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ON THE EQUIVALENCE OF TWO METHODS FOR INTERPOLATION

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Two methods for interpolation in stationary discrete processes are presented in the paper. First, the method proposed by Brubacher and Wilson, second Jaglom's method. There is given a proof in the paper that both methods give the same results.

1. INTRODUCTION AND PRELIMINARIES

Let $\{Y_t\}$ be a white noise with $\mathsf{E}Y_t = 0$, $\mathsf{E}Y_t^2 = \sigma^2$, $t = \dots, -1, 0, 1, \dots$, $\mathsf{E}Y_sY_t = 0$, $s \neq t$. Let π_j be real numbers satisfying the condition $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and assume that there exists a linear stationary discrete process $\{X_t\}$ given by

$$Y_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} . \tag{1.1}$$

Assume throughout the paper that the variables X_{s+t_i}, $i = 0, 1, \dots, n$ $(t_0 = 0)$ are missing. Very important problem is to find the best linear interpolation of X_{s+t_i}, $i = 0, 1, \dots, n$. Brubacher and Wilson proposed in [2] to minimize the sum of squares of Y_t with respect to the unknown variables X_{s+t_i}. The method given by Jaglom (see [3] and [4]) is based on the projection in the Hilbert space. It will be proved in the paper that both methods give the same results.

It is well known that the spectral density $f(\lambda)$ of $\{X_t\}$ given by (1.1) has the form

$$f(\lambda) = (\sigma^2/2\pi) \left| \sum_{j=0}^{\infty} \pi_j \, e^{-ij\lambda} \right|^{-2} . \tag{1.2}$$

Introduce numbers p_k by

$$p_k = \begin{cases} \sum_{j=0}^{\infty} \pi_j \pi_{j+k} , & k = 0, 1, 2, \dots, \\ p_{-k} , & k = -1, -2, \dots . \end{cases} \tag{1.3}$$

Introduce an $(n + 1) \times (n + 1)$ symmetric matrix

$$\mathbf{P} = \|p_{t_i - t_j}\|_{i,j=0,1,\ldots,n} .$$

Let $\mathbf{P}$ be regular.

2. BRUBACHER-WILSON'S METHOD

Brubacher and Wilson propose in [2] to minimize

$$Z_m = \sum_{t=-m}^{m} Y_t^2 = \sum_{t=-m}^{m} \left(\sum_{j=0}^{\infty} \pi_j X_{t-j}\right)^2$$

with respect to the unknown variables X_{s+t_i}, $i = 0, 1, \ldots, n$. Using

$$\frac{\partial Z_m}{\partial X_{s+t_i}} = 2 \sum_{t=s+t_i}^{m} \pi_{t-s-t_i} \left(\sum_{j=0}^{\infty} \pi_j X_{t-j}\right) = 0$$

we have for $m \to \infty$

$$(2.1) \qquad \sum_{k=-\infty}^{\infty} p_k X_{s+t_i+k} = 0 , \quad i = 0, 1, \ldots, n .$$

Let $\tilde{X}_{s+t_i}$ be a solution of the linear equations (2.1).

Denote $V_i = - \sum_{\substack{l=-\infty \\ l \neq t_0, t_1, \ldots, t_n}}^{\infty} p_{l-t_i} X_{s+l}$. Then (2.1) can be written in the form

$$\sum_{j=0}^{n} p_{t_j - t_i} \tilde{X}_{s+t_j} = V_i , \quad i = 0, 1, \ldots, n .$$

Remark 1. For $t_i = i$, $i = 0, 1, \ldots, n$ we obtain

$$(2.3) \qquad \sum_{j=0}^{n} p_{j-i} X_{s+j} = - \sum_{j \notin [0,n]} p_{j-i} X_{s+j}$$

which for $n = 0$ can be simplified to

$$\tilde{X}_s = - (1/p_0) \sum_{j \neq 0} p_j X_{s+j} .$$

3. JAGLOM'S METHOD

Let H be the Hilbert space generated by the variables $\{X_t\}_{t=-\infty}^{\infty}$. Let $K = \{s + t_0, s + t_1, \ldots, s + t_n\}$, $0 = t_0 < t_1 < \ldots < t_n$ and let H_K be the subspace of H generated by the variables X_k, $k \notin K$. The best linear interpolation $\hat{X}_{s+t_j}$ of X_{s+t_j} based on X_k, $k \notin K$, is defined as the projection of X_{s+t_j} onto H_K. Let the projection be given by

$$\hat{X}_{s+t_j} = \sum_{\substack{k=-\infty \\ k \neq t_0 - t_j, \ldots, t_n - t_j}}^{\infty} a_k X_{s+t_j+k} .$$

