } D_k \text{ is affine on } S \ \forall k = 1, \ldots, q .$$

**Theorem 7.1.** Under the Q. C. C. the V. C. C. F. P. becomes a quasiconcave program.

Proof. See [8] Theorem 51 page 62.

We see that the V. C. C. F. P. satisfying the Q. C. C., in particular the V. L. F. P., are vector quasiconcave programs on $S$. Therefore for such program the $T_1$-duality in Section 4 of [2] is applicable. For. V. C. C. F. P. where it is difficult to verify the Q. C. C. it is more convenient to use the $T_2$-duality.

In Section 8 a $T_1$-duality for V. Q. A. F. P. is given and in Section 9 a $T_2$duality for V. Q. F. P. is developed.

## 8. $T_1$-DUALITY IN VECTOR QUADRATIC-AFFINE FRACTIONAL PROGRAMMING

Let $\mathbf{B}_k$, $k = 1 \ldots, q$ be real symmetric negative semidefinite $n \times n$ matrices, $\mathbf{A}$ be a real $m \times n$ matrix, $b \in R^m$, $c_k$, $d_k \in R^n$ and $\alpha_k, \alpha_k \in R$ for $k = 1, \ldots, q$. Suppose that

$$S_+ = \{x \in R^n \mid d'_k \,.\, x + \beta_k < 0,\ \forall k = 1, \ldots, q\} \neq \emptyset$$

and put

$$Q(x) = \{(x'\mathbf{B}_1 x + c'_1 \,.\, x + \alpha_1)/(d'_1 \,.\, x + \beta_1), \ldots$$
$$\ldots, x'\mathbf{B}_q x + c'_q \,.\, x + \alpha_q)/(d'_q \,.\, x + \beta_q)\} \,.$$

The program

(8.1) $$\text{find} \quad \mathrm{Sup}^w \{Q(x) \mid x \in S_+, \mathbf{A}x \leqq b\}$$

is called a *Vector Quadratic-Affine Fractional Program* (V. Q. A. F. P.)

As usual we will find a dual for the modified program of (8.1)

(8.2) $$\text{find} \quad \mathrm{Sup}^w \bigcup_{\substack{x \leqq b \\ \mathbf{A}x \in S_+}} \mu_Q(x) = S$$

where

$$\mu_Q(x) = Q(x) - R^q_+ \,.$$

Using the transformation

(8.3) $$u_k = c'_k \,.\, x + \alpha_k \,, \quad v_k = d'_k \,.\, x + \beta_k \,, \quad k = 1, \ldots, q$$

one can write program (8.2) in the following equivalent form

(8.4) $$\text{find} \quad \mathrm{Sup}^w \bigcup_{x \in \mathscr{D}} \mu_F(x, u, v) = S$$

where

$$\mathscr{D} = \{(x, u, v) \mid x \in R^n, u \in R^q, v \in R^q, v < 0 : A_x \leqq b \,\&$$
$$\& \, c'_k \,.\, x + \alpha_k - u_k = 0 \,\&\, d'_k \,.\, x + \beta_k - v_k = 0 \ \forall k = 1, \ldots, q\} \,,$$
$$F(x, u, v) = \{(x'\mathbf{B}_1 x + u_1)/v_1, \ldots, (x'\mathbf{B}_q x + u_q)/v_q\}$$

and

$$\mu_F(x, u, v) = F(x, u, v) - R^q_+ \,.$$

For the program (8.4) we shall apply the approach introduced in Section 4 of [2] to define its $T_1$-dual. Given

$$z \in R^m \,, \quad z \leqq 0 \,, \quad \bar{u}_k, \bar{v}_k \in R \,, \quad k = 1, \ldots, q \,, \quad \bar{x} \in R^n \,, \quad \tilde{u}_k, \tilde{v}_k \in R \,,$$
$$k = 1, \ldots, q$$

we have

(8.5) $$r = r(z, \bar{u}, \bar{v}, \bar{x}, \tilde{u}, \tilde{v}) = \sup \{z'(\mathbf{A}x - b) + \sum_{k=1}^{q} \bar{u}_k(c'_k x + \alpha_k - u_k) +$$

