Mohsen Razzaghi; M. F. Habibi; R. Fayzebakhsh

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SUBOPTIMAL CONTROL OF LINEAR DELAY SYSTEMS VIA LEGENDRE SERIES

M. RAZZAGHI, M. F. HABIBI AND R. FAYZEBAKHSH

A method for finding the suboptimal control of linear delay systems with a quadratic cost functional using Legendre series is discussed. The state variable, state delay, state rate, and the control vector are expanded in the shifted Legendre series with unknown coefficients. The relation between the coefficients of the state rate with state variable is provided and the necessary condition of optimality is derived as a linear system of algebraic equations. A numerical example is included to demonstrate the validity and the applicability of the technique.

1. INTRODUCTION

The control of systems with time-delay has been of considerable concern. Delays occur frequently in biological, chemical, electronic and transportation systems [1]. Time-delay systems are therefore a very important class of systems whose control and optimization have been of interest to many investigators. The application of Pontryagin's maximum principle to the optimization of control systems with time-delays as outlined by Kharatishvili [2] results in a system of coupled two-point boundary-value (TPBV) problem involving both delay and advance terms whose exact solution, except in very special cases, is very difficult. Therefore, the main object of all computational aspect of optimal time-delays systems has been to devise a methodology to avoid the solution of the mentioned (TPBV) problem.

Inoue et al. [3] have proposed a sensitivity approach to obtain the suboptimal control for linear systems with small delay in the state. They expanded the control in a Maclaurin's series in the delay and obtained the series coefficients from the solution of simple (TPBV) problems. The method presented in [4, 5] are also sensitivity approaches in which the original system is imbedded in a class of non-delay systems using an appropriate parameter.

Recently, orthogonal functions and polynomial series have received considerable attention in dealing with various control problems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem and making it computationally plausible. The approach is based on converting the underlying differential equations into an integral equations through integration, approximating a various signals involved in the equation by truncated orthogonal series and using the operational matrix of integration P, to eliminate the integral operations. Clearly, the form of P depends on the particular choice for the orthogonal functions. Special attention has been given to applications of Walsh functions [6], Block pulse functions [7], Bessel series [8], Laguerre polynomials [9], Chebyshev polynomials [10], [11], Legendre series [12], and Fourier series [13].

The numerical methods for obtaining the optimal control of linear delay systems with a quadratic cost functional has been presented, among others, by Hwang and Shih [7], Chou and Horng [11], Perng [12] and Hwang and Chen [14]. References [7], [11] and [12] used Block pulse, shifted Chebyshev and shifted Legendre operational matrices of integration respectively to calculate the integral involved in the performance index and Reference [14] employed the integration of the product of three shifted Legendre polynomials and the integration of the product of shifted Legendre polynomials and its derivative to obtain the integral in the performance index.

In this paper, the shifted Legendre series is used for the optimal control of linear delay systems with a quadratic cost functional. The state variable $x(t)$, state delay $x(t-\tau)$, state rate $\dot{x}(t)$ and control variable $u(t)$ are expanded in the shifted Legendre series with unknown coefficients. The Legendre properties are used to relate the coefficients of state rate and state delay to the coefficients of state. Using the method, the performance index, system dynamics, and the initial condition are converted to a system of algebraic equations. A method of constrained extremum is applied which consists of adjoining the constraints equations which are derived from the given dynamical system and the initial condition to the performance index by a set of undetermined Lagrange multipliers. As a result the necessary conditions of optimality are derived as a system of linear algebraic equations in the unknown coefficients of $x(t)$, $u(t)$ and the Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed. As compared to Perng [12] and Hwang and Chen [14] the present method is simpler to use. An illustrative example is given to demonstrate the applicability of the proposed method.