Define

$$\Phi_j(\lambda) = \sum_{\substack{k=-\infty \\ k \neq t_0 - t_j, \dots, t_n - t_j}}^{\infty} a_k \, e^{ik\lambda}, \quad -\pi \leqq \lambda \leqq \pi$$

and

$$\Psi_j(\lambda) = (1 - \Phi_j(\lambda)) f(\lambda), \quad -\pi \leqq \lambda \leqq \pi. \tag{3.1}$$

The function $\Phi_j(\lambda)$ is called the spectral characteristic for interpolation. Since $f(\lambda)$ has the form (1.2) we can introduce the functions

$$f^*(e^{i\lambda}) = f(\lambda), \quad \Phi_j^*(e^{i\lambda}) = \Phi_j(\lambda),$$
$$\Psi_j^*(e^{i\lambda}) = \Psi_j(\lambda).$$

Let $U = \{z; |z| < 1\}$, $\bar{U} = \{z; |z| \leqq 1\}$.

Theorem 1. Let $\Phi_j^*(z) = \Omega_{j,0}^*(z) + \Omega_{j,1}^*(z) + \dots + \Omega_{j,n+1}^*(z)$, $j = 0, 1, \dots, n$, be functions of the complex variable satisfying the following conditions:

a) $\Omega_{j,0}^*(z)$ and $\Omega_{j,n+1}^*(z)$ are analytic functions on U^C and $\bar{U}$, respectively, and

$$\Omega_{j,i}^*(z) = \sum_{k=t_{i-1}-t_j+1}^{t_i - t_j - 1} a_k z^k, \quad i = 1, 2, \dots, n;$$

b) $\lim_{z \to \infty} z^{t_j} \Omega_{j,0}^*(z) = \lim_{z \to 0} z^{t_j - t_n} \Omega_{j,n+1}^*(z) = 0$;

c) function $\Psi_j^*(z)$ can be expressed in the form

$$\Psi_j^*(z) = \sum_{k=0}^{n} c_k z^{t_k - t_j}, \quad c_k \in \mathbb{R}. \tag{3.2}$$

Then the function $\Phi_j(\lambda) = \Phi_j^*(e^{i\lambda})$ is the spectral characteristic for interpolation of X_{s+t_j} $(j = 0, 1, \dots. n)$ based on $\{X_{s+t_j+k}\}$, $k \neq t_0 - t_j, \dots, t_n - t_j$.

Proof. We use a method similar to that given in [1] for the case of extrapolation. From a) and b) we have that $\Omega_{j,0}^*(z)$ and $\Omega_{j,n+1}^*(z)$ can be expressed in the form

$$\Omega_{j,0}^*(z) = \sum_{k=-\infty}^{t_0 - t_j - 1} a_k z^k \quad (z \in U^C), \quad \Omega_{j,n+1}^*(z) = \sum_{k=t_n - t_j + 1}^{\infty} a_k z^k \quad (z \in \bar{U}).$$

Then there exist $d \in (0, 1)$ and $d' \in (1, \infty)$ that both sums converge for $d < |z| < d'$. Thus $\sum_{k=-\infty}^{t_0 - t_j - 1} a_k e^{ik\lambda}$ and $\sum_{k=t_n - t_j + 1}^{\infty} a_k e^{ik\lambda}$ converge in the quadratic mean with respect to $f(\lambda)$.