$$+\sum_{k=1}^{q} \bar{v}_k(d'_k x + \beta_k - v_k) + \bar{x}'x + \sum_{k=1}^{q} \tilde{u}_k u_k + \sum_{k=1}^{q} \tilde{v}_k v_k \mid x \in R^n ,$$

$$u = (u_1, \ldots, u_q), v = (v_1, \ldots, v_q) \in R^q, v > 0\} =$$

$$= \sup \{[z'\mathbf{A} + \bar{x}' + \sum_{k=1}^{q} \bar{u}_k c'_k + \sum_{k=1}^{q} \bar{v}_k d'_k] x + \sum_{k=1}^{q} (\tilde{u}_k - \bar{u}_k) u_k +$$

$$+ \sum_{k=1}^{q} (\tilde{v}_k - \bar{v}_k) v_k + [-z'b + \sum_{k=1}^{q} \bar{u}_k \alpha_k + \sum_{k=1}^{q} v_k \beta_k] \mid x \in R^n, u, v \in R^q, v < 0\} =$$

$$= \begin{cases} -z'b + \sum_{k=1}^{q} \bar{u}_k \alpha_k + \sum_{k=1}^{q} \bar{v}_k \beta_k & \text{if} \quad z'\mathbf{A} + \bar{x}' + \sum_{k=1}^{q} \bar{u}_k c'_k + \sum_{k=1}^{q} \bar{v}_k d'_k = 0 \\ \tilde{u}_k = \bar{u}_k \ \& \ \tilde{v}_k \leqq \bar{v}_k , \quad k = 1, \ldots, q \\ +\infty \quad \text{otherwise} \end{cases}$$

Having denoted

$$\nu(H_{z,\bar{u},\bar{v},\bar{x},\tilde{v}}) = \{y \in R^q \mid \forall x \in R^n, u, v \in R^q, v > 0 : \bar{x}'x +$$
$$+ \sum_{k=1}^{q} \bar{u}_k u_k + \sum_{k=1}^{q} \tilde{v}_k v_k \leqq r \Rightarrow F(x, u, v) \gtreqless y\}$$

it is easy to verify that

$$\nu(H_{z,\bar{u},\bar{v},\bar{x},\tilde{v}}) \subset \nu(H_{z,\bar{u},\bar{v},\bar{x},\bar{v}}) = \nu(H_{z,\bar{u},\bar{v},\bar{x}}) \quad \forall \tilde{v} \leqq \bar{v} .$$

Then according to Section 4 of [2] the $T_1$-dual of program (8.4) is the following program

$$\operatorname{Inf}^{w} \bigcup_{H_{z,\bar{u},\bar{v},\bar{x}} \in P_0^*} \nu(H_{z,\bar{u},\bar{v},\bar{x}}) = I \tag{8.6}$$

where

$$P_0^* = \{H_{z,\bar{u},\bar{v},\bar{x}} \mid \nu(H_{z,\bar{u},\bar{v},\bar{x}}) \neq \emptyset\}$$

Further, in view of the Remark 1 of Lemma 1.4 in [1] and Lemma 4.3 in [2] we have

$$I = \operatorname{Inf}^{w} \bigcup_{H_{z,\bar{u},\bar{v},\bar{x}} \in P_0^*} \nu(H_{z,\bar{u},\bar{v},\bar{x}}) = \operatorname{Inf}^{w} \bigcup_{H_{z,\bar{u},\bar{v},\bar{x}} \in P_0^*} \operatorname{Inf}^{w} \nu(H_{z,\bar{u},\bar{v},\bar{x}}) =$$
$$= \operatorname{Inf}^{w} \bigcup_{(z,\bar{u},\bar{v},\bar{x}) \in \mathscr{L}} L(z, \bar{u}, \bar{v}, \bar{x})$$

where

$$\mathscr{L} = \{(z, \bar{u}, \bar{v}, \bar{x}) \in R^m \times R^q \times R^q \times R^n \mid z \leqq 0, z'\mathbf{A} + \bar{x}' + \tag{8.8}$$
$$+ \sum_{k=1}^{q} \bar{u}_k c'_k + \sum_{k=1}^{q} \bar{v}_k d'_k = 0 \ \&$$
$$\& \ \operatorname{Sup}^{w} \{F(x, u, v) \mid v > 0, \bar{x}'x + \bar{u}'u + \bar{v}'v \leqq r\} \neq \emptyset\}$$
$$r = r(z, \bar{u}, \bar{v}, \bar{x}) = - z'b + \sum_{k=1}^{q} \bar{u}_k \alpha_k + \sum_{k=1}^{q} \bar{v}_k \beta_k$$

and

$$(8.9)\qquad L(z, \bar{u}, \bar{v}, \bar{x}) = \operatorname{Sup}^{w} \bigcup_{\substack{x\in R^n, u,v\in R^q, v>0 \\ \bar{x}'x+\bar{u}'u+\bar{v}'v\leqq r}} \mu_F(x, u, v)\,.$$