2. PROPERTIES OF SHIFTED LEGENDRE POLYNOMIALS

The shifted Legendre polynomials, $P_n(t)$, where $0 \leq t \leq h$ are obtained from [15],

$$P_{i+1}(t) = \frac{2i+1}{i+1}\left(2\frac{t}{h}-1\right)P_i(t) - \frac{i}{i+1}P_{i-1}(t), \quad i \geq 1 \tag{1}$$

with

$$P_0(t) = 1 \tag{2}$$

$$P_1(t) = 2\frac{t}{h} - 1 \tag{3}$$

The orthogonality property is given by

$$\int_0^h P_i(t)P_j(t)\,\mathrm{d}t = \begin{cases} 0, & i \neq j \\ \frac{h}{2i+1}, & i = j. \end{cases} \tag{4}$$

Further a function, $f(t)$, which is absolutely integrable within $0 \leq t \leq h$ may be expressed in terms of shifted Legendre series as

$$f(t) = \sum_{i=0}^{\infty} f_i P_i(t) \tag{5}$$

where

$$f_i = \frac{(2i+1)}{h} \int_0^h f(t)P_i(t)\,\mathrm{d}t \tag{6}$$

if Eq. (5) is truncated up to its first m terms, then

$$f(t) \approx \sum_{i=0}^{m-1} f_i P_i(t) = f^{\mathrm{T}} P(t) \tag{7}$$

where

$$f^{\mathrm{T}} = [f_0, f_1, \ldots, f_{m-1}] \tag{8}$$

$$P^{\mathrm{T}}(t) = [P_0(t), P_1(t), \ldots, P_{m-1}(t)]. \tag{9}$$

If we assume that the derivative of $f(t)$ in (5) be described by

$$\dot{f}(t) = \sum_{i=0}^{\infty} g_i P_i(t) \tag{10}$$

then, using the recurrence formula

$$P_i(t) = \frac{h}{2(2i+1)} [\dot{P}_{i+1}(t) - \dot{P}_{i-1}(t)] \tag{11}$$

the relationship between the coefficients f_i in Eq. (5) and g_i in Eq. (10) can be obtained from [16]

$$f_i = \frac{h}{2} \left[\frac{g_{i-1}}{(2i-1)} - \frac{g_{i+1}}{(2i+3)} \right] \quad i = 1, 2, \ldots \tag{12}$$

Also, if $f(t)$ in (7) has its initial function for $t < 0$ as

$$f(t) = f_1(t) \quad -\tau \leq t < 0 \tag{13}$$

then the delay function $f(t-\tau)$ can be expressed by [12]

$$f(t-\tau) = [f^{\mathrm{T}} D(\tau) + G^{\mathrm{T}}(\tau)]\, P(t) \tag{14}$$

where $D(\tau)$ is an $m \times m$ matrix and is given in [12] and $G(\tau)$ is an m vector given by

$$G^{\mathrm{T}}(\tau) = [G_0(\tau),\, G_1(\tau), \ldots, G_{m-1}(\tau)]$$

where

$$G_i(\tau) = \frac{2i+1}{h} \int_0^{\tau} f_1(t-\tau)P_i(t)\,\mathrm{d}t, \quad i = 0, 1, \ldots, m-1. \tag{15}$$

3. PROBLEM STATEMENT

Consider the following class of linear systems with time-delay

$$\dot{x}(t) = Ax(t) + Bu(t) + Cx(t-\tau) \tag{16}$$

$$\begin{aligned} x(0) &= x_0 \\ x(t) &= x_1(t) \quad -\tau \le t < 0 \end{aligned}$$

where $x(t)$ and $u(t)$ are $n \times 1$ state and control vectors, respectively, A and B are matrices of appropriate dimensions and τ is the time-delay. The problem is to find the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, $0 \le t \le h$, satisfying (16) while minimizing the quadratic cost functional

$$J = \frac{1}{2}x^{\mathrm{T}}(h)Sx(h) + \frac{1}{2}\int_0^h [x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\,\mathrm{d}t \tag{17}$$

where T denotes transposition, S, Q, and R are matrices of appropriate dimensions, S and Q are symmetric positive semi-definite matrices and R is a symmetric positive definite matrix.

4. THE PERFORMANCE INDEX APPROXIMATION

By expanding each state vector and each control vector in shifted Legendre series of order m, we determine the following approximate solutions, i .e., for $N = 0, 1, \ldots, n-1$

$$x_N(t) = \sum_{i=0}^{m-1} a_{Ni}P_i(t) \tag{18}$$

$$u_N(t) = \sum_{i=0}^{m-1} b_{Ni}P_i(t) \tag{19}$$

where $(a_{N0}, a_{N1}, \ldots, a_{N(m-1)})$ and $(b_{N0}, b_{N1}, \ldots, b_{N(m-1)})$ are unknown.