Denote

$$\hat{X}_{s+t_j}^{(0)} = \int_{-\pi}^{\pi} e^{i(s+t_j)\lambda} \sum_{k=-\infty}^{t_0 - t_j - 1} a_k \, e^{ik\lambda} \, dZ(\lambda),$$

$$\hat{X}_{s+t_j}^{(n+1)} = \int_{-\pi}^{\pi} e^{i(s+t_j)\lambda} \sum_{k=t_n - t_j + 1}^{\infty} a_k \, e^{ik\lambda} \, dZ(\lambda),$$

where $Z(\lambda)$ is the random measure corresponding to $\{X_t\}$ (see [1]). Then we obtain

$$\hat{X}_{s+t_j}^{(0)} = \underset{N \to \infty}{\text{l.i.m.}} \sum_{k=-N}^{t_0 - t_j - 1} a_k X_{s+t_j+k},$$

$$\hat{X}_{s+t_j}^{(n+1)} = \underset{N \to \infty}{\text{l.i.m.}} \sum_{k=t_n - t_j + 1}^{N} a_k X_{s+t_j+k}.$$

Denote

$$\hat{X}^{(i)}_{s+t_j} = \int_{-\pi}^{\pi} e^{i(s+t_j)\lambda}\, \Omega_{j,i}(\lambda)\, dZ(\lambda) = \sum_{k=t_{i-1}-t_j+1}^{t_i-t_j-1} a_k X_{s+t_j+k}, \quad i = 1, 2, \ldots, n .$$

Then

$$\hat{X}^{(0)}_{s+t_j} \in H_K, \quad \hat{X}^{(n+1)}_{s+t_j} \in H_K \quad \text{and} \quad \hat{X}^{(i)}_{s+t_j} \in H_K, \quad i = 1, 2, \ldots, n .$$

Thus

$$\hat{X}_{s+t_j} = \sum_{i=0}^{n+1} \hat{X}^{(i)}_{s+t_j} \in H_K ,$$

$$(X_{s+t_j} - \hat{X}_{s+t_j}, X_{s+k}) = \mathsf{E} \int_{-\pi}^{\pi} e^{i(s+t_j)\lambda} (1 - \Phi_j(\lambda))\, dZ(\lambda) \overline{\int_{-\pi}^{\pi} e^{i(s+k)\lambda}\, dZ(\lambda)} =$$

$$= \int_{-\pi}^{\pi} e^{i(t_j-k)\lambda}\, \Psi_j(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \sum_{l=0}^{n} c_l e^{i(t_l-k)\lambda}\, d\lambda .$$

Hence

$$(X_{s+t_j} - \hat{X}_{s+t_j}, X_{s+k}) = 0 \quad \text{for} \quad k \neq t_0, t_1, \ldots, t_n .$$

This implies $(X_{s+t_j} - \hat{X}_{s+t_j}) \perp H_K$. Thus we have proved that $\hat{X}_{s+t_j}$ is the projection of X_{s+t_j} onto H_K. □

Remark 2. In the special case when $K = \{s\}$ we have

$$\Psi_0^*(z) = (1 - \Phi^*(z)) f^*(z) = c_0 \quad \text{where } c_0 \text{ is a real constant} .$$

Remark 3. For finding the best linear interpolation $\hat{X}_{s+t_j}$ it is sufficient to determine numbers $c_0, c_1, \ldots, c_n$ (depending on j) in (3.2) and then to express $\Phi_j^*(z)$ from (3.1) in the form

$$\Phi_j^*(z) = 1 - (1/f^*(z)) \sum_{k=0}^{n} c_k z^{t_k - t_j} . \tag{3.3}$$

Especially if $K = \{s\}$ then

$$\Phi_0^*(z) = 1 - (c_0/f^*(z)) .$$

Theorem 2. Define a vector $\mathbf{e}_j = (\delta_{0,j}, \ldots, \delta_{n,j})'$, where $\delta_{k,j}$ is Kronecker's δ. Then

$$\hat{X}_{s+t_j} = - \sum_{\substack{l=-\infty \\ l \neq 0, t_1, \ldots, t_n}}^{\infty} X_{s+l}(c_0^* p_l + c_1^* p_{l-t_1} + \ldots + c_n^* p_{l-t_n}) , \tag{3.4}$$

where $\mathbf{c}^* = (c_0^*, c_1^*, \ldots, c_n^*)'$ is a solution of the equations

$$\mathbf{P}\mathbf{c}^* = \mathbf{e}_j , \quad j = 0, 1, \ldots, n . \tag{3.5}$$

Proof. We can write $f(z) = (\sigma^2/2\pi) / \sum_{l=-\infty}^{\infty} p_l z^l$.