**Lemma 8.1.** Let $\bar{x} \in R^n$, $u, v \in R^q$ and $r \in R$, If

$$(8.10)\qquad \operatorname{Sup}^{w}\{F(x, u, v) \mid x \in R^n, u, v \in R^q, v > 0 : \bar{x}'x + \bar{u}'u + \bar{v}'v \leqq r\} \neq \emptyset$$

then $u \gtrsim 0$.

Proof. Let $(x^0, u^0, v^0)$ be an arbitrary feasible solution of (8.10) If $u = 0$ or there is an index $j \in \{1, \ldots, q\}$ such that $u_j < 0$, one can choose a sequence $\{u^l\}_{l=1}^{\infty}$ such that

$$u_k^l \to \infty \quad \text{for} \quad l \to \infty \quad \text{and} \quad \forall k = 1, \ldots, q$$

and

$$\bar{x}'x^0 + \bar{u}'u^l + \bar{v}v^0 \leqq r\,.$$

Then we obtain $F(x^0, u^l, v^0) \to \infty$ for $l \to \infty$ which contradicts with (8.10). □

Now from the nonnegativity of the vector $u$ we see that $\mathscr{L}$ and $(L(z, \bar{u}, \bar{v}, \bar{x})$ remain unchanged if in (8.8) and (8.9) instead of the inequality sign we replace the equality one. So we have

$$(8.11)\qquad \mathscr{L} = \{(z, \bar{u}, \bar{v}, \bar{x}) \in R^m \times R^q \times R^q \times R^n \mid z \leqq 0\,,\quad z'\mathbf{A} + \bar{x}' + \\ + \sum_{k=1}^{q} \bar{u}_k c_k' + \sum_{k=1}^{q} \bar{v}_k d_k' = 0\,\& \\ \&\ \operatorname{Sup}^{w}\{F(x, u, v) \mid x \in R^n, u, v \in R^q, v > 0 : \bar{x}'x + \bar{u}'u + \bar{v}'v = r\} \neq \emptyset$$

and

$$(8.12)\qquad L(z, \bar{u}, \bar{v}, \bar{x}) = \operatorname{Sup}^{w} \bigcup_{\substack{x\in R^n, u,v\in R^q, v>0 \\ \bar{x}'x+\bar{u}'u+\bar{v}'v=r}} \mu_F(x, u, v)\,.$$

Summarizing the foregoing results we obtain

**Theorem 8.1.** The program

$$(8.13)\qquad \text{find} \quad \operatorname{Inf}^{w} \bigcup_{(z,\bar{u},\bar{v},\bar{x})\in\mathscr{L}} L(z, \bar{u}, \bar{v}, \bar{x}) = I$$

where $\mathscr{L}$ and $L(z, \bar{u}, \bar{v}, \bar{x})$ are defined in (8.11) and (8.12) is the $T_1$-dual of the program (8.2). If the primal (8.2) is feasible then the strong duality principle holds, i.e. $S = I$.

Now we shall show that our $T_1$-dual is a generalization of the scalar one, given by Schaible in [5]. Consider thus the program

$$(8.14)\qquad \text{find} \quad \sup\{Q(x) = x'\mathbf{B}x + c'x + \alpha)/(d'x + \beta) \mid x \in S_0, \mathbf{A}x \leqq b\}$$

where $\mathbf{B}$ is a symmetric, negative semidefinite $n \times n$ matrix, $\mathbf{A}$ is a real $m \times n$ matrix, $c, d \in R^n$, $\alpha, \beta \in R$ and $S_0 = \{x \in R^n \mid d'x + \beta_0 > 0\} \neq \emptyset$.