Let

$$\alpha = (a_0\ a_1\ \ldots\ a_{m-1})^{\mathrm{T}} = \begin{pmatrix} (a_{00} & a_{01} & \cdots & a_{0(m-1)})^{\mathrm{T}} \\ \vdots & & & \\ (a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)(m-1)})^{\mathrm{T}} \end{pmatrix}, \tag{20}$$

$$\beta = (b_0\ b_1\ \ldots\ b_{m-1})^{\mathrm{T}} = \begin{pmatrix} (b_{00} & b_{01} & \cdots & b_{0(m-1)})^{\mathrm{T}} \\ \vdots & & & \\ (b_{(n-1)0} & b_{(n-1)1} & \cdots & b_{(n-1)(m-1)})^{\mathrm{T}} \end{pmatrix}, \tag{21}$$

and

$$\hat{P}(t) = \begin{pmatrix} P^{\mathrm{T}}(t) & & 0 \\ & \ddots & \\ 0 & & P^{\mathrm{T}}(t) \end{pmatrix} \tag{22}$$

Note that α, β, and $\hat{P}(t)$ are matrices of order $nm\times 1$, $nm\times 1$, and $n\times nm$ respectively.

Then using (18) and (19) the state and control vector can be expressed as

$$x(t) = \hat{P}(t)\alpha \tag{23}$$

$$u(t) = \hat{P}(t)\beta \tag{24}$$

substituting (23) and (24) in (17) we get

$$J = \frac{1}{2}\alpha^{\mathrm{T}}\hat{P}^{\mathrm{T}}(h)S\hat{P}(h)\alpha + \frac{1}{2}\alpha^{\mathrm{T}}\left[\int_0^h \hat{P}^{\mathrm{T}}Q\hat{P}(t)\,\mathrm{d}t\right]\alpha + \frac{1}{2}\beta^{\mathrm{T}}\left[\int_0^h \hat{P}^{\mathrm{T}}(t)R\hat{P}(t)\,\mathrm{d}t\right]\beta \tag{25}$$

Equation (25) can be computed more efficiently by writing J as

$$J = \frac{1}{2}\alpha^{\mathrm{T}}[P(h)P^{\mathrm{T}}(h)\otimes S]\alpha + \frac{1}{2}\alpha^{\mathrm{T}}(D\otimes Q)\alpha + \frac{1}{2}\beta^{\mathrm{T}}(D\otimes R)\beta \tag{26}$$

where

$$D = \int_0^h P(t)P^{\mathrm{T}}(t)\,\mathrm{d}t = h\left(\mathrm{diag}\left[1, \frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2m-1}\right]\right). \tag{27}$$

In (26), $\otimes$ denotes Kronecker product [17].

5. APPROXIMATION OF THE TIME DELAY SYSTEM

By expanding the derivative of each of the n state vector in equation (16) by shifted Legendre series, we get

$$\dot{x}_N = \sum_{i=0}^{m-2} C_{N_i}P_i(t), \quad N = 0, 1, \dots, n-1 \tag{28}$$

Let

$$(A(x(t))_N = \sum_{i=0}^{m-1} y_{Ni}P_i(t) \tag{29}$$

$$(Bu(t))_N = \sum_{i=0}^{m-1} z_{Ni}P_i(t) \tag{30}$$

$$(Cx(t-\tau))_N = \sum_{i=0}^{m-1} w_{Ni}P_i(t) \tag{31}$$

Using (29) – (31) for each N, $N = 0, \ldots, n-1$, the right hand side of (16) has the form

$$\sum_{i=0}^{m-1} (y_{Ni} + z_{Ni} + w_{Ni}) P_i(t) \tag{32}$$

which is a polynomial of degree $m-1$ while the left hand side is a polynomial of degree $m-2$. By equating the coefficients of same-order shifted Legendre polynomials, we obtain

$$y_{Ni} + z_{Ni} + w_{Ni} = \begin{cases} C_{Ni}, & i = 0, 1, \ldots, m-2 \\ 0 & i \geq m-1 \end{cases} \tag{33}$$