Using (3.3) we get further

$$\Phi_j^*(z) = 1 - \sum_{l=-\infty}^{\infty} p_l z^l \sum_{k=0}^{n} c_k^* z^{t_k - t_j} = \tag{3.6}$$

$$= 1 - \sum_{l=-\infty}^{\infty} \sum_{k=0}^{n} p_l c_k^* z^{l + t_k - t_j} ,$$

where $c_k^* = (2\pi/\sigma^2)\, c_k$.

To fulfil the conditions from Theorem 1 the coefficient by $z^{t_i - t_j}$ must be equal to $\delta_{i,j}$. For $k = m$ $(m = 0, 1, \ldots, n)$ and $l = t_i - t_m$ we have $z^{l + t_k - t_j} = z^{t_i - t_j}$ and the coefficient standing by $z^{t_i - t_j}$ is equal to $\sum_{k=0}^{n} c_k^* p_{t_i - t_k}$. Hence we have the linear equations $\sum_{k=0}^{n} c_k^* p_{t_i - t_k} = \delta_{i,j}$ which are equivalent to (3.5). But the formula (3.6) can be written in the form

$$\Phi_j^*(z) = - \sum_{\substack{l=-\infty \\ l \neq t_0 - t_j, \ldots, t_n - t_j}}^{\infty} \sum_{k=0}^{n} p_{l - t_k + t_j} c_k z^l$$

and thus

$$\hat{X}_{s+t_j} = \sum_{\substack{l=-\infty \\ l \neq t_0 - t_j, \ldots, t_n - t_j}}^{\infty} \sum_{k=0}^{n} p_{l - t_k + t_j} c_k^* X_{s + t_j + l} .$$

Substituting $l' = t_j + l$ we obtain (3.4). □

Remark 4. If $K = \{s\}$ we have

$$\hat{X}_s = -(1/p_0) \sum_{k=1}^{\infty} p_k (X_{s+k} + X_{s-k}) .$$

Denote P_{ij}^* the algebraic complement of p_{ij} in the matrix $\mathbf{P}$. Then

$$\hat{X}_{s+t_j} = -(1/\det \mathbf{P}) \sum_{\substack{l=-\infty \\ l \neq 0, t_1, \ldots, t_n}}^{\infty} X_{s+l} \sum_{k=0}^{n} P_{jk}^* p_{l - t_k} . \tag{3.7}$$

Remark 5. Especially for $t_i = i$, $i = 0, 1, \ldots. n$ we get

$$\hat{X}_{s+t_j} = -(1/\det \mathbf{P}) \left[\sum_{l=1}^{\infty} X_{s+n+l} \sum_{k=0}^{n} P_{jk}^* p_{l+n-k} + \sum_{l=1}^{\infty} X_{s-l} \sum_{k=0}^{n} P_{jk}^* p_{l+k} \right] .$$

4. COMPARISON OF BOTH METHODS

In this section we use the notation from the previous sections. The following theorem is the main result of our paper.

Theorem 4. Let $K = \{s + t_0, \ldots, s + t_n\}$. Then $\tilde{X}_{s+t_j} = \hat{X}_{s+t_j}$, $j = 0, 1, \ldots, n$.

Proof. Denote $\tilde{\mathbf{X}} = (\tilde{X}_{s+t_0}, \ldots, \tilde{X}_{s+t_n})'$ and $\mathbf{V} = (V_0, \ldots, V_n)'$. Using the notation from Section 2 we can write (2.2) in the form $\mathbf{P}\tilde{\mathbf{X}} = \mathbf{V}$. Thus $\tilde{\mathbf{X}} = \mathbf{P}^{-1}\mathbf{V}$. From here we obtain

$$\tilde{X}_{s+t_j} = (1/\det \mathbf{P}) \sum_{i=0}^{n} P_{ji}^* V_i .$$

Hence

$$\tilde{X}_{s+t_j} = -(1/\det \mathbf{P}) \sum_{\substack{l=-\infty \\ l \neq t_0, t_1, \ldots, t_n}}^{\infty} X_{s+l} \sum_{i=0}^{n} P_{ji}^* p_{l - t_i} ,$$

which corresponds to (3.7). Thus $\tilde{X}_{s+t_j} = \hat{X}_{s+t_j}$. □

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