According to (8.11) and (8.12) we have

$$(8.15)\quad \mathscr{L} = \{(z, \bar{u}, \bar{v}, \bar{x}) \in R^m \times R \times R \times R^n \mid z \leqq 0, z'\mathbf{A} + \bar{x}' + \bar{u}c' + \bar{v}d' = 0\ \& \\ \&\ \sup\{F(x, u, v) = (x'\mathbf{B}x + u)/v \mid x \in R^n, u, v \in R, v > 0, \bar{x}'x + \bar{u}u + \\ + \bar{v}v = r = -z'b + \bar{u}\alpha + \bar{v}\beta\} \neq \{-\infty, +\infty\}\}$$

and

$$(8.16)\quad L(z, \bar{u}, \bar{v}, \bar{x}) = \sup\{F(x, u, v) \mid x \in R^n, u, v \in R, v < 0, \bar{x}'x + \bar{u}u + \bar{v}v = r\}.$$

It is easy to check that in the scalar case the $T_1$-dual (8.13) remains unchanged if we assume the additional condition $\bar{u} = 1$ (see Lemma 8.1). Consider now the problem

$$(8.17)\quad L(z, \bar{v}, \bar{x}) = \sup\{(x'\mathbf{B}x + u)/v \mid x \in R^n, u, v \in R, v > 0, \bar{x}'x + \bar{u}u + \bar{v}v = r\}.$$

Having replaced $u = r - \bar{x}'x - \bar{v}v$ in $F(x, u, v)$ we obtain

$$(8.18)\quad L(z, \bar{v}, \bar{x}) = \sup\{x'\mathbf{B}x + r - \bar{x}'x - \bar{v}v)/v \mid x \in R^n, v \in R, v > 0\} = \\ = \sup\{r - \bar{x}'x + x'\mathbf{B}x)/v \mid x \in R^n, v \in R, v > 0\} - \bar{v} = \\ = \begin{cases} -\bar{v} & \text{if} \quad r \leqq \inf_{x \in R^n}\{\bar{x}'x - x'\mathbf{B}x\} \\ +\infty & \text{otherwise} \end{cases}$$

Put $\mathscr{B} = \{2\mathbf{B}w \mid w \in R^n\}$. If $\bar{x} \notin \mathscr{B}$ then $\bar{x}$ can be expressed as follows

$$\bar{x} = x^1 + x^2, \quad x^1 \in \mathscr{B}, \quad x^2 \perp \mathscr{B}, \quad x^2 \neq 0.$$

We have then

$$\bar{x}'(tx^2) - (tx^2)'\,\mathbf{B}(tx^2) = t\langle x^2, x^2\rangle \to -\infty \quad \text{for} \quad t \to -\infty.$$

If $\bar{x} = 2\mathbf{B}w$ for some $w \in R^n$ then, for the negative semidefinity,

$$\bar{x}'x - x'\mathbf{B}x = 2w'\mathbf{B}x - x'\mathbf{B}x \geqq w'\mathbf{B}w$$

and the equality is attained when $x = w$.

From (8.18) and the above consideration we obtain

$$(8.19)\qquad L(z, v, x) = -\bar{v} \Leftrightarrow \exists w \in R^n, \bar{x} = 2\mathbf{B}w\ \&\ r \leqq w'\mathbf{B}w.$$

Finally, in view of (8.13), (8.15), (8.16), (8.18) and (8.19) the $T_1$-dual of the program (8.14) can be formulated as following

$$(8.20)\qquad \begin{array}{c} \text{find} \quad \inf -\lambda \\ \text{subject to} \quad z'\mathbf{A} + 2w'\mathbf{B} + c' + \lambda d' = 0 \\ -z'b + \alpha + \lambda\beta \leqq w'\mathbf{B}w \\ z \in R^m, \quad z \leqq 0, \quad w \in R^n, \quad \lambda \in R. \end{array}$$