Equation (12), (18), and (33) give the following relationship

$$\begin{aligned} F_{i-1} = h\left[(2i+3)C_{N(i-1)} - (2i-1)C_{N(i+1)}\right] \\ - 2(2i-1)(2i+3)a_{Ni} = 0, \qquad i = 1, 2, \ldots, m-1 \end{aligned} \tag{34}$$

$$F_{i-1} = (2i+3)C_{N(i-1)} - (2i-1)C_{N(i+1)} = 0, \qquad \text{for } i \geq m, \tag{35}$$

with

$$C_{N(m-1)} = C_{N(m)} = 0 \tag{36}$$

Using (18), the initial condition $x(0) = x_0$, can be replaced by

$$F_m = \sum_{i=0}^{m-1} a_{Ni} P_i(0) = \sum_{i=0}^{m-1} (-1)^i a_{Ni} = x_N(0), \qquad N = 0, 1, \ldots, n-1 \tag{37}$$

Further the relation between w_{Ni} and a_{Ni} can be obtained by using (14).

6. THE SHIFTED LEGENDRE COEFFICIENTS FOR $x(t)$ AND $u(t)$

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find α and β so that $J(\alpha, \beta)$ is minimized subject to the constraints (34) – (37).

Let

$$L(\alpha, \beta) = J(\alpha, \beta) + \sum_{j=0}^{m} \lambda_j F_j(\alpha, \beta) \tag{38}$$

where $\lambda = (\lambda_0, \lambda_1, \ldots \lambda_m)$ represents the unknown Lagrange multipliers, then the necessary conditions for stationarity are given by

$$\frac{\partial L}{\partial a_i} = \frac{\partial J}{\partial a_i} + \sum_{j=0}^{m} \lambda_j \frac{\partial F_j}{\partial a_i} = 0, \quad i = 0, 1, \ldots, m-1 \tag{39}$$

$$\frac{\partial L}{\partial b_i} = \frac{\partial J}{\partial b_i} + \sum_{j=0}^{m} \lambda_j \frac{\partial F_j}{\partial b_i} = 0, \quad i = 0, 1, \ldots, m-1 \tag{40}$$

$$F_j = 0, \qquad j = 0, 1, \ldots, m \tag{41}$$

7. ILLUSTRATIVE EXAMPLE

Consider the linear system with time delay

$$\dot{x}(t) = u(t) + x(t-1) \quad 0 \le t \le 2 \tag{42}$$

$$x(t) = 1 \quad -1 \le t < 0 \tag{43}$$

with the cost functional

$$J = \frac{1}{2}\left[10^5 x^2(2) + \int_0^2 u^2(t)\,\mathrm{d}t\right]. \tag{44}$$

The problem is to find the optimal control $u(t)$ which minimizes (44) subject to (42) and (43). We determine the shifted Legendre approximation for $m = 6$.