**Remark.** If $\mathbf{B} = 0$ for all $k = 1, \ldots, q$ the program (8.1) becomes a *Vector Linear*

*Fractional Program* (V. L. F. P.), the $T_1$-dual of which will be as following

$$\text{find} \quad \operatorname{Inf}^{w} \bigcup_{(z,\bar{u},\bar{v}) \in \mathscr{L}} L(z, \bar{u}, \bar{v}) = I \tag{8.21}$$

where

$$\mathscr{L} = \{(z, \bar{u}, \bar{v}) \in R^m \times R^q \times R^q \mid z \leqq 0,\ z'\mathbf{A} + \sum_{k=1}^{q} \bar{u}_k c'_k + \sum_{k=1}^{q} \bar{v}_k d'_k = 0\ \& \tag{8.22}$$

$$\& \operatorname{Sup}^{w} \{F(u, v) \mid u, v \in R^q, v > 0,\ \bar{u}u + \bar{v}'v = r\} = \emptyset\}$$

$$F(u, v) = \{u_1/v_1, \ldots, u_q/v_q\}$$

$$r = r(z, \bar{u}, \bar{v}) = -z'b + \sum_{k=1}^{q} \bar{u}_k \alpha_k + \sum_{k=1}^{q} \bar{v}_k \beta_k$$

$$L(z, \bar{u}, \bar{v}) = \operatorname{Sup}^{w} \bigcup_{\substack{u,v \in R, v > 0 \\ \bar{u}'u + \bar{v}'v = r}} F(u, v) - R^q_+ \tag{8.23}$$

The scalar linear fractional program

$$\text{find} \quad \sup \{c'x + \alpha)/(d'x + \beta) \mid x \in R^n,\ d'x + \beta < 0,\ \mathbf{A}x \leqq b\} \tag{8.24}$$

is a particular case of program (8.14), where $\mathbf{B} = 0$. Consequently according to (8.20) it has the following dual

$$\inf - \lambda \tag{8.25}$$

$$\text{subject to} \quad z'\mathbf{A} + c' + \lambda d' = 0$$

$$-z'b + \alpha + \lambda\beta \leqq 0$$

$$z \in R^m, \quad z \leqq 0, \quad \lambda \in R$$

## 9. $T_2$-DUALITY IN VECTOR QUADRATIC FRACTIONAL PROGRAMMING

In this section we assume that $\mathbf{C}_k$, $\mathbf{D}_k$, $k = 1, \ldots, q$, are real symmetric $n \times n$ matrices, negative, positive semidefinite respectively, $c_k$, $d_k \in R^n$ and $\alpha_k, \beta_k \in R$, $k = 1, \ldots, q$. Consider the program

$$\text{find} \quad \operatorname{Sup}^{w} \{Q(x) \mid x \in \mathscr{D}\} \tag{9.1}$$

where

$$Q(x) = [(x'\mathbf{C}_k x + c'_k x + \alpha_k)/(x'\mathbf{D}_k x + d'_k x + \beta_k)]_{k=1}^{q}$$

and $\mathscr{D}$ is a nonempty convex set of $R^n$ on which the denominators $x'\mathbf{D}_k x + d'_k x + \beta_k$ are positive for all $k = 1, \ldots, q$.

The function

$$q_k(x) = (x'\mathbf{C}_k x + c'_k x + \alpha_k)/(x'\mathbf{D}_k x + d'_k x + \beta_k)$$

is 0-quasiconcave, resp. $-\infty$-quasiconcave, if $\mathbf{D}_k \neq \emptyset$, resp. $\mathbf{D}_k = 0$, so that for this program one can apply the $T_2$-duality theory introduced in Section 6. Put

$$\mathscr{L} = \{L = (l^1, \ldots, l^q) \mid l^k \in R^n, k = 1, \ldots, q, \sum_{k=1}^{q} l^k = 0, \exists j \in \{1, \ldots, q\} : l^j \neq 0\}$$

$$\Lambda = \{(\lambda_1, \ldots, \lambda_q) = \lambda \in R^q \mid \sum_{k=1}^{q} \lambda_k \leqq 0\}$$

$$\mathscr{D}^* = \{(L, \lambda) \in \mathscr{L} \times \Lambda \mid \exists j \in \{1, \ldots, q\} : \mathscr{D} \cap \{x \in R^n \mid \langle l^j, x\rangle > \lambda_j\} \neq \emptyset\}$$

$$\mathscr{S}_k(l^k, \lambda_k) =$$

$$= \sup \{(x'\mathbf{C}_k x + c_k' x + \alpha_k)/(x'\mathbf{D}_k x + d_k' x + \beta_k) \mid x \in \mathscr{D}, \langle l^k, x\rangle > \lambda_k\}$$

and

$$\Phi(L, \lambda) = [\mathscr{S}_1(l^1, \lambda_1), \ldots, \mathscr{S}_q(l^q, \lambda_q)] .$$

Then a $T_2$-dual of program (9.1), or more exact, of its modified program, is

$$\text{(9.2)} \qquad \text{find } \operatorname{Inf}^w \bigcup_{(L,\lambda)\in\mathscr{D}^*} v_\Phi(L, \lambda) = I$$

where

$$v_\Phi(L, \lambda) = \Phi(L, \lambda) + R^q_+ .$$

As a consequence of Theorem 6.3 we have

**Theorem 9.1.** If ri $\mathscr{D} \neq \emptyset$, then

$$S \cap R_r = M \cap R_r = I \cap R_r$$

where

$$S = \operatorname{Sup}^w \bigcup_{x\in\mathscr{D}} (Q(x) - R^q_+)$$

$$M = \operatorname{Min}^w \bigcup_{(L,\lambda)\in\mathscr{D}^*} v_\Phi(L, \lambda)$$

and

$$r = (r_1, \ldots, r_q) \quad \text{with} \quad r_k = \begin{cases} 0 & \text{if } D_k \neq 0 \\ -\infty & \text{if } D_k = 0 . \end{cases}$$

Now applying our approach we derive the results obtained by Schaible in [4] for the scalar quadratic fractional programming. Consider the following program

$$\text{(9.3)} \qquad \text{find } \sup_{x\in S} Q(x) = (x'\mathbf{C}x + c'x + \alpha)/(x'\mathbf{D}x + d'x + \beta)$$

$$S = \{x \in S_+ \mid \mathbf{A}x \leqq b\}$$

where $S_+ \subset R^n$ is a nonempty, open and convex set on which the denominator of $Q(x)$ is positive, $\mathbf{C}$, $\mathbf{D}$ are real symmetric $n \times n$ matrices, negative and positive semidefinite respectively, $c, d \in R^n$, $b \in R^m$, $\alpha, \beta \in R$ and $\mathbf{A}$ is an $m \times n$ matrix. Suppose

$$\text{(9.4)} \qquad -\infty < \bar{q} = \sup_{x\in S} Q(x) < +\infty$$

For any $\lambda \in R$ we have

(9.5)
$$\lambda \geqq \bar{q} \Leftrightarrow \sup_{x \in S} [Q(x) - \lambda] \leqq 0 \Leftrightarrow$$
$$\Leftrightarrow \sup_{x \in S} [x'(\mathbf{C} - \lambda \mathbf{D}) x + (c - \lambda d)' x + \alpha - \lambda \beta] \leqq 0$$

**Lemma 9.1.** If

(9.6)
$$\{x \in R^n \mid \mathbf{A}x \leqq b\} \subset S_+$$

then $\lambda \geqq \bar{q}$ if and only if there is $z \in S_+$, $u \in R^m_+$ such that

$$-2(\mathbf{C} - \lambda \mathbf{D}) z + \mathbf{A}'u - (c - \lambda d) = 0$$
$$z'(\mathbf{C} - \lambda \mathbf{D}) z + b'u - (\alpha - \lambda \beta) \geqq 0 .$$

Proof. Since $S$ is a polytope the supremum

(9.7)
$$\sup_{x \in S} [x'(\mathbf{C} - \lambda \mathbf{D}) x + (c - \lambda d)'x]$$

is attained (see [8] Theorem 23, 13, 3, page 217), say at $z \in S$. According to [8] Chapter 8 $z$ is an optimal solution of (9.7) if and only if there is $u \in R^m_+$ such that

$$-2(\mathbf{C} - \lambda \mathbf{D}) z + \mathbf{A}'u - (c - \lambda d) = 0$$

and

$$u'(\mathbf{A}z - b) = 0 .$$

Consequently we have

$$\sup_{x \in S} \{x'(\mathbf{C} - \lambda \mathbf{D}) x + (c - \lambda d)' x + \alpha - \lambda \beta\} = z'(\mathbf{C} - \lambda \mathbf{D}) z +$$
$$+ (c - \lambda d)' z + \alpha - \lambda \beta = z'(\mathbf{C} - \lambda \mathbf{D}) z + z'[-2(\mathbf{C} - \lambda \mathbf{D}) z + \mathbf{A}']u +$$
$$+ \alpha - \lambda \beta = - z'(\mathbf{C} - \lambda \mathbf{D}) z + b'u + \alpha - \lambda \beta$$