Let

$$x(t) = \sum_{i=0}^{5} a_i P_i(t) = a^{\mathrm{T}} P(t) \tag{45}$$

$$u(t) = \sum_{i=0}^{5} b_i P_i(t) = b^{\mathrm{T}} P(t) \tag{46}$$

$$x(t-1) = \sum_{i=0}^{5} d_i P_i(t) = d^{\mathrm{T}} P(t) \tag{47}$$

$$\dot{x}(t) = \sum_{i=0}^{4} c_i P_i(t) = c^{\mathrm{T}} P(t) \tag{48}$$

using (33) and (42) we have

$$c_i = b_i + d_i \quad i = 0, 1, \ldots, 4 \tag{49}$$

$$c_5 = b_5 + d_5 = 0 \tag{50}$$

Further, using (14) we obtain

$$a^{\mathrm{T}} D(\tau) + G(\tau) = d^{\mathrm{T}} \tag{51}$$

where, $\tau = 1$,

$$D(\tau) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & 0 & -\frac{7}{16} & 0 & \frac{11}{32} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{5}{16} & \frac{7}{16} & -\frac{3}{32} & -\frac{11}{32} \\ 0 & -\frac{3}{16} & -\frac{7}{16} & -\frac{7}{32} & \frac{9}{32} & \frac{77}{256} \\ \frac{1}{16} & \frac{3}{16} & \frac{5}{32} & -\frac{5}{32} & -\frac{99}{256} & -\frac{33}{256} \\ 0 & \frac{1}{32} & \frac{5}{32} & \frac{77}{256} & \frac{53}{256} & -\frac{77}{512} \\ -\frac{1}{32} & -\frac{3}{32} & -\frac{35}{256} & -\frac{21}{256} & \frac{63}{512} & \frac{157}{512} \end{pmatrix}, \tag{52}$$

and

$$G(\tau) = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & 0 & \frac{7}{16} & 0 & \frac{11}{32} \end{pmatrix}^{\mathrm{T}} \tag{53}$$

By applying (33)−(37) the unknown coefficients must satisfy the constraints

$$\begin{aligned} F_0 &= 10c_0 - 2c_2 - 10a_1 = 0 \\ F_1 &= 14c_1 - 6c_3 - 42a_2 = 0 \\ F_2 &= 18c_3 - 10c_4 - 90a_3 = 0 \\ F_3 &= 22c_3 - 14c_5 - 254a_4 = 0 \\ F_4 &= 26c_4 - 234a_5 = 0 \\ F_5 &= 15c^5 = 0 \end{aligned} \tag{54}$$

$$F_6 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 - 1 = 0 \tag{55}$$

Using (26) we obtain the following approximation for J

$$J = \frac{1}{2}\left[10^5 \sum_{i=1}^{5} a_i^2 + \sum_{i=1}^{5} \frac{2}{2i+1} b_i^2\right] \tag{56}$$

Equations (39)−(41) give 19 equations from which $x(t)$ and $u(t)$ in (45) and (46) can be calculated.

In Table (1), a comparison is made between the values of $x(t)$ and $u(t)$ using present method with $m = 6$, method of [11] with $m = 8$ and the exact solution.

Table 1. Estimated and Exact values of $x(t)$ and $u(t)$.

	$x(t)$			$u(t)$		
t	Method of [11] $m = 8$	Present $m = 6$	Exact	Method of [11] $m = 8$	Present $m = 6$	Exact
0.0	1.000043	1.000000	1.000000	-2.114431	-2.108100	-2.100000
0.2	0.800846	0.801121	0.801000	−1.893601	−1.890831	−1.890000
0.4	0.644449	0.644048	0.644000	−1.676797	−1.679410	−1.680000
0.6	0.528564	0.529163	0.529000	−1.475604	−1.475431	−1.470000
0.8	0.456059	0.456019	0.456000	−1.250573	−1.257838	−1.260000
1.0	0.424890	0.424948	0.425000	−1.078828	−1.055281	−1.050000
1.2	0.394360	0.394385	0.394400	−1.040670	−1.052383	−1.050000
1.4	0.328484	0.328397	0.328200	−1.057799	−1.054334	−1.050000
1.6	0.234327	0.234542	0.234800	−1.046507	−1.051482	−1.050000
1.8	0.122659	0.122576	0.122600	−1.053616	−1.053163	−1.050000
2.0	0.000182	0.000010	0.000000	−1.064356	−1.054316	−1.050000

8. CONCLUSIONS

In the present work, a technique has been developed for obtaining the optimal control of linear delay systems with a quadratic cost functional using shifted Legendre polynomials. The method is based upon reducing a linear delay quadratic optimization problem to a set of linear equations. The unity of the weight function of orthogonality for shifted Legendre series and the simplicity of the approximated performance index are merits that make the approach very attractive. Moreover, only a small number of shifted Legendre series are needed to obtain a satisfactory solution. The given numerical example supports this claim.

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Dr. M. Razzaghi, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, U.S.A. and Department of Mathematics, Amirkabir University, Tehran, Iran.

Dr. M. F. Habibi and Ms. R. Fayzebakhsh, Department of Physics, Amirkabir University, Tehran. Iran.