Hence in view of (9.5) we obtain the assertion of the lemma. □

On the basis of Lemma 9.1 one can formulate a dual of program (9.3) as following

(9.8)
$$\text{find} \quad \inf \lambda$$
$$\text{subject to} \quad -2(\mathbf{C} - \lambda \mathbf{D}) z + \mathbf{A}'u - (c - \lambda d) = 0$$
$$z'(\mathbf{C} - \lambda \mathbf{D}) z - b'u - (\alpha - \lambda \beta) \geqq 0$$
$$z \in S_+, u \in R^m_+, \lambda \in R$$
$$(\lambda \geqq 0 \quad \text{if} \quad \mathbf{D} \neq 0)$$

For this *dual* we have $\bar{q} = \inf \lambda$.

## 10. APPENDIX: SOME $T_1$-DUALITY THEOREMS

In the appendix, for completeness, we formulate and prove some $T_1$-duality theorems for quasiconcave programming.

**Theorem 10.1.** (Direct $T_1$-Duality Theorem.)

Given a vector quasiconcave program with affine constraints (I) and its $T_1$-*dual* (I*), defined in Section 2 of [2], assume that the set

$$P = \{(z, w) \in Z \times W \mid \exists x \in D : z = G(x), w = F_1(x)\}$$

is a polytope. Then if $\bar{x}$ is an optimal solution of the primal (I), there is an optimal solution $(\bar{z}^*, \bar{w}^*)$ of the $T_1$-dual such that

$$F(\bar{x}) \in L(\bar{z}^*, \bar{w}^*) .$$

Proof. Let $\bar{x}$ be an optimal solution of the primal (I). It means that the set $P$ does not meet the set

$$N = \{(z, w) \in Z \times W \mid z = 0, F_2(w) > F(\bar{x})\}$$

By Theorem 3.3 of [2] there exists an $H \in P^*$ such that

$$P \subset H \;\&\; H \cap N = \emptyset$$

or equivalently

$$H \in Q_y^* \quad \forall y < F(\bar{x})$$

By definition there is a $(\bar{z}^*, \bar{w}^*) \in \mathscr{L}$ such that

$$H = H_{\bar{z}^*, \bar{w}^*, r(\bar{z}^*, \bar{w}^*)} .$$

It is evident that $F(x) \in L(\bar{z}^*, \bar{w}^*)$. The proof is complete. □

**Remark 1.** This direct $T_1$-duality theorem does not generally hold if the constraints are not affine. Consider the example (5.26) of [2].

If $(a, b) \in \Gamma \setminus \Gamma_0$ then $(a, b)$ is obviously an optimal solution of the primal (5.26), but there is no $n$ and $(z, w^*) \in \mathscr{L}_n$ such that

$$\begin{pmatrix} a_3 \\ b^3 \end{pmatrix} \in L(z, w^*) .$$

**Theorem 10.2.** (Converse $T_1$-Duality Theorem.)

Suppose that in the program (I) of Section 4 of [2] the operators $F_2$ and $G$ are continuous, dim $W < +\infty$, $P$ is a polytope and the set $F_1$ is closed. Let $(\bar{z}^*, \bar{w}^*)$ be an optimal solution of the $T_1$-dual (I*). Then if the set

$$M = \{w \in F_1(D) \mid \langle \bar{w}^*, w \rangle \leqq r(\bar{z}^*, \bar{w}^*) \;\&\; F_2(w) \in L(\bar{z}^*, \bar{w}^*) \cap I_{I^*}\}$$

is nonempty anf bounded, there exists an optimal solution $\bar{x}$ of the primal (I) such that

$$F(\bar{x}) \in L(\bar{z}^*, \bar{w}^*) .$$

Proof. Let $(\bar{z}^*, \bar{w}^*)$ be an optimal solution of the dual (I*). We remind that

$$L(\bar{z}^*, \bar{w}^*) = \mathrm{Sup}^w \{\mu_F(w) \mid w \in F_1(D), \langle \bar{w}^*, w \rangle \leqq r(\bar{z}^*, \bar{w}^*)\} .$$

Choose an arbitrary $w_0 \in M$ and a sequence $\{y_n\}_{n=1}^{\infty} \subset Y$ such that

$$y_n < y = F_2(w_0) \quad \forall n \quad \text{and} \quad y_n \uparrow y .$$

Put

$$M_n = \{w \in F_1(D) \mid \langle \bar{w}^*, w \rangle \leqq r(\bar{z}^*, \bar{w}^*) \,\&\, F_2(w) \geqq y_n\}$$

we have $M_n \subset M_{n+1}$ $\forall n$ and

$$\emptyset \neq \bigcap_{n=1}^{\infty} M_n = \{w \in F_1(D) \mid \langle \bar{w}^*, w \rangle \leqq r(\bar{z}^*, \bar{w}^*) \,\&\, F_2(w) \geqq y\} \subset M .$$

Hence in view of the first part of the proof of Lemma 5.9 in [2] there is an integer $N$ such that $N_n$ are bounded for all $n \geqq N$. Since $y \in \mathrm{Sup}^w \bigcup_{x \in D, G(x)=0} \mu_F(x)$ (see Theorem 4.1 of [2]) we have

$$\forall n \quad \exists x_n \in D,\, G(x_n) = 0 \,\&\, F(x_n) \geqq y_n .$$

Put $w_n = F_1(x_n) \in M_n$. Then in view of the boundedness of $M_n$ for all $n \geqq N$ and the closedness of $F_1(D)$ there exists an $\bar{x} \in D$ such that $G(\bar{x}) = 0$ and $F_1(\bar{x}) = \lim_{n \to \infty} w_n$. It is evident that $F(\bar{x}) \geqq y$ and hence $F(\bar{x}) \in L(\bar{x}^*, \bar{w}^*)$. The proof is complete. □

**Remark.** The assumption that the set $M$ is nonempty and bounded cannot be skipped. Consider the following example

$$\sup \{-x_2/x_1 \mid x_1 < 0, x_2 \in R : x_1 \geqq 1, x_2 \geqq 1\} = 0 .$$

Since it is a linear fractional program, one can apply the result of Section 8 and obtain the following dual

$$\begin{aligned} &\text{find} \quad \inf - \lambda \\ \text{subject to} \quad & -z_1 + \lambda = 0 \\ & -z_2 - 1 = 0 \\ & z_1 + z_2 \leqq 0 \\ & z_1 \leqq 0, z_2 \leqq 0 \end{aligned}$$

which is reduced to

$$\begin{aligned} &\inf - \lambda = 0 \\ &\lambda \leqq 1 . \quad \lambda \leqq 0 , \end{aligned}$$

We see that $\lambda = 0$ is the optimal solution of the dual, but the primal has no optimal solution. In this case the corresponding set $M$ is not bounded.

(Received November 1, 1983.)

## REFERENCES


[1] Tran Quoc Chien: Duality in vector optimization. Part I. Abstract duality scheme. Kybernetika *20* (1984), 4, 304—314.

[2] Tran Quoc Chien: Duality in vector optimization. Part II. Vector quasiconcave programming. Kybernetika *20* (1984), 5, 386—404.

[3] S. Schaible: A Survey of Fractional Programming: Generalized Concavity in Optimization and Economics. Wiley, New York 1981.

[4] S. Schaible: Duality in fractional programming: a unified approach. Oper Res. *24* (1976) 3, 452—461.

[5] S. Schaible: Fractional programming I: duality. Management Sci. *22* (1976), 8, 858—867.

[6] S. Schaible: Analyse und Anwendungen von Quotientenprogrammen. Hein-Verlag, Meisenheim 1978.

[7] U. Passy: Pseudoduality in mathematical programs with quotients and ratios. J. Optim. Theory Appl. *33* (1981) 3, 349—374.

[8] B. Martos: Nonlinear Programming, Theory and Methods. Akadémiai Kiadó, Budapest 1975.



*RNDr. Tran Quoc Chien, matematicko-fyzikální fakulta UK (Faculty of Mathematics and Physics — Charles University), Malostranské nám. 25, 118 00 Praha 1. Czechoslovakia.* Permanent address: *Department of Mathematics, Polytechnical School of Da-Nang. Vietnam.*